13.1   FUNCTIONS OF TWO OR MORE VARIABLES

This section presents many of the precalculus concepts relating to functions of several variables, and the focus is on using formulas and tables of data to create and interpret graphical representations of functions of two variables.

Definition:

> A **function f of two variables** is a rule that assigns to each ordered pair (x, y) in the domain of the function a unique real number z. This can be written $z = f(x,y)$.

As with a function of one variable, a function of two variables is typically given by a table of values, a graph, or a formula.

Tables of Values

If we have data or perform measurements about the elevation of the ground (above sea level) at several locations, it is natural to record the elevation measurements using several columns of numbers as in Fig. 1.

Example 1:   Describe the progression of elevations for the data in Fig. 1 if we start at the location $(2,1)$, keep the x–values constant at $x = 2$, and proceed to increase the values of y: list the values of $f(2,1), f(2,2), f(2,3), f(2,4),$ ...

Solution: $f(2,1) = 4.3, f(2,2) = 5.6, f(2,3) = 6.7,$
    $f(2,4) = 7.1, f(2,5) = 6.7,$ and $f(2,6) = 5.6$. As y increases from 1 to 4, the z–values increase to a maximum of $z = 7.1$. As the y–values increase from 4 to 6, the z–values then decrease.

Practice 1:   Describe the progression of elevations for the data in Fig. 1 if we start at the location $(2,1)$, keep the y–values constant at $y = 1$, and proceed to increase the values of x: list the values of $f(2,1), f(3,1), f(4,1), f(5,1),$ ...

x	y	z		x	y	z		x	y	z
0	0	1.6		3	0	3.8		6	0	1.6
0	1	1.8		3	1	5.3		6	1	1.8
0	2	2.0		3	2	7.1		6	2	2.0
0	3	2.1		3	3	9.1		6	3	2.1
0	4	2.2		3	4	10.0		6	4	2.2
0	5	2.1		3	5	9.1		6	5	2.1
0	6	2.0		3	6	7.1		6	6	2.0
1	0	2.4		4	0	3.3				
1	1	2.9		4	1	4.3				
1	2	3.3		4	2	5.6				
1	3	3.7		4	3	6.7				
1	4	5.9		4	4	7.1				
1	5	3.7		4	5	6.7				
1	6	3.3		4	6	5.6				
2	0	3.3		5	0	2.4				
2	1	4.3		5	1	2.9				
2	2	5.6		5	2	3.3				
2	3	6.7		5	3	3.7				
2	4	7.1		5	4	3.8				
2	5	6.7		5	5	3.7				
2	6	5.6		5	6	3.3				

Fig. 1: z is the elevation at location (x,y)

Many people find it difficult to recognize patterns and shapes from lists of numbers, and it is common to arrange the information in a table as in Fig. 2. With this table arrangement it is easier to answer the questions in Example 1 and Practice 1 as well as more complicated questions.

Example 2:   Describe the progression of elevations

for the data in Fig. 2 if we start at the location (2,1) and

move "southeast" (x and y both increase at the same rate):

list the values of f(2,1),

f(3,2), f(4,3), f(5,4), ...

y x	0	1	2	3	4	5	6
0	1.6	1.8	2.0	2.1	2.2	2.1	2.0
1	2.4	2.9	3.3	3.7	5.9	3.7	3.3
2	3.3	4.3	5.6	6.7	7.1	6.7	5.6
3	3.8	5.3	7.1	9.1	9.9	9.1	7.1
4	3.3	4.3	5.6	6.7	7.1	6.7	5.6
5	2.4	2.9	3.3	3.7	3.8	3.7	3.3
6	1.6	1.8	2.0	2.1	2.2	2.1	2.0

Solution: f(2,1) =4.3, f(3,2) = 7.1, f(4,3) = 6.7,

f(5,4) = 3.8, and f(6,5) = 2.1 .

Fig. 2: Elevations arranged in a table

Graphs

Once the data is arranged in a table, it seems natural to create a partial graph of the function by plotting the

elevations as points at the appropriate elevations above the xy–plane, but such a point graph is usually

difficult to "read." Instead of a point graph, we could build a surface by plotting a small platform at each point

(Fig. 3) or by connecting the points to nearby points (Fig. 4).

For a function f of two variables, the graph of the surface defined by f is the set of points $z = f(x, y)$ where

