# 13.3 PARTIAL DERIVATIVES

For a function $y = f(x)$ of one variable, the derivative $\frac{dy}{dx}$ measured the rate of change of the variable $y$ with respect to the variable $x$. For a function $z = f(x,y)$ of two variables we can ask about the rate of change of $z$ with respect to the variable $x$ or the variable $y$: how do changes in $x$ effect $z$, and how do changes in $y$ effect $z$? In scientific and economic settings with many variables, it is common to try to determine the effect of each variable by holding all of the other variables constant and then measuring the outcomes as that single variable in allowed to vary.

**Example 1**: The table of data in Fig. 1 shows the number of thousands of gallons of drinks sold at a sports stadium as a function of the temperature at the beginning of the game and the number of people attending the game. At a game with 30,000 people on a $70^{\circ}$ day,

(a) what is the average rate of change of drink sales as the temperature rises to $80^{\circ}$?

(b) what is the average rate of change of drink sales as the attendance increases to 40,000 people?

| Attendance (1000s) \ Temperature ($^{\circ}$F) | 50 | 60 | 70 | 80 | 90 |
|---|---|---|---|---|---|
| 10 | 5 | 2 | 2 | 5 | 5 |
| 20 | 7 | 5 | 3 | 8 | 10 |
| 30 | 10 | 8 | 6 | 15 | 20 |
| 40 | 12 | 10 | 12 | 20 | 25 |
| 50 | 15 | 12 | 15 | 25 | 30 |
| 60 | 20 | 15 | 20 | 30 | 30 |

Fig. 1: Gallons (1000s) of drinks sold

Solution: (a) In this situation the attendance is constant at 30,000 people, and the temperature changes from $70^{\circ}$ to $80^{\circ}$. The average rate of change is

$$\frac{f(30,80^{\circ}) - f(30,70^{\circ})}{80^{\circ} - 70^{\circ}} = \frac{15000 - 6000 \text{ gallons}}{10^{\circ}} = 900 \text{ gallons per degree rise in temperature.}$$

(b) In this case the temperature is constant at $70^{\circ}$, and the attendance changes from 30,000 people to 40,000 people. The average rate of change is

$$\frac{f(40,70) - f(30,70)}{40000 - 30000} = \frac{12000 - 6000 \text{ gallons}}{10000 \text{ people}} = 0.6 \text{ gallons per additional person in attendance.}$$

Note that these rates of change depend on the starting attendance and temperature as well as on the variable that is allowed to change. You should also notice that the units of the two answers are different — one is "gallons/degree" and the other is "gallons/person."

**Practice 1**: Using the data in Fig. 1 and at a game with 20,000 people on a $80^{\circ}$ day,

(a) what is the average rate of change of drink sales as the temperature rises to $90^{\circ}$?

(b) what is the average rate of change of drink sales as the attendance increases to 30,000 people?

The definition of a partial derivative follows from this idea of holding one variable constant and measuring the rate of change as the other variable changes.

> Definition:
>
> The **partial derivative of f(x,y) with respect to x at the point (a,b)** is
>
> $$\mathbf{f_x(a,b)} = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h} \quad \text{(if the limit exists and is finite).}$$
>
> Meaning: $f_x(x,y)$ measures the ***instantaneous rate of change*** of f at the point (x,y) in the direction of increasing x values.
>
> To calculate $f_x(x,y)$ when $z = f(x,y)$ is given by a formula,
>
> **treat y as a constant and differentiate with respect to x.**

**Example 2**: (a) For $f(x,y) = 3x^2 + 7y^2 - 10xy$, find $f_x(x,y)$ , $f_x(1,2)$ and $f_x(3,1)$.

(b) For $g(x,y) = \sin(\ 3xy\ ) + \ln(\ 5y\ ) + x^3y^5$ , find $g_x(x,y)$ and $g_x(1,2)$ .

Solution: (a) $f_x(x,y) = 6x + 0 - 10y = 6x - 10y$ and $f_x(1,2) = 6(1) - 10(2) = -14$. $f_x(3,1) = 6(3) - 10(1) = 8$.

(b) $g_x(x,y) = 3y \cdot \cos(\ 3xy\ ) + 0 + 3x^2y^5$ and

$g_x(1,2) = 3(2)\cos(\ 3(1)(2)\ ) + 3(1)^2(2)^5 = 6\cos(6) + 96 \approx 101.8$ .

**Practice 2**: (a) For $f(x,y) = x^3 + 4y^2 + 5x^2y$, find $f_x(x,y)$ and $f_x(2,5)$.

(b) For $g(x,y) = e^{xy} + \frac{x}{y}$ , find $g_x(x,y)$ and $g_x(0,2)$ .

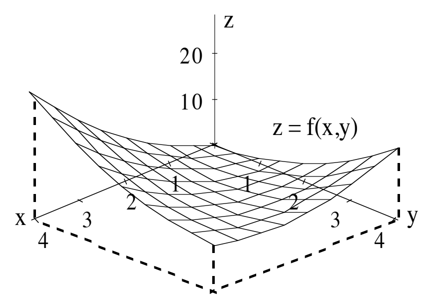


Fig. 2

We can also interpret the partial derivatives graphically. The graph of $z = f(x,y)$ is typically a surface (Fig. 2) and the graph of "y = a constant" is a plane, so the graph of "z = f(x,y) with y held constant" is the curve resulting from the intersection of the surface and the plane. Fig. 3 shows such a surface and plane and their curve of intersection when $y = 2$. $f_x(1,2)$ is the slope of the line tangent to this curve at the point (1,2) as shown in Fig. 4.

