13.2 LIMITS AND CONTINUITY

Our development of the properties and the calculus of functions $z = f(x,y)$ of two (and more) variables parallels the development for functions $y = f(x)$ of a single variable, but the development for functions of two variables goes much quicker since you already understand the main ideas of limits, derivatives, and integrals. In this section we consider limits of functions of two variables and what it means for a function of two variables to be continuous. In many respects this development is similar to the discussions of limits and continuity in Chapter One and many of the results we state in this section are merely extensions of those results to a new setting. It may be a good idea to spend a little time now in Chapter One rereading the main ideas and results for limits of functions of one variable and reworking a few limit problems.

The main focus of this section is on functions of two variables since it is still possible to visualize these functions and to work geometrically, but the end of this section includes extensions to functions of three and more variables.

Limits of Functions of Two Variables

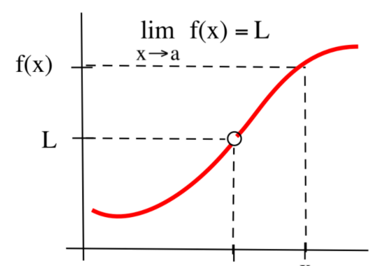

Fig. 1

When we considered limits of functions of one variable, $\lim_{x\to a} f(x)$, we were interested in the values of $f(x)$ when x was close to the point a in the domain of f (Fig. 1), and we often read the symbols "$x\to a$" as "x approaches a."

For the limit of function of two variables, $\lim_{(x,y)\to(a,b)} f(x,y)$, we are interested in the values of $f(x,y)$ when the point (x,y) is close to the point (a,b) in the domain of f (Fig. 2).

> **Definition:**
>
> Let f be a function of two variables defined for all points "near" (a,b) but possibly not defined at the point (a,b). We say the **limit of $f(x,y)$ as (x,y) approaches (a,b) is L**, written as
>
> $$\lim_{(x,y)\to(a,b)} f(x,y) = L ,$$
>
> if the distance from $f(x,y)$ to L, $|f(x,y) - L|$, can be made arbitrarily small by taking (x,y) sufficiently close to (a,b),
> (if $\sqrt{(x-a)^2 + (y-b)^2} = |\langle x-a, y-b \rangle|$ is sufficiently small).

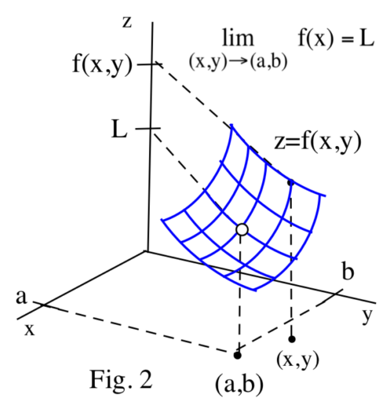

Fig. 2

All of the limit properties in the Main Limit Theorem (Section 1.2) are also true for limits of functions of two variables, and many limits of functions of two variables are easy to calculate.

Example 1: Calculate the following limits:

(a) $\lim_{(x,y)\to(1,2)} \frac{xy}{x^2+y^2}$

(b) $\lim_{(x,y)\to(0,2)} \cos(xy^2) + \frac{x+6}{y}$

(c) $\lim_{(x,y)\to(5,3)} \sqrt{x^2-y^2}$

Solution:

(a) $\lim_{(x,y)\to(1,2)} \frac{xy}{x^2+y^2} = \frac{1\bullet 2}{1^2+2^2} = \frac{2}{5}$

(b) $\lim_{(x,y)\to(0,2)} \cos(xy^2) + \frac{x+6}{y} = \cos(0\bullet 2) + \frac{0+6}{2} = 4$

(c) $\lim_{(x,y)\to(5,3)} \sqrt{x^2-y^2} = \sqrt{5^2-3^2} = \sqrt{16} = 4$

Practice 1: Calculate the following limits:

(a) $\lim_{(x,y)\to(3,1)} \frac{xy}{x^2-y^2}$

(b) $\lim_{(x,y)\to(0,2)} \cos(x^2y) + \frac{x+9}{y+1}$

(c) $\lim_{(x,y)\to(3,2)} \sqrt{x^2-y^3}$

At the end of this section we consider some examples of more complicated situations with functions whose limits do not exist. And we also extend the idea of limits to functions of three or more variables.

Continuity of Functions of Two Variables

A function of one variable is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$. Geometrically that means that the graph of f is connected at the point $(a, f(a))$ and does not have a hole or break there (Fig. 3). The definition and meaning of continous for functions of two variables is quite similar.