the ordered pairs (x,y) are in the domain of f.
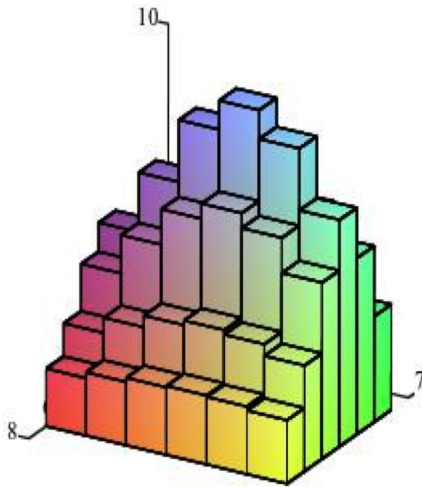
Fig. 3: A "graph" of the data from Fig. 2
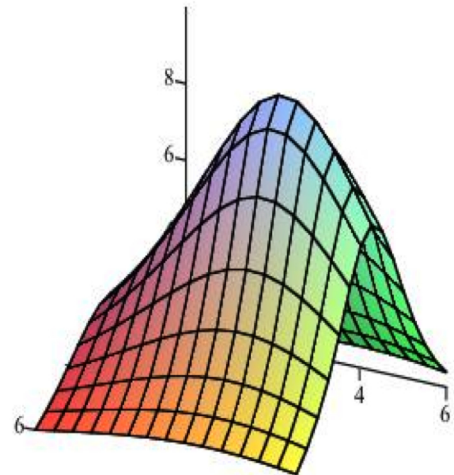
Fig. 4: A surface from the data in Fig. 2

Note: Figures 3 and 4 were drawn by computer using the language Maple. The Maple commands for drawing

these and the other 3–dimensional surfaces in this section are given after the problem set.

If we "slice" the surface in Fig. 4 with the plane $z = 8$,
the points where the plane cuts the surface are those points
where the elevation of the surface is 8 units above the
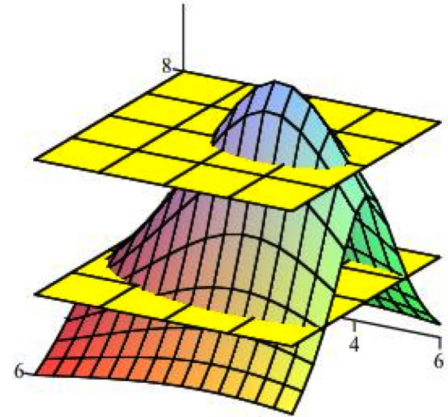xy–plane. Fig. 5 shows the surface being sliced by the
planes $z = 8$ and $z = 4$.

Fig. 5: The planes z=4 and z=8 "slicing" the surface

Fig. 6 shows the results of slicing the surface with
several planes at different elevations. The thicker curves
are those points where the planes $z = 4$ and $z = 8$
intersect the surface.

Definition:

    The **k level curve** of a function f of two variables is
    the set of points (x,y) that satisfy the equation
    $f(x,y) = k$, where k is a constant.
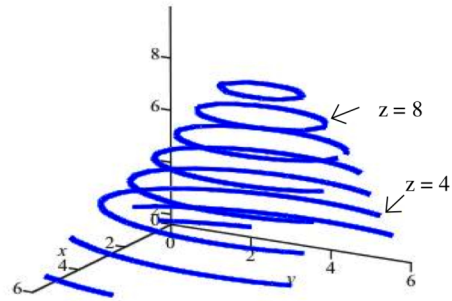
Fig. 6: Intersections of the surface and several planes

If we move all of the curves in Fig. 6 to the xy–plane
(or, equivalently, view them from directly overhead), the result
is a 2–dimensional graph of the level curves
of the original surface. A graph of several level curves of the
surface is shown in Fig. 7. We call such a graph a **level curve
graph of f** or a **contour graph of f.**

Fig. 7: Several level curves of the surface in Fig. 3

Level curve graphs are an effective and efficient method of
presenting a great deal of information about a surface or function
of two variables in a 2–dimensional way, and such graphs are very commonly used. Weather maps showing
temperatures or barometric pressures over a region use a variation of contour graphs (Fig. 8). Hikers are
familiar with topographic maps that show elevations of the terrain over which they plan to hike (Fig. 9). And
almost every issue of scientific journals such as *Science* or *Nature* contains a variety of surface and contour
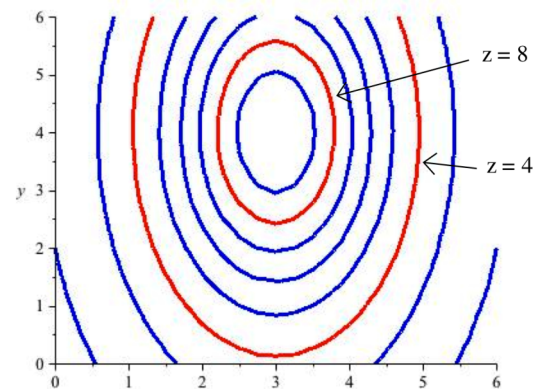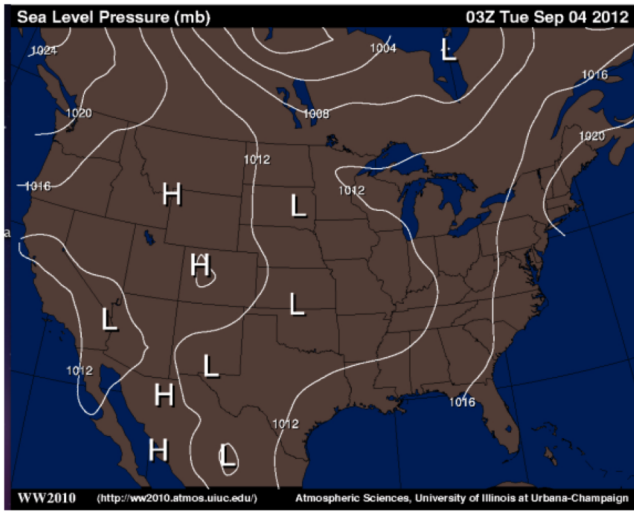graphs.