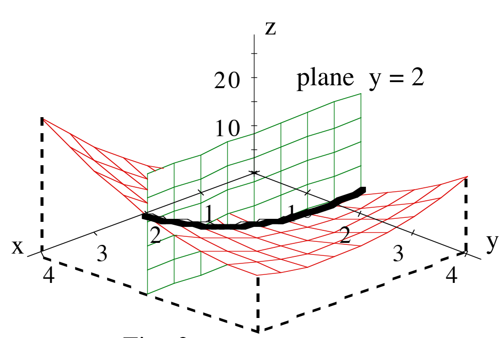


Fig. 3

**Example 3**: Use the information in Fig. 4 to estimate the value of $f_x(1,2)$.

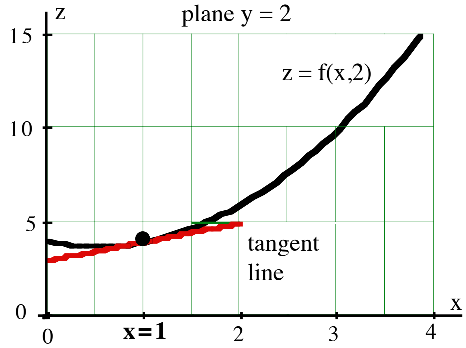


Fig. 4

Solution: We can estimate the slope of the tangent line in Fig. 4 by picking two points on the line and calculating the slope of the line connecting the two points. It looks like (0,3) and (2,5) are points on the tangent line, and the slope of the segment between those two points is $\frac{5-3}{2-0} = 1$. Then we estimate $f_x(1,2) \approx 1$.

**Practice 3**: Use the information in Fig. 4 to estimate the values of $f(3,2)$ and $f_x(3,2)$.

The partial derivative with respect to y is similar, but now we treat x as the constant.

Definition:

The **partial derivative of f(x,y) with respect to y at the point (a,b)** is

$$\mathbf{f_y(a,b)} = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h} \quad \text{(if the limit exists and is finite).}$$

Meaning: $f_y(x,y)$ measures the *instantaneous rate of change* of f at the point (x,y) in the direction of increasing y values.

To calculate $f_y(x,y)$ when $z = f(x,y)$ is given by a formula,

**treat x as a constant and differentiate with respect to y**.

**Example 4**: (a) For $f(x,y) = 3x^2 + 7y^2 - 10xy$, find $f_y(x,y)$, $f_y(1,2)$, and $f_y(3,1)$.

(b) For $g(x,y) = \sin(3xy) + \ln(5y) + x^3y^5$, find $g_y(x,y)$ and $g_y(1,2)$.

Solution: (a) $f_y(x,y) = 0 + 14y - 10x = 14y - 10x$. Then $f_y(1,2) = 14(2) - 10(1) = 18$ and $f_y(3,1) = -16$.

(b) $g_y(x,y) = 3x\cdot\cos(3xy) + \frac{5}{5y} + 5x^3y^4$. Then

$$g_y(1,2) = 3(1)\cdot\cos(3(1)(2)) + \frac{5}{5(2)} + 5(1)^3(2)^4 \approx 2.88 + 0.5 + 80 = 83.38 .$$

**Practice 4**: (a) For $f(x,y) = x^3 + 4y^2 + 5x^2y$, find $f_y(x,y)$ and $f_y(2,5)$.

(b) For $g(x,y) = e^{xy} + \frac{x}{y}$, find $g_y(x,y)$ and $g_y(0,2)$.

**Notations**: The following notations are all commonly used to represent partial derivatives of $z = f(x,y)$

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = D_x f \qquad \text{Partial derivative of } f \text{ with respect to } x$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = D_y f \qquad \text{Partial derivative of } f \text{ with respect to } y$$

**Example 5**: Use the information in Figures 5 and 6 to estimate the value of $f_y(1,2)$.

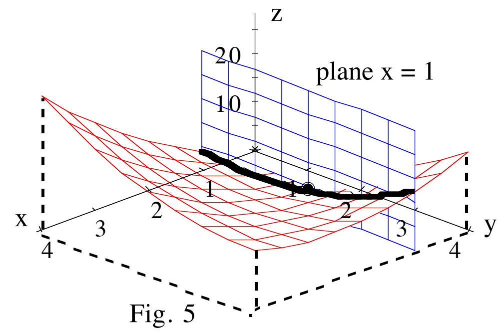


Fig. 5

Solution: Fig. 5 shows the surface $z = f(x,y)$ and the plane $x = 1$, but it is difficult to estimate the value of $f_y(1,2)$ from it. Fig. 6 shows the intersection of the surface graph with the plane, and the tangent line at the point $(1,2)$ is included. We can estimate the value of $f_y(1,2)$ by picking two points on the tangent line and calculating the slope between them. It looks like $(1,2)$ and $(3,6)$ are points on the tangent line, and the slope of the segment is $\frac{6-2}{3-1} = 2$. Then we estimate $f_y(1,2) \approx 2$.