> Definition:
>
> A function of two variables defined at the point (a,b) and for all points near (a,b) is
>
> **continuous at (a,b)** if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. (Fig. 4)

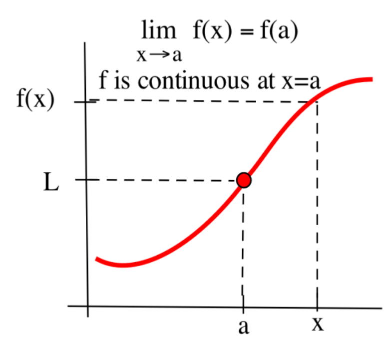

Fig. 3

And just as we talked about a function of one variable being continuous on an interval (or even on the entire real number line), we can talk about a function of two variables being continuous on a set D in the xy–plane or even on the entire xy–plane.

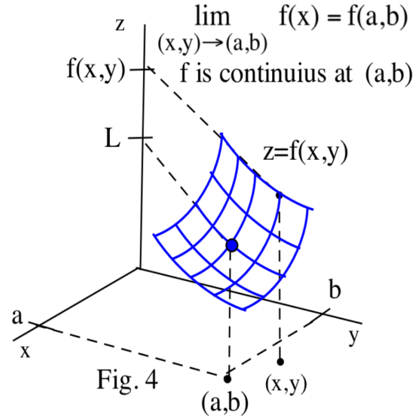

Fig. 4

> Definition:
>
> If a function of two variables is continuous at every point (a,b) in a set D, we say that the function is **continuous on D**.

Most of the functions we will work with are continuous either everywhere (at all points (x,y) in the plane) or continuous everywhere except at a "few" places.

- A **polynomial** function of two variables is **continuous everywhere**, at every point (x,y).
- A **rational** function of two variables is **continuous everywhere in its domain** (everywhere except where division by zero would occur).
- If f(x,y) is continuous at (a,b), then **sin(** f(x,y)), **cos(** f(x,y)), and $\mathbf{e}^{f(x,y)}$ are continuous at (a,b).
- More generally, if f (a function of two variables) is continuous at (a,b) and g (a function of one variable) is continuous at f(a,b), then g(f(x,y)) = gof (x,y) is continuous at (a,b).

Geometrically "f(x,y) is continuous at (a,b)" that means that the surface graph of f is connected at the point (a,b, f(a,b)) and does not have a hole or break there. Fig. 5 shows the surface graphs of several continuous functions of two variables.

Similar definitions and results are used for functions of three or more variables.