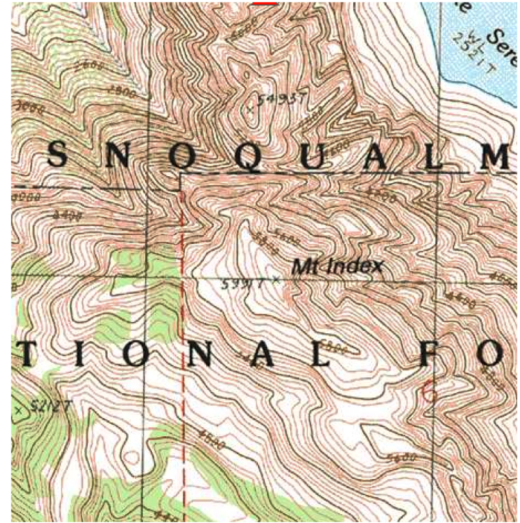Fig. 8: Baromatric pressure map showing isobars

Fig. 9: Topographic map showing level contours

Practice 2:   Fig. 10 shows a level curve graph of the temperature at each location on a table. On Fig. 11 sketch the a graph of the temperatures a bug experiences as it moves from point A to point B along a straight line. On Fig. 12 sketch the a graph of the temperatures a bug experiences as it moves from point C clockwise along the ellipse and returns to point C.
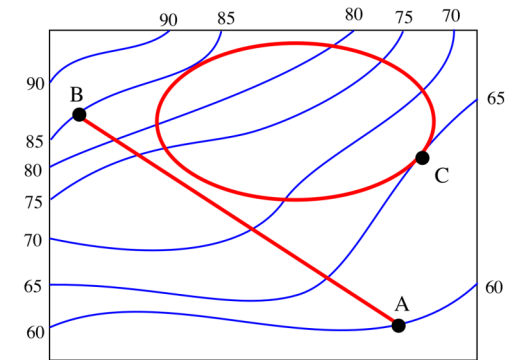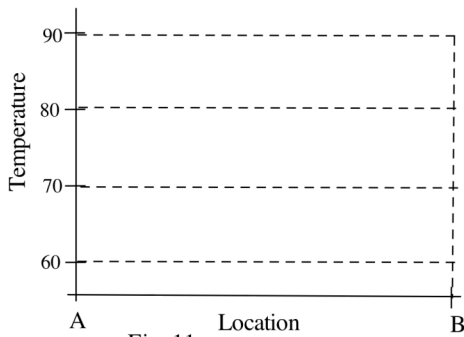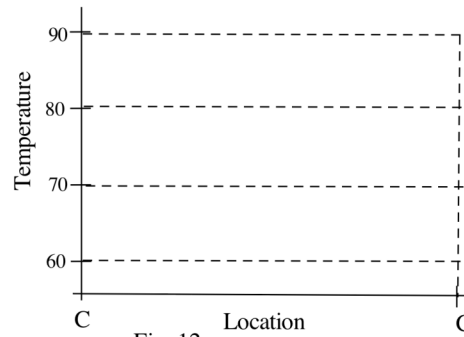

Fig. 10: Temperatures on a table

Fig. 11

Fig. 12

Sketching Level Curves of a Surface From Data

In general, it is difficult to sketch good surface graphs (such as Fig. 4) from data or even from an equation for a surface, and such surface graphs are typically done by computers. It is much easier, however, to sketch level curves of a surface.

If our surface is described by data (by z–values at given locations (x,y)) and if we can assume that the surface does not have any holes or jumps, then there is a straightforward method for sketching crude level curves for the surface.

"Crude" Level Curve z = k Algorithm for Surface Data (Fig. 13)

   Step 1:   "Triangulate" the data locations by connecting adjacent locations with light (dotted) lines. The triangulation should be done so no dotted lines cross (but they can meet at the data locations).

   Step 2:   Plot a point (or small box) on each dotted line segment whose endpoints "surround" the value k (one endpoint is larger than k and one endpoint is smaller).

   Step 3:   In each triangle, connect the boxes with a line segment.