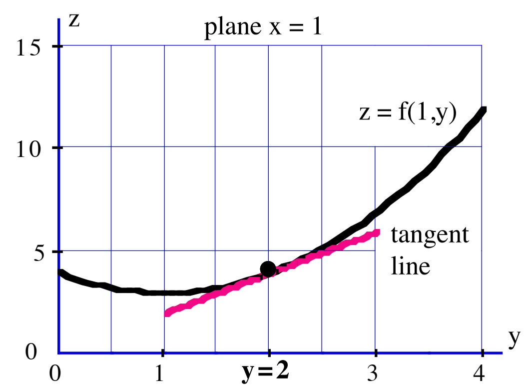


Fig. 6

**Practice 5**: Use the information in Figures 5 and 6 to estimate the signs (positive, negative or zero) of (a) $f_y(1,3)$, (b) $f_y(1,1)$, and (c) $f_y(1,4)$.

## Partial Derivatives in Context

Of course it is very important to be able to calculate partial derivatives, but it also important to understand and to be able to communicate what they mean and measure. And you need to be able to attach the correct units to your answers.

**Example 6:** The surface area A (square inches) of a small child is a function of the length L (inches) and the weight W (pounds) of the child: A = A( L, W). Explain (in clear English sentences) the meaning of the following. Be sure to include units.

(a) A( 26, 46 ) = 164 (b) $\frac{\partial A(26,46)}{\partial W} = 7$ (c) $\frac{\partial A(26,46)}{\partial L} = 5$

Solution: (a) A( 26, 46 ) = 164 square inches. A child who is 26 inches long and weighs 46 pounds will have a surface area of 164 square inches.

(b) $\frac{\partial A(26,46)}{\partial W} = 7$ square inches per pound . The surface area of this child (length 26 inches, weight 46 pounds, area 164 square inches) is **increasing at an INSTANTANEOUS RATE of** 7 square inches per each additional pound of weight if the length stays constant. Units: (square inches)/pound

(c) $\frac{\partial A(26,46)}{\partial L} = 5$ square inches per inch. The surface area of this child (length 26 inches, weight 46 pounds, area 164 square inches) is **increasing at an INSTANTANEOUS RATE of** 5 square inches per each additional inch of length if the weight stays constant. Units: (square inches)/inch

**Practice 6:** A certain biotech process using bacteria to produce a vaccine V (in grams) depends on the number B of bacteria and the temperature T (in $^{o}C$) of the laboratory: V = V( B, T). Explain (in clear English sentences) the meaning of the following. Be sure to include units.

(a) $V(2000,40) = 8.9$ (b) $\dfrac{\partial V(2000,40)}{\partial B} = 0.003$ (c) $\dfrac{\partial V(2000,40)}{\partial T} = -1.4$

## Partial Derivatives and Level Curves

Level curves of a surface $z = f(x,y)$ give us information about $f$ and also about the rate of change of $f$ as $x$ and $y$ increase, the partial derivatives $f_x$ and $f_y$.

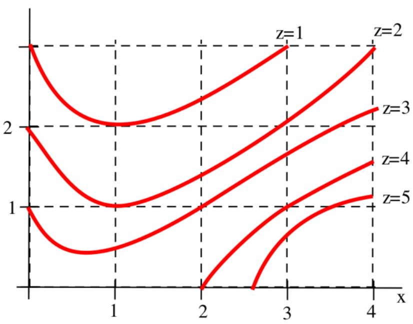
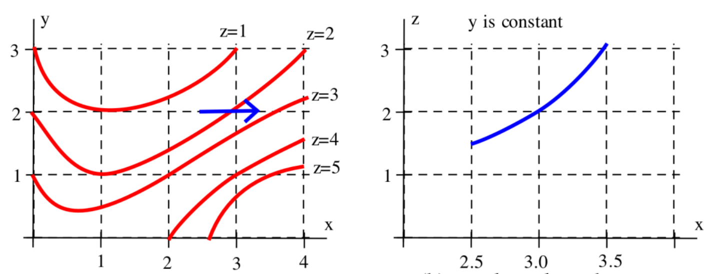


Fig. 7: Level curves for z=f(x,y)

**Example 7**: Use the information in Fig. 7 to estimate the signs (positive, negative or zero) of
(a) $f_x(3,2)$ and (b) $f_y(3,2)$.

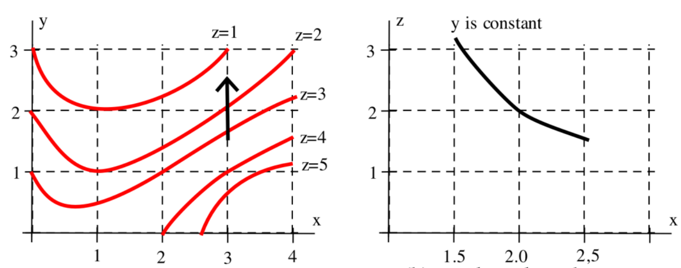
Solution: (a) As we move through the point (3,2) in the increasing $x$ direction (Fig. 8a), the level curves are increasing in value so $f_x(3,2)$ is positive. Fig. 8b shows the graph of $z$ along the line segment of increasing x–values ($y$ is constantly 2), and the slope of the tangent line to this graph is positive when $x = 3$ so $f_x(3,2)$ is positive.