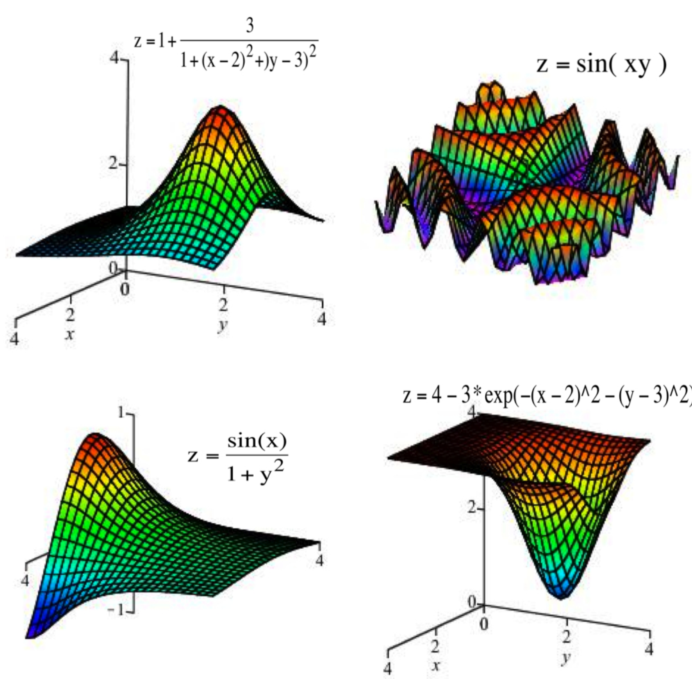

Fig. 5

Limits That Do Not Exist

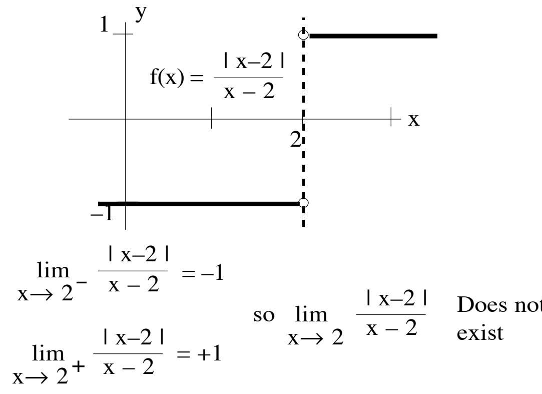

Fig. 6

Most of the functions we work with will have limits and will be continuous, but not all of them. A function of one variable did not have a limit if its left limit and its right limit had different values (Fig. 6). Similar situations can occur with functions of two variable as shown graphically in Fig. 7. For the function f(x,y) in Fig. 7, if (x,y) approaches the point (1,2) along path 1 (x = 1 and $y \to 2^-$) then the values of f(x,y) approach 2. But if (x,y) approaches the point (1,2) along path 2 (x = 1 and $y \to 2^+$) then the values of f(x,y) approach 1. Since two paths to the point (1,2) result in two different limiting values for f, we say that the limit of f(x,y) as (x,y) approaches (1,2) does not exist.

Showing a limit does not exist:

If there are two paths so that $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along path 1, and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along path 2, and $L_1 \neq L_2$,

then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

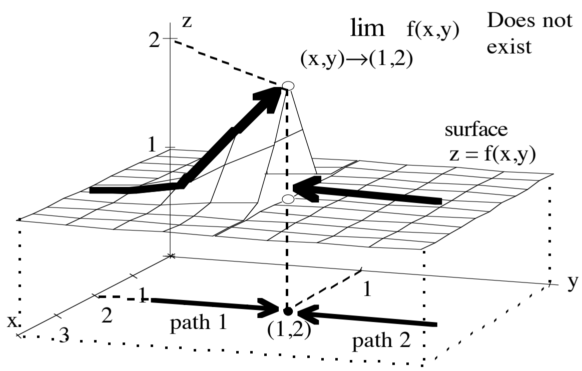

Fig. 7

Example 2: Find $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

Solution: Let path 1 be the x–axis, so y = 0.

Then $\frac{xy}{x^2+y^2} = \frac{0}{x^2+0} = 0$ and the limit of $\frac{xy}{x^2+y^2}$ as $(x,y) \to (0,0)$ along path 1 is 0.

However, if we take the path 2 to be the line y = x,

then $\frac{xy}{x^2+y^2} = \frac{x^2}{x^2+x^2} = \frac{1}{2}$ so the limit of $\frac{xy}{x^2+y^2}$ as $(x,y) \to (0,0)$ along path 2 is $\frac{1}{2}$.

Since the limits of f as $(x,y) \to (0,0)$ along two different paths is two different numbers, the limit of this f(x,y) as $(x,y) \to (0,0)$ **does not exist**.

Fig 8 shows these two paths and the different limits of f along them.

Practice 2: Find the limit of $f(x,y) = \frac{xy}{x^2+y^2}$ as $(x,y) \to (0,0)$ along the path y = 3x.

Since the limit of $f(x,y) = \frac{xy}{x^2+y^2}$ as $(x,y) \to (0,0)$ does not exist, this function is not continuous at (0,0).

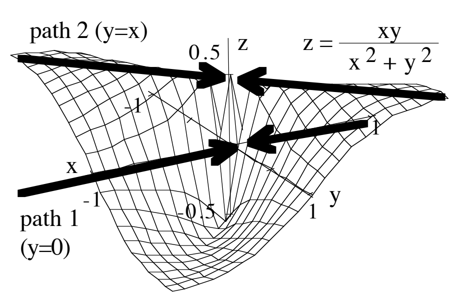

Fig. 8

Practice 3: Show that $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Note: The "path method" only shows that a limit does not exist. Even if the limit of a function as $(x,y)\to(a,b)$ is the same value along two or three paths (or even along an infinite number of paths) we still cannot validly conclude that the limit exists.

Example 3: Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$ along the paths (a) the y–axis (x=0), (b) the x–axis (y=0), (c) the lines $y = mx$ for all values of $m \neq 0$, and (d) along the parabola $y = x^2$.

Solution: (a) Since x=0, $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{y\to 0} \frac{0}{0 + y^2} = 0$.

(b) Since y=0, $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0} \frac{0}{x^4 + 0} = 0$.