   Step 4:   Remove the light (dotted) lines.
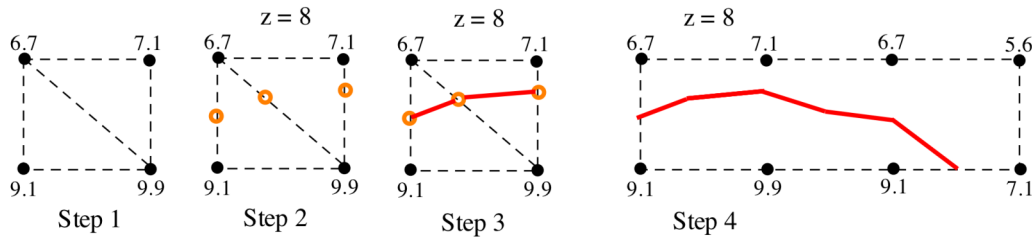
Fig. 13: Steps for drawing a (crude) level curve z=8 from data

Fig. 14 shows the result when this algorithm is used in the data from Fig. 2 to approximate the level curves z = 4 and z = 8.

x \ y	0	1	2	3	4	5	6
0	1.6	1.8	2.0	2.1	2.2	2.1	2.0
1	2.4	2.9	3.3	3.7	5.9	3.7	3.3
2	3.3	4.3	5.6	6.7	7.1	6.7	5.6
3	3.8	5.3	7.1	9.1	9.9	9.1	7.1
4	3.3	4.3	5.6	6.7	7.1	6.7	5.6
5	2.4	2.9	3.3	3.7	3.8	3.7	3.3
6	1.6	1.8	2.0	2.1	2.2	2.1	2.0

Fig. 14: Crude level curves z=4 and z=8 from data

If the data points are far apart, the resulting level curves may be only crude approximations of the actual level curves of the surface. However, if the data points are relatively close together and if the surface does not change elevation rapidly, then the algorithm results in a good approximation of the actual level curves of the surface. Many computer programs use variations of this algorithm to create level curves of surfaces.

Note: After a bit of practice most people don't bother actually sketching the dotted lines, but they still need to "think triangles."

It is possible to work with functions of three or more variables, but they are difficult (or impossible) to graph. The idea of a level curve for a function of 2 variables generalizes to a **level surface** for a function of 3 variables.

Sketching Level Curves of a Surface From A Formula

If the surface is described by a formula of the form $z = f(x,y)$,
then we can often use algebra and our 2–dimensional graphing
skills to create level curves for the surface.

Example 3:   Sketch the level curve $z = 4$ for the surface $z = x^2 + 4y^2$ (Fig. 15).

Solution: The graph of the level curve $z = 4$ is the

graph of $4 = x^2 + 4y^2$ or $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$.

From Chapter 9, we know that the graph of

$\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$ is an ellipse. The graph of

the level curve $z = 4$ and other level curves are

shown in Fig. 16.

Practice 3:   Sketch the level curves $z = 4$ and $z = -8$

for the surface $z = xy$.

Fig. 15: $z = x^2 + 4y^2$ surface

Fig. 16: $z = x^2 + 4y^2$ contours

Functions of Three Variables: $w = f(x, y, z)$

A function f of three variables is a rule that assigns to each ordered
triple (x, y, z) in the domain of the function a unique real number w.
This can be written $w = f(x,y,z)$. For example, the temperature w at
each location (x,y,z) in a classroom is a function of the three
variables $x, y,$ and z that specify the location. The cost of a new car
is a function of the make of the car, the options included in the car, and
the time of year (and probably several more variables). Unfortunately,
the graph of $w = f(x,y,z)$ requires four axes and four dimensions.
However, a level surface of $w = f(x,y,z)$ is the set of points (x,y,z)
such that $w = k$ is constant, and a level surface only requires three
dimensions. The $w = 75^{o}$ level surface in a classroom
is the set of locations (x,y,z) at which the temperature is 75°, and
sometimes we can graph such level surfaces (Fig. 17).

We will examine how to differentiate and integrate functions
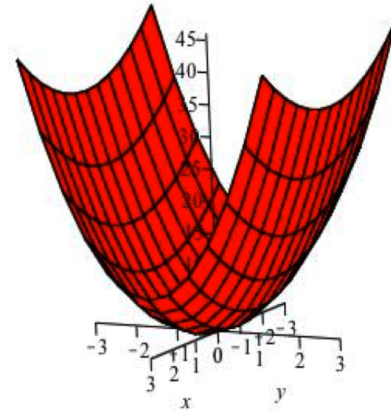of three or more variables, but we will do very little with their graphs.
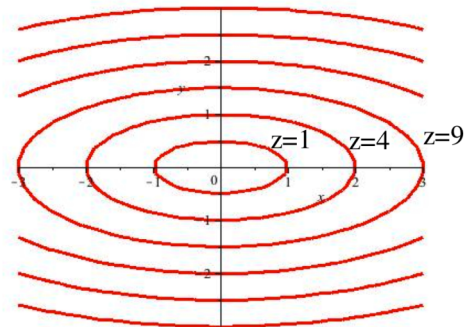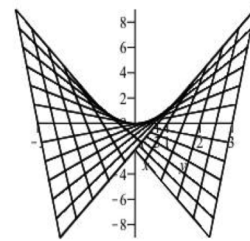

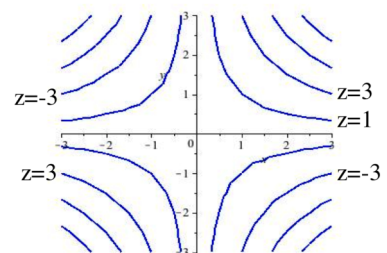
Fig. 17: $z = xy$ surface

Fig. 18: $z = xy$ contours

Problems

1. The table of data in Fig. 19 shows the number thousands of gallons of drinks sold at a sports stadium as a function of the temperature at the beginning of the game and the number of people attending the game.