(a) level curves for z=f(x,y) (b) z-values along the arrow

Fig. 8: the sign of $f_x(3,2)$

(b) As we move through the point (3,2) in the increasing $y$ direction (Fig. 9a), the level curves are decreasing in value so $f_y(3,2)$ is negative. Fig. 9b shows the graph of $z$ along the line segment of increasing y–values ($x$ is constantly 3), and the slope of the tangent line to this graph is negative when $y = 2$ so $f_y(3,2)$ is negative.



(a) level curves for z=f(x,y) (b) z-values along the arrow

Fig. 9: the sign of $f_y(3,2)$

Note: If the level curves are close together in our direction of movement, then the z–values are changing rapidly in that direction and the magnitude (the absolute value of the magnitude) of the rate of change in that direction is large. For the function described by the level curves in Fig. 7, $| f_x(3,2) | > | f_y(3,2) |$ because the level curve lines are closer together as when we move from (3,2) in the x–direction than when we move in the y–direction.

**Practice 7**: Use the information in Fig. 7 to estimate the signs (positive, negative or zero) of $f_x(3,1)$ , $f_y(3,1)$ , $f_x(1,1)$ and $f_y(1,1)$ . Which of these partial derivatives has the largest absolute value?

## Second Partial Derivatives

For a function $y = f(x)$ of one variable, the second derivative $f''(x) = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2 f}{dx^2}$ is the rate of change of the rate of change of f, and it measures the concavity of the graph of f. The situation for $z = f(x,y)$ is similar. The second derivative of a function of one variable was used to determine whether a critical point was a local maximum or minimum, and the second partial derivatives will be used to help determine whether critical points of functions of two variables are local maximums or minimums.

Definition: Second Partial Derivatives of $z = f(x,y)$:

$$f_{xx}(x,y) = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} \qquad \text{differentiate twice with respect to } x$$

$$f_{yy}(x,y) = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2} \qquad \text{differentiate twice with respect to } y$$

$f_{xx}(x,y)$ measures the concavity of the graph of f in the x–direction. $f_{yy}(x,y)$ measures the concavity in the y–direction.

We can also differentiate first with respect to one variable and then differentiate the result with respect to the other variable.

Definition: Second Mixed Partial Derivatives of $z = f(x,y)$:

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \qquad \text{differentiate first with respect to x, then with respect to y}$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \qquad \text{differentiate first with respect to y, then with respect to x}$$

$f_{xy}(x,y)$ measures the rate of change in the y–direction of the rate of change in the x–direction. This is more difficult to interpret graphically.

Note how order of the x and y changes depending on the notation: $f_{\mathbf{xy}} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial \mathbf{y} \partial \mathbf{x}}$ and $f_{\mathbf{yx}} = \frac{\partial^2 f}{\partial \mathbf{x} \partial \mathbf{y}}$ .

**Example 8**: For $f(x,y) = 3x^3 + 7y^4 - 10x^2y$, calculate $f_{xx}$ , $f_{yy}$ , $f_{xy}$ and $f_{yx}$ .

Solution: $f_x = 9x^2 - 20xy$ and $f_y = 28y^3 - 10x^2$ . Then

$$f_{xx} = \frac{\partial}{\partial x}(\, 9x^2 - 20xy\,) = 18x - 20y. \qquad f_{yy} = \frac{\partial}{\partial y}(\, 28y^3 - 10x^2\,) = 84y^2 \ .$$

$$f_{xy} = \frac{\partial}{\partial y}(\, 9x^2 - 20xy\,) = -20x \ . \qquad f_{yx} = \frac{\partial}{\partial x}(\, 28y^3 - 10x^2\,) = -20x \ .$$

**Practice 8**: For $g(x,y) = e^{xy} + \frac{x}{y}$ , calculate $g_{xx}$ , $g_{yy}$ , $g_{xy}$ and $g_{yx}$ .

In the previous Example and Practice it turned out that the mixed partials were equal: $f_{xy} = f_{yx}$ and $g_{xy} = g_{yx}$ . The next theorem says this is always the case for "nice" (sufficiently smooth) surfaces.

Clairaut's Theorem:

If $f(x,y)$ is defined and continuous at (a,b) and for all points near (a,b) and $f_{xy}$ and $f_{yx}$ are both continuous at all points near (a,b),

then $f_{xy}(a,b) = f_{yx}(a,b)$ .

We can also define higher partial derivatives in a natural way such as $f_{xyy} = (\, f_{xy}\,)_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial y \partial y \partial x}$ .

These higher partial derivatives are sometimes useful in physics and other areas , but we will not use them.

## Partial Derivatives Implicitly

In all of the previous examples we knew z explicitly as a function of x and y. But sometimes it is not possible to algebraically isolate z in order to calculate a partial derivative. In that case we can still determine the partial derivatives, but we need to do so implicitly.

**Example 8:** $xy + yz = xz$ . Determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in general and at the point ( 3, 2, 6).

Solution: In this case we can calculate the partial derivatives both explicitly and implicitly.