(c) Since $y = mx$ $(m\neq 0)$, $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0} \frac{x^2 (mx)}{x^4 + (mx)^2} = \lim_{x\to 0} \frac{x^2}{x^2} \cdot \frac{mx}{x^2 + m^2} = 0.$

From parts (a), (b), and (c) we know that the limit of $\frac{x^2 y}{x^4 + y^2}$ is 0 as $(x,y)\to(0,0)$ along **every straight line path**. But that is not enough to conclude that the limit along every path is 0.

(d) Along the parabolic path $y = x^2$, $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0} \frac{x^2 (x^2)}{x^4 + (x^2)^2} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}$.

Part (d) together with any one of parts (a), (b), or (c) lets us conclude that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2} \text{ does not exist.}$$

Fig. 9 shows this surface.

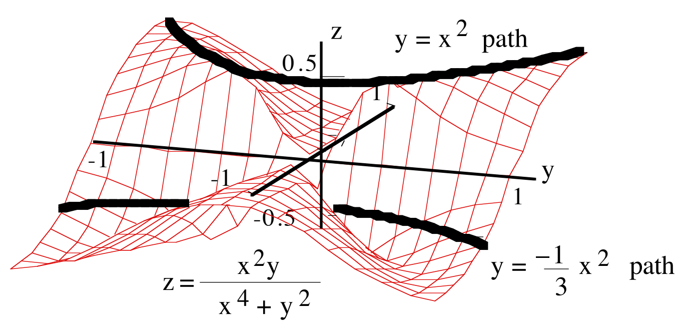

Fig. 9

Functions of More Than Two Variables: Limits and Continuity

Once we have made the adjustments to extend the ideas and definitions of limits and continuity to functions of two variables, it is straightforward to extend them to functions of three or more variables.

Definition:

Let f be a function of three variables defined for all points "near" (a,b,c) but possibly not defined at the point (a,b,c). We say the

limit of f(x,y,z) as (x,y,z) approaches (a,b,c) is L, written as

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L,$$

if the distance from f(x,y,z) to L, $| f(x,y,z) - L |$, can be made arbitrarily small by taking (x,y,z) sufficiently close to (a,b,c),

(if $\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = | \langle x-a, y-b, z-c \rangle |$ is sufficiently small).

Definition:

A function of three variables defined at the point (a,b,c) and for all points near (a,b,c) is **continuous at (a,b,c)** if $\displaystyle\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c)$.

PROBLEMS

In Problems 1 – 4, the level curves of functions are given. Use the information from these level curves to determine the limits. (Since only a few level curves are shown, you need to make reasonable assumptions about the behavior of the functions.)

1. The level curves of z = f(x,y) are shown in Fig. 10.

   (a) $\displaystyle\lim_{(x,y)\to(1,2)} f(x,y)$ (b) $\displaystyle\lim_{(x,y)\to(1,1)} f(x,y)$

   (c) $\displaystyle\lim_{(x,y)\to(2,1)} f(x,y)$ (d) $\displaystyle\lim_{(x,y)\to(3,2)} f(x,y)$