   (a) What are the minimum and maximum number of thousands of gallons sold?

   (b) When the attendance is 30,000 people, describe what happens to sales as the temperature increases from 50° to 90°.

   (c) When the temperature is 90°, describe what happens to sales as the attendance increases from 10,000 to 60,000.

	Temperature (^oF)				
Attendance (1000s)	50	60	70	80	90
10	5	2	2	5	5
20	7	5	3	8	10
30	10	8	6	15	20
40	12	10	12	20	25
50	15	12	15	25	30
60	20	15	20	30	30

Fig. 19: Gallons (1000s) of drinks sold

2. The table of data in Fig. 20 shows the usual gas mileage (miles per gallon) for a truck hauling different loads and traveling at different speeds.

   (a) What are the minimum and maximum mileages for the truck?

   (b) When the truck has a load of 15,000 pounds, describe what happens to the mileage as the speed increases from 20 miles per hour (mph) to 60 mph.

   (c) When the truck is traveling at 50 mph, describe what happens to the mileage as the weight of the load varies from 20,000 pounds to 5,000 pounds.

	Speed (miles per hour)				
Weight of load (pounds)	20	30	40	50	60
5,000	14	18	20	22	20
10,000	11	14	17	19	17
15,000	9	12	16	17	13
20,000	7	11	15	14	10

Fig. 20: Gas Mileage of a Truck (miles per gallon)

3. Fig. 21 shows the depth of a lake at several locations.

   (a) What is the maximum depth of the lake in this region?

   (b) Describe the changing depth a fish swimming along the bottom would experience if the fish started at the location (2,1) and swam east.

   (c) Describe the changing depth a fish swimming along the bottom would experience if the fish started at the location (2,1) and swam south.

East →					
x \ y	1	2	3	4	5
1	3	4	5	4	3
2	5	8	4	5	4
3	3	5	6	8	6
4	2	4	7	10	7
5	1	3	5	8	5

(South ↓)

Fig. 21: Depth of a lake (meters)

4. Fig. 22 shows a map of several level elevation curves.

    (a) What is the maximum elevation in this region?

    (b) Sketch a graph of the elevation of a hiker moving
         along the straight path from A to B.

    (c) Sketch a graph of the elevation of a hiker moving
         along the curved path from C to D.

    (d) Sketch a path from E to F so that the
         path is never very steep.

    (e) At the point A, which directions can the
         hiker move (a short distance) without
         changing elevation?
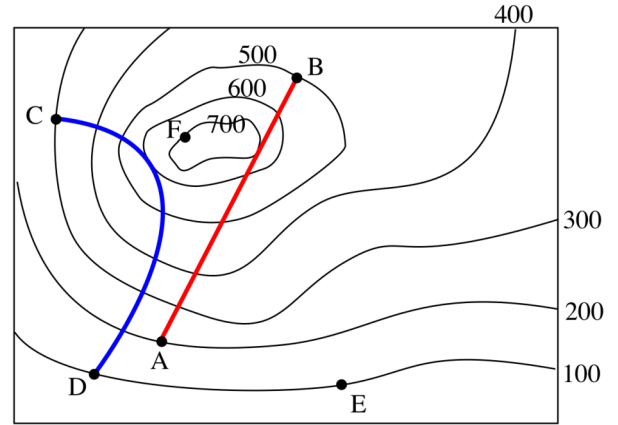
Fig. 22: Level elevation curves (topographic map)

5. Fig. 23 shows the nutrient concentration levels for a
    region at the bottom of the ocean.

    (a) Label the location of the highest nutrient
         concentration with an X.

    (b) Sketch a graph of the nutrient concentration level for
         an animal moving along the path from A to B.

    (c) Sketch a graph of the nutrient concentration level
         for an animal moving along the curved path from
         C to D.
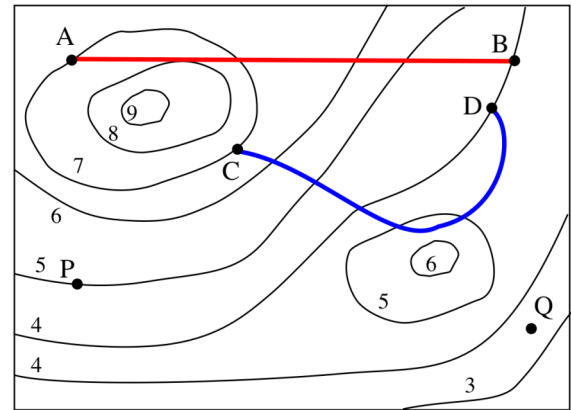
Fig. 23: Nutrient concentrations (parts per million)

    (d) Suppose an animal can sense the nutrient levels in
         the water near its location and always moves to increase the nutrient level. For an animal that starts
         at the point P, sketch several nutrient–increasing paths for the animal. Along which path does the
         nutrient concentration level seem to increase most rapidly?

