13.4 TANGENT PLANES and DIFFERENTIALS

In Section 2.8 we were able to use the derivative f ' of a function y = f(x) of one variable to find the equation of the line tangent to the graph of f at a point (a, f(a)) (Fig. 1): $y = f(a) + f'(a) \cdot (x - a)$. And then we used this tangent line to approximate values of f near the point (a, f(a)), and we introduced the idea if the differential $df = f'(a) \cdot dx$ of the function f . In this section we extend these ideas to functions z = f(x,y) of two variables. But here we will find tangent planes (Fig. 2) rather than tangent lines, and we will use the tangent plane to approximate values of f(x,y). Finally, we will extend the concept of a differential to functions of two variables.

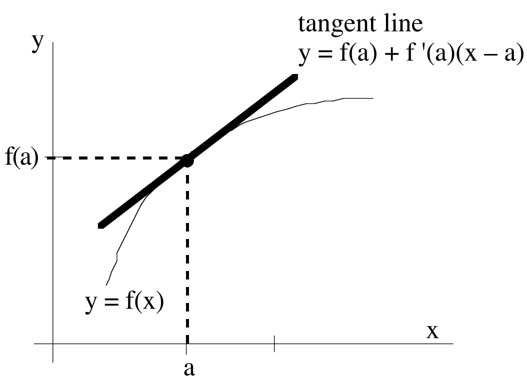


Fig. 1: Tangent line to y = f(x) at x = a

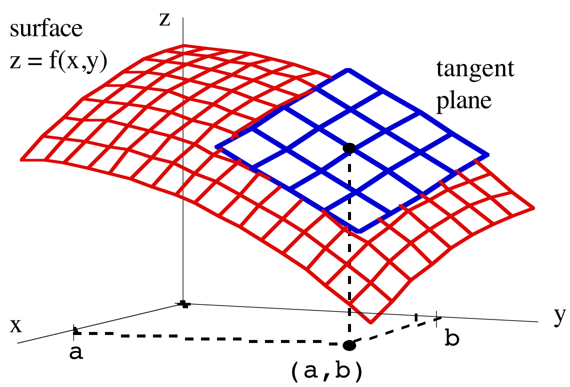


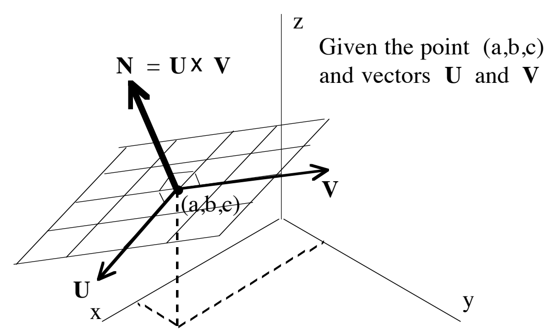
Fig. 2: Tangent plane to z = f(x,y) at (a,b)

Tangent Planes

In Section 11.6 we saw how to use a point (a, b, c) and two (nonparallel) vectors to determine the equation of the plane through the point and containing lines parallel to the given vectors (Fig. 3):

(1) we used the cross product of the two given vectors to find a normal vector $\mathbf{N} = \langle n_1, n_2, n_3 \rangle$ to the plane, and then

(2) we used the normal vector **N** and the point to write the equation of the plane as $n_1(x - a) + n_2(y - b) + n_3(z - c) = 0$.



Equation of the tangent plane is
$n_1(x–a) + n_2(y–b) + n_3(z–c) = 0$

Fig. 3

This approach also works when we need the equation of a plane tangent to a surface z = f(x,y), but we will use the formula for the surface to find the two needed vectors.

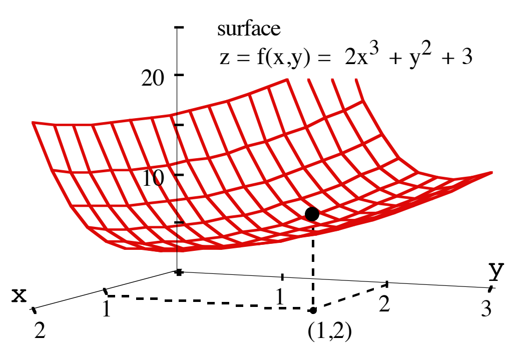


Fig. 4

Example 1: Find the equation of the plane tangent to the surface $f(x,y) = 2x^3 + y^2 + 3$ at the point $P = (1, 2, f(1,2)) = (1, 2, 9)$ on the surface (Fig. 4).

Solution: We are given a point (1, 2, 9) on the plane, but we need two vectors. These vectors are the rates of change of the surface f(x,y) in the x and y directions. The rate of change of f(x,y) in the x–direction is $f_x(x,y) = 6x^2$, and at the point (1,2,9) we have $f_x(1,2) = 6(1)^2 = 6$. Similarly, the rate of change of f(x,y) in the y–direction is $f_y(x,y) = 2y$, and at the point (1,2,9) we have $f_y(1,2) = 2(2) = 4$.