Explicitly: Solving for z we get $xy = xz - yz$ so $z = \frac{xy}{x-y}$. Then, using the quotient rule,

$$\frac{\partial z}{\partial x} = \frac{(x-y)\cdot \frac{\partial(xy)}{\partial x} - xy\cdot \frac{\partial(x-y)}{\partial x}}{(x-y)^2} = \frac{(x-y)\cdot y - xy\cdot 1}{(x-y)^2} = \frac{-y^2}{(x-y)^2} \ .$$

At ( 3, 2, 6), $\frac{\partial z}{\partial x} = -4$. Similarly, $\frac{\partial z}{\partial y} = \frac{x^2}{(x-y)^2}$ which equals 9 at ( 3, 2, 6).

Implicitly: Taking the partial derivative of each side, $\frac{\partial}{\partial x}(xy + yz) = \frac{\partial}{\partial x}(xz)$, we get

$\left[ x \cdot \frac{\partial y}{\partial x} + y \cdot \frac{\partial x}{\partial x} \right] + \left[ y \cdot \frac{\partial z}{\partial x} + z \cdot \frac{\partial y}{\partial x} \right] = x \cdot \frac{\partial z}{\partial x} + z \cdot \frac{\partial x}{\partial x}$. But $\frac{\partial y}{\partial x} = 0$ (why?) and

$\frac{\partial x}{\partial x} = 1$ so the previous equation simplifies to $[0 + y] + \left[ y \cdot \frac{\partial z}{\partial x} + 0 \right] = x \cdot \frac{\partial z}{\partial x} + z$.

Then $\frac{\partial z}{\partial x} = \frac{z - y}{y - x}$ which equals -4 at ( 3, 2, 6), the same result we got differentiating explicitly. Similarly, $\frac{\partial z}{\partial u} = \frac{x + z}{x - y} = \frac{9}{1}$ at ( 3, 2, 6).

**Practice 9:** $xy^2 + \sin(z) + 3 = 2x + 3z$. Determine $\frac{\partial z}{\partial x}$ in general and at the point ( 3, 1, 0).

## A Final Comment!

Partial derivatives are used extensively in the remaining sections on multivariate calculus, and it is vital that you understand what they measure and that you become able to calculate partial derivatives quickly and accurately. Extra practice now will save you time (and points) in the rest of the course.

## PROBLEMS

1. For $f(x,y) = 16 - 4x^2 - y^2$, find $f_x(1,2)$ and $f_y(1,2)$ and interpret these numbers as slopes. Illustrate with sketches.

2. For $f(x,y) = \sqrt{4 - x^2 - 4y^2}$ , find $f_x(1,0)$ and $f_y(1,0)$ and interpret these numbers as slopes. Illustrate with sketches.

In problems 3 – 11, find the indicated partial derivatives.

3. $f(x,y) = x^3y^5$ ; $f_x(3,-1)$

4. $f(x,y) = xe^{-y} + 3y$ ; $\frac{\partial}{\partial y}(1,0)$

5. $z = \frac{x^3 + y^3}{x^2 + y^2}$ ; $\frac{\partial z}{\partial x}$ , $\frac{\partial z}{\partial y}$

6. $z = \frac{x}{y} + \frac{y}{x}$ ; $\frac{\partial z}{\partial x}$

7. $xy + yz = xz$ ; $\frac{\partial z}{\partial x}$ , $\frac{\partial z}{\partial y}$

8. $\sin(x) + y \cdot e^z = z$ ; $\frac{\partial z}{\partial x}$ , $\frac{\partial z}{\partial y}$

9. $y^2 + yz^2 = zx^2$; $\frac{\partial z}{\partial x}$ , $\frac{\partial z}{\partial y}$

10. $x^2 + y^2 - z^2 = 2x( y + z)$ ; $\frac{\partial z}{\partial x}$ , $\frac{\partial z}{\partial y}$

11. $u = xy \sec( xy )$ ; $\frac{\partial u}{\partial x}$

12. $f(x,y,z) = xyz$ ; $f_y(0,1,2)$

13. $u = xy + yz + zx$ ; $u_x, u_y, u_z$

In problems 14 – 29, find the first partial derivatives of the given functions.

14. $f(x,y) = x^3y^5 - 2x^2y + x$

15. $f(x,y) = x^4 + x^2y^2 + y^4$

16. $f(x,y) = \frac{x-y}{x+y}$

17. $f(x,y) = e^x \tan(x - y)$

18. $f(s,t) = \sqrt{2 - 3s^2 - 5t^2}$

19. $f(u,v) = \arctan(u/v)$

20. $g(x,y) = y \tan(x^2y^3)$

21. $z = \ln(x + \sqrt{x^2 + y^2})$

22. $z = \sinh(\sqrt{3x + 4y})$

23. $f(x,y) = \int_x^y e^{(t^2)}\, dt$

24. $f(x,y,z) = x^2yz^3 + xy - z$

25. $f(x,y,z) = x^{yz}$

26. $u = z \sin\left(\frac{y}{x+z}\right)$

27. $f(x,y,z,t) = \frac{x-y}{z-t}$

28. $u = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$

29. Use the definition of partial derivatives as limits to find $f_x(x,y)$ and $f_y(x,y)$ when $f(x,y) = 5x + x^2 - xy + 3y$ .

30. Use the definition of partial derivatives as limits to find $f_x(x,y)$ and $f_y(x,y)$ when $f(x,y) = x^2 - xy + 2y^2$ .

In problems 31 – 33, find $\partial z/\partial x$ and $\partial z/\partial y$ .

31. $z = f(x) + g(y)$

32. $z = f(x + y)$

33. $z = f(x/y)$

In problems 34 – 36, find all of the second partial derivatives.