    (e) Suppose the animal at location Q is at the nutrient concentration level that is best for it.
         Sketch an approximate path for the animal to move to stay at this nutrient concentration level.

6. Fig. 24 shows a level curve elevation graph for a piece of
    property with a river. Which direction (left–to–right or
    right–to–left) does the river flow? (The elevations of the level
    curves are intentionally unlabeled. You should be able to
    answer the question from the shape and location of the curves.)
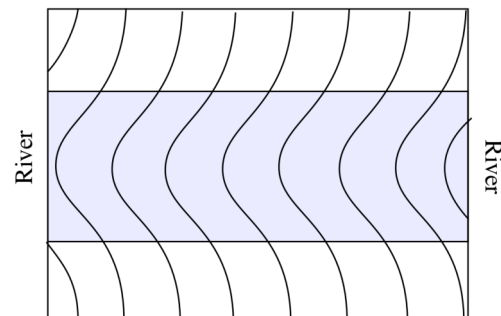
Fig. 24: River property

7. For the data in Fig. 19 sketch the approximate level curves for selling

    (a) 4,000 gallons of drinks.   (b) 11,000 gallons of drinks   (c) 15,000 gallons of drinks.

8. For the depth data in Fig. 23 sketch the approximate level curves

    (a) for the depth 5.5 .   (b) for the depth 6.5 .   (c) for the depth 7.5 .

9. Find and sketch the level curves $z = 1, 4$, and 9 for $z = x^2 + y^2$.

10. Find and sketch the level curves $z = 1, 4$, and 9 for $z = x^2 - y^2$.

11. Find and sketch the level curves $z = 0, 7, 12$ and 16 for $z = 16 - x^2 - y^2$.

12. Find and sketch the level curves $z = 1, 4$ and 5 for $z = \dfrac{5}{1 + (x-3)^2 + (y-4)^2}$

13. Find and sketch the level curves $z = 0, 1, 4$ and 9 for $z = x^2 - y^2$.

14. Sketch several level curves for a surface with a hill of height 600 feet at the location $(4, 6)$

15. Sketch several level curves for a surface with a hill of height 600 feet at the location $(4, 6)$ so that if we move south or east from the peak the path is very steep, but if we move north or west from the peak the path is not steep.

16. Sketch several level curves for a surface with a hill of height 600 feet at the location $(4, 6)$ and a hill of height 400 feet at the location $(2,2)$.

In problems 17 – 23, a surface is given. Match the surface with one of the level curve graphs A – G.

17. The surface in Fig. 25.   18. The surface in Fig. 26.

19. The surface in Fig. 27.   20. The surface in Fig. 28.

21. The surface in Fig. 29.   22. The surface in Fig. 30.

23. The surface in Fig. 31.

24. A new manufacturing process uses a computer–controlled laser light and a vat of liquid plastic. The part of the plastic exposed to the laser light becomes solid to a depth of 1 cm. The process begins with a platform 1 cm below the liquid level of the plastic, and the laser light moves over a region of plastic which hardens. Then the platform is lowered 1 cm, and the laser light moves over another region of the plastic. This process of hardening a region of plastic and then lowering the platform is repeated until the solid plastic object is complete. How is this process related to the level curves of the object?
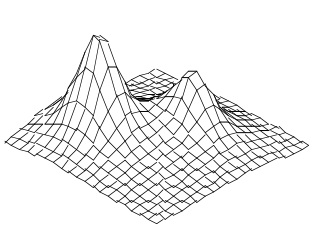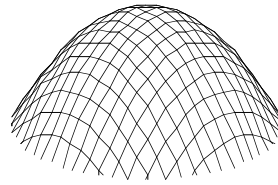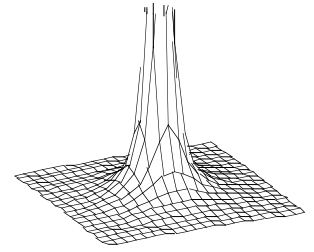
Fig. 25
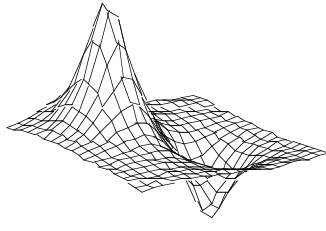
Fig. 26