Then a "rate of change vector in the x–direction" is $\mathbf{U} = \langle 1, 0, 6 \rangle$ formed by taking 1 "step" in the x–direction, taking 0 "steps" in the y–direction (y is constant), and then taking 6 "steps" in the z–direction ($6 = f_x(1,2)$ = rate of change of z with respect to increasing x–values). Similarly, a "rate of change" vector in the y–direction is $\mathbf{V} = \langle 0, 1, 4 \rangle$. These vectors are shown in Fig. 5.

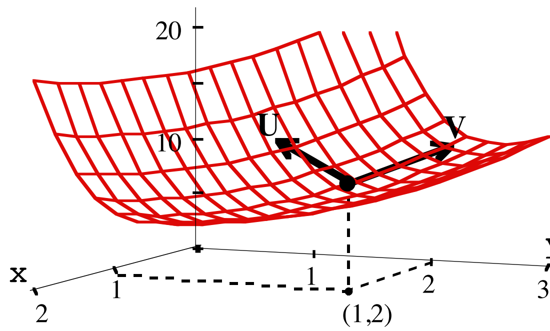


Fig. 5: Surface and tangent vectors **U** and **V**

Now a normal vector **N** to the plane we want is formed by taking

$$\mathbf{N} = \mathbf{V} \times \mathbf{U} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 4 \\ 1 & 0 & 6 \end{vmatrix} = (6)\mathbf{i} - (-4)\mathbf{j} + (-1)\mathbf{k} = 6\mathbf{i} + 4\mathbf{j} - 1\mathbf{k}$$

(Note: taking $\mathbf{N} = \mathbf{U} \times \mathbf{V} = -6\mathbf{i} - 4\mathbf{j} + 1\mathbf{k}$ also works.)

Finally, using the point P = (1, 2, 9) and the normal vector $\mathbf{N} = \mathbf{V} \times \mathbf{U} = 6\mathbf{i} + 4\mathbf{j} - 1\mathbf{k}$, we know that the equation of the plane is

$$\mathbf{6(x - 1) + 4(y - 2) - 1(z - 9) = 0} \quad \text{or} \quad \mathbf{z = 9 + 6(x - 1) + 4(y - 2)}$$

Looking at the equation $z = 9 + 6(x - 1) + 4(y - 2)$ of the plane, you should notice that the 9 is the z–coordinate of our original point, that the coefficient of the x variable, 6, is $f_x(1,2)$, and that the coefficient of the y variable, 4, is $f_y(1,2)$. Fig. 6 shows the surface and the tangent plane.

Fortunately we do not need to go through all of those calculations every time we need the equation of a plane tangent to a surface at a point: the pattern that we noted about the coefficients of the variables in the tangent plane equation and the values of the partial derivatives holds for every differentiable function.

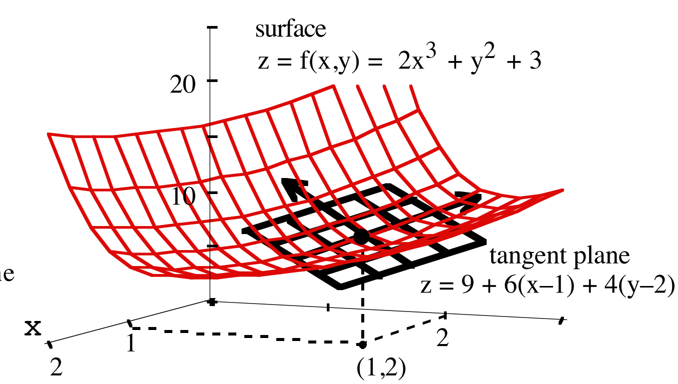


Fig. 6: Surface and tangent plane

Equation for a Tangent Plane

If $f(x,y)$ is differentiable at the point $(a, b, f(a,b))$,

then the equation of the plane tangent to the surface $z = f(x, y)$ at the point $P(a, b, f(a,b))$

is $\mathbf{z = f(a,b) + f_x(a,b)(x - a) + f_y(a,b)(y - b)}$.

Proof: The proof simply involves the steps we went through for Example 1. $\mathbf{U} = \langle 1, 0, f_x(a,b) \rangle$ is formed by taking 1 "step" in the x–direction, taking 0 "steps" in the y–direction (y is constant), and then taking $f_x(a,b)$ "steps" in the z–direction . Similarly, $\mathbf{V} = \langle 0, 1, f_y(a,b) \rangle$. Then

$$\mathbf{N} = \mathbf{V} \times \mathbf{U} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & f_y(a,b) \\ 1 & 0 & f_x(a,b) \end{vmatrix} = (f_x(a,b))\mathbf{i} - (-f_y(a,b))\mathbf{j} + (-1)\mathbf{k} = f_x(a,b)\,\mathbf{i} + f_y(a,b)\,\mathbf{j} - 1\mathbf{k} .$$

Finally, using the point $(a, b, f(a,b))$ and $\mathbf{N} = \mathbf{V} \times \mathbf{U} = f_x(a,b)\,\mathbf{i} + f_y(a,b)\mathbf{j} - 1\mathbf{k}$, we have that the equation of the plane is

$$f_x(a,b)(x - a) + f_y(a,b)(y - b) - 1(z - f(a,b)) = 0 \text{ or } z = f(a,b) + f_x(a,b)(x - a) + f_y(a,b)(y - b) .$$

Example 2: Find the plane tangent to the surface $z = 2x^2y^3 + \ln(xy) + 7$ at the point $(1, 1, 9)$.