34. $f(x,y) = x^2y + x\sqrt{y}$

35. $z = (x^2 + y^2)^{3/2}$

36. $z = t \cdot \arcsin(\sqrt{x})$

In problems 37 and 38, verify that the conclusion of Clairaut's Theorem holds, that is, $u_{xy} = u_{yx}$ .

37. $u = x^5y^4 - 3x^2y^3 + 2x^2$

38. $u = \arcsin(xy^2)$

39. Verify that the function $u = e^{-a^2k^2t} \sin(kx)$ is a solution of the heat equation $u_t = a^2 u_{xx}$ .

40. The total resistance R produced by three conductors with resistances $R_1, R_2, R_3$ connected in a parallel electrical circuit is given by the formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$ . Find $\partial R/\partial R_1$ .

## Practice Answers

**Practice 1:** (a) ( 2000 gallons )/( 10 degrees ) = 200 gallons/degree

(b) ( 7000 gallons )/( 10,000 people ) = 0.7 gallons/person

**Practice 2:** (a) $f_x(x,y) = 3x^2 + 10xy$ , $f_x(2,5) = 112$

(b) $g_x(x,y) = y \cdot e^{xy} + \frac{1}{y}$ , $g_x(0,2) = 2.5$

**Practice 3:** $f(3,2) \approx 9$ , $f_x(3,2) \approx 9$

**Practice 4:** (a) $f_y(x,y) = 8y + 5x^2$ , $f_y(2,5) = 60$

(b) $g_y(x,y) = x\,e^{xy} - x/y^2$ , $g_y(0,2) = 0$

**Practice 5:** (a) $f_y(1,3)$ is positive (b) $f_y(1,1)$ is approximately 0 (c) $f_y(1,4)$ is positive

**Practice 6:** (a) This process will produce 8.9 grams of vaccine when we have 2000 bacteria and the laboratory temperature is 40 ºC.

(b) The amount of vaccine produced (when we have 2000 bacteria at a temperature of 40 ºC) will **increase at an INSTANTANEOUS RATE of** 0.003 grams for each additional bacteria when the temperature stays constant. Units: grams/bacteria

(c) The amount of vaccine produced (when we have 2000 bacteria at a temperature of 40 ºC) will **decrease at an INSTANTANEOUS RATE of** 1.4 grams for each degree increase in temperature when the number of bacteria stays constant. Units: grams/ºC

**Practice 7:** $f_x(3,1)$ is positive , $f_y(3,1)$ is negative ,

$f_x(1,1)$ is zero (z has alocal max for increasing x values) , and $f_y(1,1)$ is negative .

I estimate that $f_y(3,1)$ has the largest absolute value (contour lines are closest together).

**Practice 8:** If $g(x,y) = e^{xy} + \frac{x}{y}$ , then

$g_x = y \cdot e^{xy} + \frac{1}{y}$ $\qquad g_y = x \cdot e^{xy} - \frac{x}{y^2}$ .

$g_{xx} = \frac{\partial}{\partial x}\left( y \cdot e^{xy} + \frac{1}{y} \right) = y^2 \cdot e^{xy}$ $\qquad g_{yy} = \frac{\partial}{\partial y}\left( x \cdot e^{xy} - \frac{x}{y^2} \right) = x^2 \cdot e^{xy} + \frac{2x}{y^3}$

$$g_{xy} = \frac{\partial}{\partial y}\left( y \cdot e^{xy} + \frac{1}{y} \right) = xy \cdot e^{xy} + e^{xy} - \frac{1}{y^2}$$

$$g_{yx} = \frac{\partial}{\partial x}\left( x \cdot e^{xy} - \frac{x}{y^2} \right) = xy \cdot e^{xy} + e^{xy} - \frac{1}{y^2}$$

**Practice 9:** $\frac{\partial}{\partial x}(x \cdot y^2 + \sin(z) + 3) = \frac{\partial}{\partial x}(2x + 3z)$ so

$$\left[ x \cdot 2y \cdot \frac{\partial y}{\partial x} + y^2 \cdot \frac{\partial x}{\partial x} \right] + \cos(z) \cdot \frac{\partial z}{\partial x} + 0 = 2 \cdot \frac{\partial x}{\partial x} + 3 \cdot \frac{\partial z}{\partial x}$$ which simplifies to

$\left[ y^2 \right] + \cos(z) \cdot \frac{\partial z}{\partial x} = 2 + 3 \cdot \frac{\partial z}{\partial x}$ so $\frac{\partial z}{\partial x} = \frac{2 - y^2}{\cos(z) - 3}$ which equals $\frac{1}{-2}$ at ( 3, 1, 0).