Fig. 27

Fig. 28
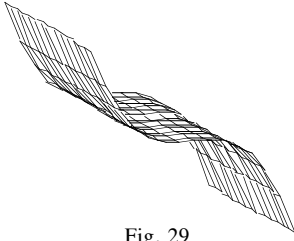
Fig. 29
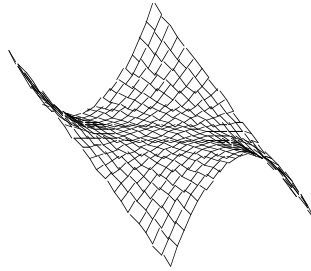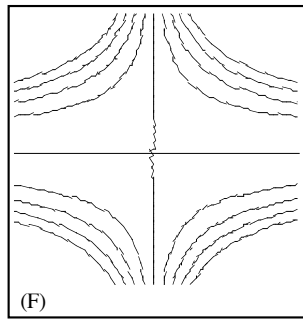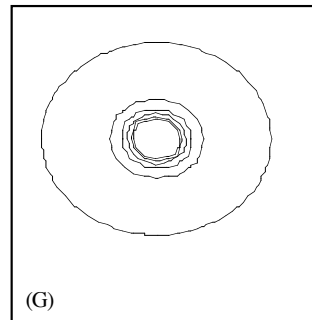

Fig. 30
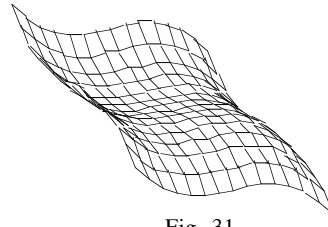
Fig. 31
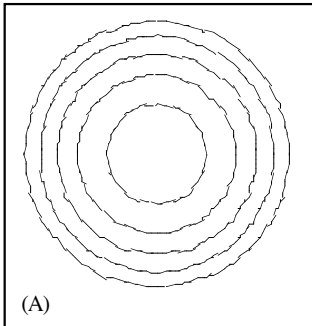
(A)
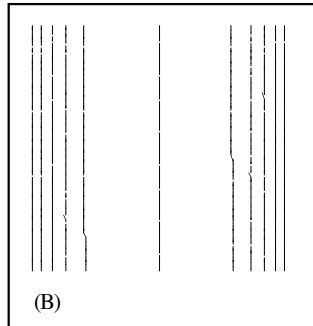
(B)
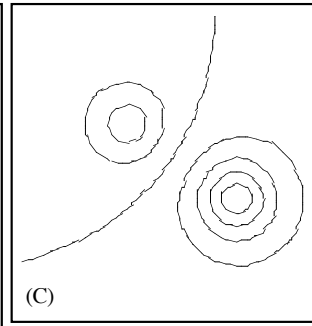
(C)
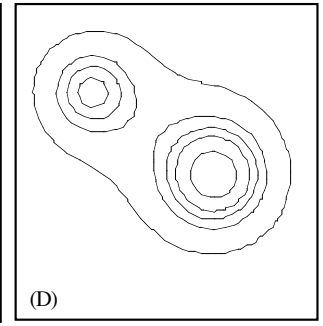
(D)
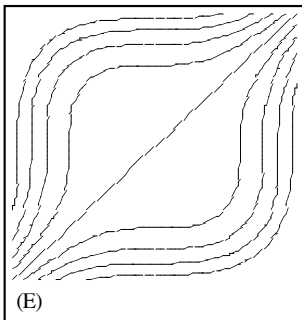
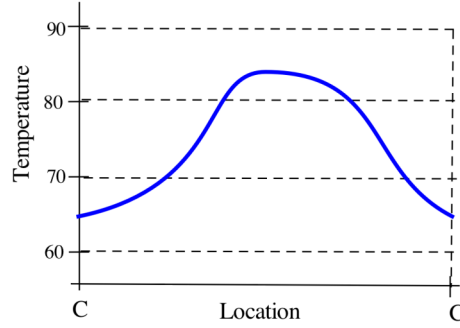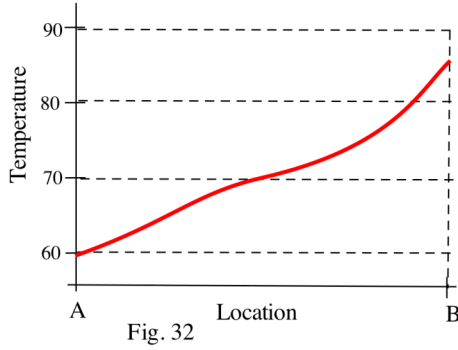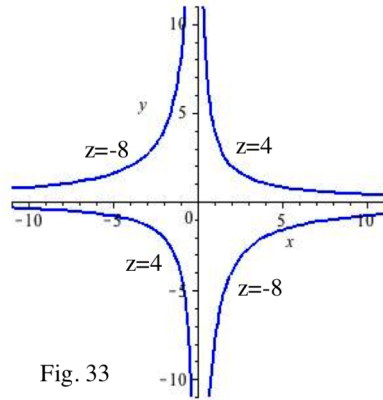
(E)

(F)

(G)

Practice Answers

Practice 1: $f(2,1) = 4.3$, $f(3,1) = 5.3$, $f(4,1) = 4.3$, $f(5,1) = 2.9$, $f(6,1) = 1.8$

As y is held constant at 1 and the x values increase from 2 to 6, the values of f rise to a

maximum of 5.3 (when x = 3) and then decrease to a minimum of 1.8 (when x = 6).