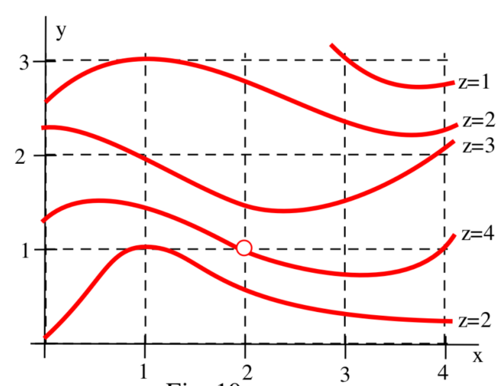

Fig. 10

2. The level curves of $z = g(x,y)$ are shown in Fig. 11.

(a) $\lim_{(x,y)\to(2,2)} g(x,y)$ (b) $\lim_{(x,y)\to(2,1)} g(x,y)$

(c) $\lim_{(x,y)\to(1,2)} g(x,y)$ (d) $\lim_{(x,y)\to(3,2)} g(x,y)$

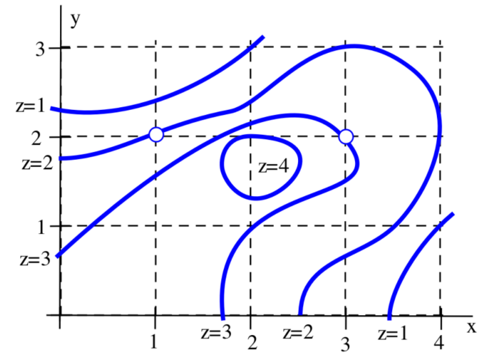

Fig. 11

3. The level curves of $z = S(x,y)$ are shown in Fig. 12.

(a) $\lim_{(x,y)\to(1,2)} S(x,y)$ (b) $\lim_{(x,y)\to(2,1)} S(x,y)$

(c) $\lim_{(x,y)\to(1,1)} S(x,y)$ (d) $\lim_{(x,y)\to(3,2)} S(x,y)$

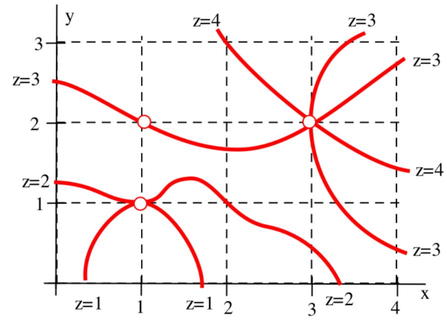

Fig. 12

4. The level curves of $z = T(x,y)$ are shown in Fig. 13.

(a) $\lim_{(x,y)\to(3,3)} T(x,y)$ (b) $\lim_{(x,y)\to(2,2)} T(x,y)$

(c) $\lim_{(x,y)\to(1,2)} T(x,y)$ (d) $\lim_{(x,y)\to(4,1)} T(x,y)$

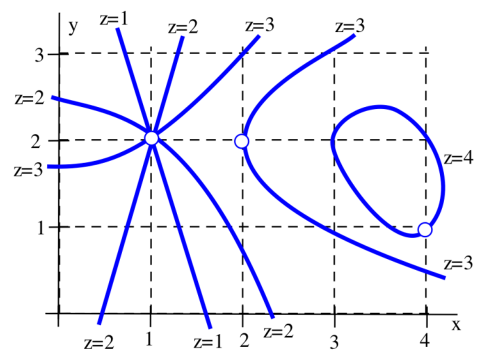

Fig. 13

In Problems 5 – 26, determine the limit if it exists or show that the limit does not exist.

5. $\lim_{(x,y)\to(2,3)} (x^2y^2 - 2xy^5 + 3y)$

6. $\lim_{(x,y)\to(-3,4)} (x^3 + 3x^2y^2 - 5y^3 + 1)$

7. $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^3 + x^3y^2 - 5}{2 - xy}$

8. $\lim_{(x,y)\to(-2,1)} \dfrac{x^2 + xy + y^2}{x^2 - y^2}$

9. $\lim_{(x,y)\to(\pi,\pi)} x \cdot \sin\left(\dfrac{x+y}{4}\right)$

10. $\lim_{(x,y)\to(1,4)} e^{(\sqrt{x+2y})}$

11. $\lim_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2}$

12. $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x + y}$

13. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y}{x^2+y^2}$

14. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

15. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{8x^2y^2}{x^4+y^4}$

16. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+xy^2}{x^2+y^2}$

17. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$

18. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$

19. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sqrt{xy}}{\sqrt{x^2+y^2}}$

20. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2+3xy+4y^2}{3x^2+5y^2}$

21. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy+1}{x^2+y^2+1}$

22. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$

23. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$

24. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^2+y^2}$

25. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

26. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2+1}-1}{x^2+y^2}$

27. $\displaystyle\lim_{(x,y)\to(0,1)} \frac{xy-x}{x^2+y^2-2x+2y+2}$

28. $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{x^2+y^2-2x-2y}{x^2+y^2-2x+2y+2}$

29. $\displaystyle\lim_{(x,y,z)\to(1,2,3)} \frac{xz^2-y^2z}{xyz-1}$

30. $\displaystyle\lim_{(x,y,z)\to(2,3,0)} \{ xe^x + \ln(2x-y) \}$

31. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2-y^2-z^2}{x^2+y^2+z^2}$

32. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+zx}{x^2+y^2+z^2}$

33. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2}$

34. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^2}{x^2+y^2+z^2}$

35. The function f whose level curves are shown in Fig. 10 is not defined at (2,1). Define a value for f(2,1) so f is continuous at (2,1).

36. The function g whose level curves are shown in Fig. 11 is not defined at (1,2) and (3,2). Can we define values for g(1,2) and g(3,2) so g is continuous at (1,2) and (3,2)?