## Appendix: Maple commands to graph surfaces and planes — figures 2 – 6

> with(plots); (enter) *Loads plots routines which enable 3D graphics commands*

Plots the surface in **Fig. 2**

```
> plot3d(3*x+(x-y)^2, x=0..4, y=0..4, axes=normal, grid=[9,9], orientation=[45,50], tickmarks=[4,4,3]);
```

Calculates and plots the surface, plane and intersection curve in **Fig. 3**

```
> AX:=plot3d(3*x+(x-y)^2,x=0..4,y=0..4,axes=normal, grid=[9,9], color=red):                    (return)
BX:=plot3d([u,2,v],u=0..4,v=0..25,axes=normal, grid=[9,6],color=green, thickness=0):    (return)
CX:=spacecurve([u,2,3*u+(u-2)^2],u=0..4, color=black, thickness=2):                     (return)
DX:=spacecurve([u,2,.6+3*u+(u-2)^2],u=0..4, color=black, thickness=2):                  (return)
PX:=pointplot( {[1,2,4]}, color=black, symbol=circle):                                  (return)
LX:=spacecurve( [u,2,3+u], u=0..2, color=orange, thickness=2):                          (return)
display3d( {AX,BX,DX,PX,LX}, orientation=[45,50], tickmarks=[4,4,6] );                  (enter)
```

Plots the plane, curve and tangent line in **Fig. 4**

```
> display3d( {BX,CX,PX,LX}, orientation=[-90,90], view=0..15, tickmarks=[4,4,3]);       (enter)
```

Calculates and plots **Fig. 5** and **Fig. 6**

```
> AY:=plot3d(3*x+(x-y)^2,x=0..4,y=0..4,axes=normal, grid=[9,9], color=red):
BY:=plot3d([1,u,v],u=0..4,v=0..25,axes=normal, grid=[9,6],color=blue, thickness=0):
CY:=spacecurve([1,v,3+(1-v)^2],v=0..4, color=black, thickness=2):
DY:=spacecurve([1,v,.5+3+(1-v)^2],v=0..4, color=black, thickness=2):
PY:=pointplot( {[1,2,4]}, color=black, symbol=circle):
LY:=spacecurve( [1,v,2*v], v=1..3, color=magenta, thickness=2):
display3d( {AY,BY,DY,PY,LY}, orientation=[45,50], tickmarks=[4,4,6] );

> display3d( {BY,CY,PY,LY}, orientation=[0,90],view=0..15,tickmarks=[4,4,3]);
```

Similar to the previous commands but at the point (3,2)

```
> AY:=plot3d(3*x+(x-y)^2,x=0..4,y=0..4,axes=normal, grid=[9,9], color=red):
BY:=plot3d([3,u,v],u=0..4,v=0..25,axes=normal, grid=[9,6],color=blue, thickness=0):
CY:=spacecurve([3,v,9+(3-v)^2],v=0..4, color=black, thickness=2):
DY:=spacecurve([3,v,.5+9+(3-v)^2],v=0..4, color=black, thickness=2):
PY:=pointplot( {[3,2,10]}, color=black, symbol=circle):
LY:=spacecurve( [3,v,14-2*v], v=1..3, color=magenta, thickness=2):
display3d( {AY,BY,DY,PY,LY}, orientation=[45,50], tickmarks=[4,4,6] );

> display3d( {BY,CY,PY,LY}, orientation=[0,90],view=0..15,tickmarks=[4,4,3]);
```

```
> AX:=plot3d(3*x+(x-y)^2,x=0..4,y=0..4,axes=normal, grid=[9,9], color=red):
BX:=plot3d([u,2,v],u=0..4,v=0..25,axes=normal, grid=[9,6],color=green, thickness=0):
CX:=spacecurve([u,2,3*u+(u-2)^2],u=0..4, color=black, thickness=2):
DX:=spacecurve([u,2,.6+3*u+(u-2)^2],u=0..4, color=black, thickness=2):
PX:=pointplot( {[3,2,10]}, color=black, symbol=circle):
LX:=spacecurve( [u,2,5*u-5], u=2..4, color=orange, thickness=2):
display3d( {AX,BX,DX,PX,LX}, orientation=[45,50], tickmarks=[4,4,3] );

> display3d( {BX,CX,PX,LX}, orientation=[-90,90], view=0..15, tickmarks=[4,4,3]);
```

**Maple commands to input a matrix of heights and then graph their surface and contours**

```
> with(plots);      (then press ENTER)     Loads routines which enable 3D graphics commands
> with(linalg);     (then press ENTER)     Loads routines which enable matrix commands
```

Defines a matrix of z–values for a surface

```
> surfmat2:=matrix( [                          (press RETURN)
[3.0,2.8,2.6,2.4,2.2,1.8,1.5,1.2,1.0],         (press RETURN)
[3.3,3.0,2.8,2.4,2.1,1.7,1.3,1.0,0.5],         (press RETURN)
[3.4,3.2,3.0,2.5,2.0,1.5,1.0,0.5,0.0],         (press RETURN)
[3.7,3.5,3.2,2.6,2.1,1.6,0.9,0.4,0.0],         (press RETURN)
[4.0,3.8,3.6,2.7,2.2,1.7,0.8,0.2,0.0],         (press RETURN)
[4.3,4.3,4.1,3.3,2.6,2.1,1.3,0.8,0.5],         (press RETURN)
[4.7,4.8,4.9,4.0,3.0,2.4,1.8,1.4,1.0],         (press RETURN)
[5.1,5.4,5.6,4.8,4.0,3.5,2.9,2.2,1.6],         (press RETURN)
[5.2,5.7,6.1,5.3,4.5,4.3,3.5,2.8,1.9],         (press RETURN)
[5.1,5.4,5.7,5.1,4.3,4.1,3.4,2.6,1.8],         (press RETURN)
[4.7,4.8,5.0,4.6,4.2,3.5,2.7,2.1,1.5]          (press RETURN)
]);                                            (press ENTER)
```