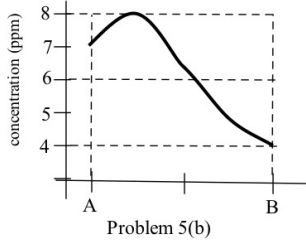**Practice 2:**   The graphs are given below.

Fig. 32

Practice 3:   The level curves are given below.
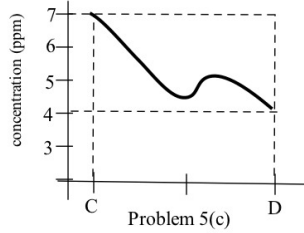
Fig. 33

13.1 Selected Answers

1. (a) minimum = 2, maximum = 30   (b) down 2, down 2 more, up 9 and up 5 more
    (c) up 5, up 5, up 5 up 5, then no change

3. (a) maximum = 10 m   (b) depth increases by 3, decreases by 4, increase 1, decreases 1
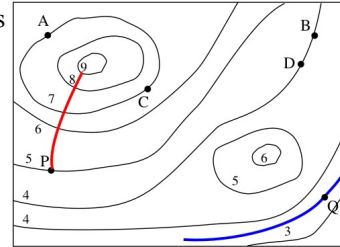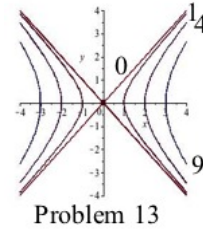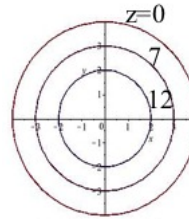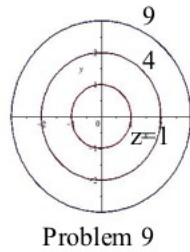   (c) depth decreases 2, decreases 1, decreases 1

5. (a)   el   ures

Problem 5(b)

Problem 5(c)

Problem 5(d) and (e)

7 to 15. See figures below

Problem 7 (a) and (b)

Problem 9

Problem 11

Problem 13

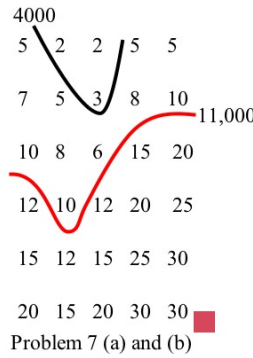Problem 15

17. level curves D   18. level curves C
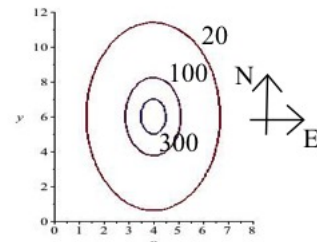
19. level curves A   20. level curves G

21. level curves B   22. level curves F

23. level curves E

Appendix: MAPLE commands to create 3D graphics for Section 13.1

> with(plots);   *loads additional graphics commands*

> with(linalg);   *loads additional matrix commands*

> f:=(i,j)->10/(1+0.4*(i-3)^2+0.1*(j-4)^2);   *defines the function f*

> M:=matrix(6,6,f);   *loads values of f into the matrix M*

> setoptions3d(axes=normal, tickmarks=[4,4,5], orientation=[30,70], thickness=0);

       sets options for the following graphs

       orientation=[30,70] puts viewing position at theta=30 and phi=70 (spherical

  coordinates)

> matrixplot(M);   *plots matrix M values as a surface*

> matrixplot(M, heights=histogram, gap=0);   *plots matrix M values as boxes*   **(Fig. 3)**

> plot3d(10/(1+0.4*(x-3)^2+0.1*(y-4)^2), x=0..6, y=0..6, grid=[15,15]);

               plots surface **(Fig. 4)**

> plot3d({ 10/(1+0.4*(x-3)^2+0.1*(y-4)^2), 4, 8 }, x=0..6, y=0..6, grid=[15,15]);

             plots surface and planes z=8 and z=4 **(Fig. 5)**

> contourplot(10/(1+0.4*(x-3)^2+0.1*(y-4)^2), x=0..6, y=0..6, grid=[25,25],

    contours=[1,2,3,4,5,6,7,8,9], orientation=[30,70]);   *plots surface contours* **(Fig. 6)**

> contourplot(10/(1+0.4*(x-3)^2+0.1*(y-4)^2), x=0..6, y=0..6, grid=[25,25],

    contours=[1,2,3,4,5,6,7,8,9], orientation=[0,–1]);   *plots surface contours* **(Fig. 7)**

The following sequence of commands illustrates how to

    put data into a matrix C,

    convert the matrix values into a list of points (x,y,z) = (i , j, C(i,j)), and then

    graph the surface with the points (i , j, C(i,j)).

      > C:=matrix([[5,4,3,4,5,7,8], [6,3,2,1,3,5,4], [5,4,5,7,8,5,4], [2,3,2,3,2,1,1],

       [1,1,2,2,3,1,1]]);

      > Cdata:=[seq([seq([i,j,C[i,j]] , i=1..5)], j=1..6)] ;

      > surfdata(Cdata , axes=normal);