37. The function S whose level curves are shown in Fig. 12 is not defined at (1,1), (1,2) and (3,2). Can we define values for S(1,1), S(1,2) and S(3,2) so S is continuous at each of those points?

38. The function T whose level curves are shown in Fig. 13 is not defined at (1,2), (2,2) and (4,1). Can we define values for T(1,2), T(2,2) and T(4,1) so T is continuous at eaach of those points?

In Problems 39 – 51, determine where the given function is not continuous.

39. $f(x,y) = \dfrac{x^2 + y^2 + 1}{x^2 + y^2 - 1}$

40. $f(x,y) = \dfrac{x^6 + x^3y^3 + y^6}{x^3 + y^3}$

41. $g(x,y) = \ln(2x + 3y)$

42. $S(x,y) = e^{xy}\sin(x + y)$

43. $T(x,y) = \sqrt{x + y} - \sqrt{x - y}$

44. $T(x,y) = 2^{x\tan(y)}$

45. $F(x,y) = x\ln(yz)$

46. $F(x,y) = x + y\sqrt{x + z}$

Practice Answers

Practice 1: (a) $\displaystyle\lim_{(x,y)\to(3,1)} \frac{xy}{x^2 - y^2} = \frac{3\bullet 1}{3^2 - 1^2} = \frac{3}{8}$

(b) $\displaystyle\lim_{(x,y)\to(0,2)} \cos(x^2y) + \frac{x+9}{y+1} = \cos(0^2\bullet 2) + \frac{0+9}{2+1} = 1 + \frac{9}{3} = 4$

(c) $\displaystyle\lim_{(x,y)\to(3,2)} \sqrt{x^2 - y^3} = \sqrt{3^2 - 2^3} = \sqrt{1} = 1$

Practice 2: Along the path $y = 3x$, $\dfrac{xy}{x^2 + y^2} = \dfrac{x(3x)}{x^2 + (3x)^2} = \dfrac{3x^2}{10x^2} .= \dfrac{3}{10}$ for $x \ne 0$.

Then the limit of $\dfrac{xy}{x^2 + y^2}$ as $(x,y) \to (0,0)$ along the path $y = 3x$ is $\dfrac{3}{10}$.

Practice 3:

Along the path $y = x$, $\dfrac{x^2 - y^2}{x^2 + y^2} = \dfrac{x^2 - x^2}{x^2 + x^2} = \dfrac{0}{2x^2} = 0$ for $x \ne 0$, so the limit along this path is 0.

Along the x–axis $y = 0$, so $\dfrac{x^2 - y^2}{x^2 + y^2} = \dfrac{x^2}{x^2} = 1$ for $x \ne 0$, so the limit along this path is 1.

(Also, along the y–axis $x = 0$, so $\dfrac{x^2 - y^2}{x^2 + y^2} = \dfrac{-y^2}{y^2} = -1$ for $y \ne 0$, so the limit along this path is –1.)

13.2 Odd Answers

1. (a) 3 (b) 2 (c) 4 (d) about 2.4 3. (a) 3 (b) 2 (c) dne (d) dne

5. -921 7. -5/2 9. Π

11. dne If y=0 then limit = 1. If y= -x then the function is undefined.

13. dne Paths y=0 and y=x give different limit values.

15. dne Paths y=0 and y=x give different limit values.

17. dne Paths y=0 and y=x give different limit values.

19. dne Paths y=0 and y=x give different limit values.

21. 1 23. dne Paths y=0 and y= x^2 give different limit values.

25. 2 Try rationalizing the denominator.

27. 0 The function equals $\frac{x(y-1)}{(x-1)^2+(y-1)^2}$ which is easier to analyze.

29. -3/5

31. dne If x=y=0 and z->0 then limit = -1. If y=z=0 and x->0 then limit = 1.

33. dne If y=z=0 and x->0 then limit = 0. If z=0 and yx=x ->0 then limit = ½..

35. Define f(2,1) = 4

37. No value of f(1,2) or f(3,2) will make f continuous at those points.
f(1,2) = 3 makes f continuous at (1,2).

39. Not continuous (not defined) on the circle where $x^2+y^2=1$.

41. Not continuous (not defined) if 2x+3y ≤ 0 or y ≤ - 2x/3 .

43. In order for T to be continuous at (x,y) we need both x+y≥0 (so y≥ -x) and x-y≥0 (so x≥y). That requires x≥0 and -x ≤ y ≤ x.

45. F is continuous at (x,y,z) if yz >0.