Plots the contours

```
> matrixplot(surfmat2, view=0..7, style=CONTOUR, axes=normal, orientation=[-90,0],
contours=[0,.5,1,1.5,2,2.5,3,3.5,4,4.5,5,5.5,6]);    (press ENTER)
```

Plots the surface

```
> matrixplot(surfmat2, view=0..7, style=HIDDEN, axes=normal, orientation=[65,45]);   (press ENTER)
```

Plots the surface and contours on one graph

```
> A:=matrixplot(surfmat2, view=0..7, style=CONTOUR, axes=normal,
contours=[0,.5,1,1.5,2,2.5,3,3.5,4,4.5,5,5.5,6], orientation=[65,45], color=black):      (press
RETURN)
B:=matrixplot(surfmat2, view=0..7, style=HIDDEN, axes=normal):                           (press
RETURN)
display3d( {A, B}, orientation=[65,45]);                                                 (press ENTER)
```

## 13.3 Selected Answers

1. $f_x(1,2) = -8,\ f_y(1,2) = -4$

2. $f_x(1,0) = -1/\sqrt{3}\ ,\ f_y(1,0) = 0$

3. $f_x(3,-1) = -27$

4. 2

5. $\frac{\partial z}{\partial x} = (x^4 + 3x^2y^2 - 2xy^3)/(x^2+y^2)^2,\ \frac{\partial z}{\partial y} = (y^4 + 3x^2y^2 - 2x^3y)/(x^2+y^2)^2$

6. $(1/y) - (y/x^2)$

7. $(y-z)/(x-y),\ (x+z)/(x-y)$

8. $\frac{\cos(x)}{1-y\cdot e^z},\ \frac{e^z}{1-y\cdot e^z}$

9. $\frac{\partial z}{\partial x} = \frac{2xz}{2yz - x^2}$ $\qquad \frac{\partial z}{\partial y} = \frac{2y+z^2}{x^2 - 2yz}$

11. $y\sec(xy) + xy^2\sec(xy)\tan(xy)$

12. 0

13. $y+z,\ x+z,\ x+y$

14. $f_x(x,y) = 3x^2y^5 - 4xy + 1,\ f_y(x,y) = 5x^3y^4 - 2x^2$

15. $f_x(x,y) = 4x^3 + 2xy^2,\ f_y(x,y) = 2x^2y + 4y^3$

16. $f_x(x,y) = 2y/(x+y)^2,\ f_y(x,y) = -2x/(x+y)^2$

17. $f_x = e^x\{\tan(x-y) + \sec^2(x-y)\},\ f_y = -e^x\sec^2(x-y)$

18. $f_s = -3s/\sqrt{2 - 3s^2 - 5t^2}\ ,\ f_t = -5t/\sqrt{2-3s^2-5t^2}$

19. $f_u = v/(u^2+v^2)\ ,\ f_v = -u/(u^2+v^2)$

20. $g_x = 2xy^4\sec^2(x^2y^3)\ ,\ g_y = \tan(x^2y^3) + 3x^2y^3\sec^2(x^2y^3)$

21. $\frac{\partial z}{\partial x} = 1/\sqrt{x^2+y^2}\ ,\ \frac{\partial z}{\partial y} = y/(x^2 + y^2 + x\sqrt{x^2+y^2})$

22. $\frac{\partial z}{\partial x} = \frac{3}{2}\cosh(\sqrt{3x+4y})/\sqrt{3x+4y}\ ,\ \frac{\partial z}{\partial y} = 2\cosh(\sqrt{3x+4y})/\sqrt{3x+4y}$

23. $f_x = 2xyz^3 + y,\ f_y = x^2z^3 + x,\ f_z = 3x^2yz^2 - 1$

24. $f_x = yz\,x^{yz-1}\ ,\ f_y = z\,x^{yz}\ln(x)\ ,\ f_z = y\,x^{yz}\ln(x)$

25. $u_x = -yz\cos(y/(x+z))/(x+z)^2\ ,\ u_y = z\cos(y/(x+z))/(x+z)\ ,$
$u_z = \sin(y/(x+z)) - yz\cos(y/(x+z))/(x+z)^2$

26. $f_x = 1/(z-t),\ f_y = -1/(z-t)\ ,\ f_z = -(x-y)/(z-t)^2\ ,\ f_t = (x-y)/(z-t)^2$

31. $\frac{\partial z}{\partial x} = f'(x)\ ,\ \frac{\partial z}{\partial y} = g'(y)$

32. $\frac{\partial z}{\partial x} = f'(x+y),\ \frac{\partial z}{\partial y} = f'(x+y)$

33. $\frac{\partial z}{\partial x} = f'(x/y)(1/y)\ ,\ \frac{\partial z}{\partial y} = f'(x/y)(-x/y^2)$

34. $f_{xx} = 2y\ ,\ f_{xy} = 2x + 1/(2\sqrt{y}) = f_{yx}\ ,\ f_{yy} = -x/(4y\sqrt{y})$

35. $z_{xx} = 3(2x^2 + y^2)/\sqrt{x^2 + y^2}$ , $z_{xy} = 3xy/\sqrt{x^2 + y^2} = z_{yx}$ , $z_{yy} = 3(x^2 + 2y^2)/\sqrt{x^2 + y^2}$