Solution: $f_x(x, y) = 4xy^3 + \frac{1}{xy}(y) = 4xy^3 + \frac{1}{x}$ and $f_y(x, y) = 6x^2y^2 + \frac{1}{xy}(x) = 6x^2y^2 + \frac{1}{y}$

so $f_x(1, 1) = 5$ and $f_y(1, 1) = 7$.

Then the equation of the tangent plane is $z = 9 + 5(x - 1) + 7(y - 1)$ or $\mathbf{z = 5x + 7y - 3}$.

(This is much quicker than the "from scratch" method of Example 1.)

Fig. 7 shows two views of this surface and the tangent plane — notice that in this case the tangent plane cuts through the surface.

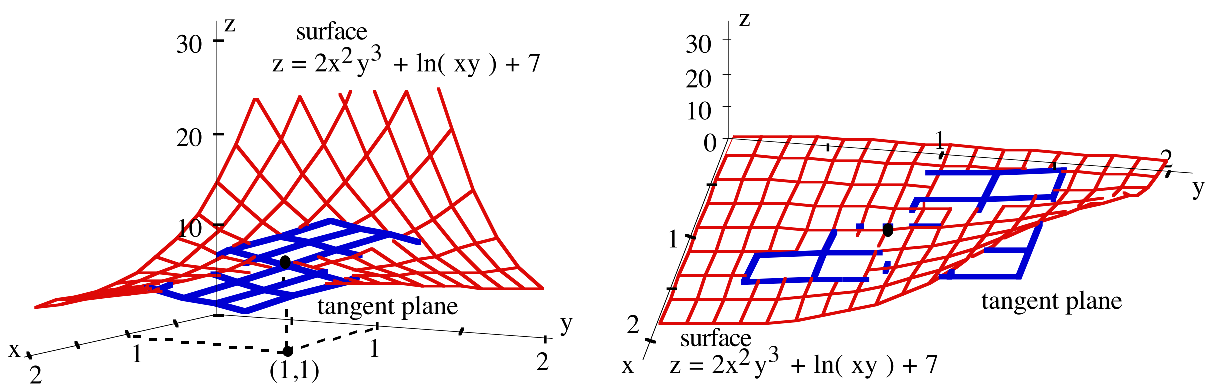


Fig. 7: Two views of the surface and tangent plane

Practice 1: Find the plane tangent to the surface $z = 5xy^2 + 7y + \sin(xy) - 2$ at the point $(0, 1, 5)$.

Differentials

The following "boxed" material summarizes, from Section 2.8, the definition and results about the differential dy of a function $y = f(x)$ of one variable.

Definition for $y = f(x)$: The **differential** of $y = f(x)$ is $\mathbf{dy = f\,'(x)\,dx} = \frac{df}{dx}\,dx$.

Meaning of dy: **dy** is the change in the y–value, **along the tangent line** to f obtained by a step of dx in the x–value.

Result: If f is differentiable at $x = a$ and dx is "small" then $f(a + dx) - f(a) \approx \mathbf{dy}$ or $f(a + dx) \approx f(a) + \mathbf{dy}$.

Meaning of the Result: For a small step dx, the actual change in f is approximately equal to the change along the tangent line: $f(a + dx) \approx f(a) + f\,'(a)\,dx$.

The extension to functions of two variables is given in the next "box."

Definition for $z = f(x,y)$: The **differential** (or **total differential**) of $z = f(x,y)$ is

$$\mathbf{dz = f_x(x,y)\,dx + f_y(x,y)\,dy} = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy .$$

Meaning of dz: **dz** is the change in the z–value, **along the tangent plane** to f, obtained by a step of dx in the x direction and a step of dy in the y direction. (Fig. 8)

Result: If $z = f(x,y)$ is differentiable at the point (a, b), then $f(a + dx, b + dy) - f(a,b) \approx \mathbf{dz} = f_x(a,b)\,dx + f_y(a,b)\,dy = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$.

Meaning of the Result: For a small step of dx in the x direction and a small step dy in the y direction, the change in f is approximately equal to the change along the tangent plane:

$$f(a + dx, b + dy) \approx f(a,b) + dz = f(a,b) + f_x(a,b)\,dx + f_y(a,b)\,dy$$

Example 3: Find the differential of $z = 5 + 3x^2y^3$ (a) in general, and (b) at the point $(x,y) = (2, 1)$.

Solution: (a) $\mathbf{dz} = f_x(a,b)\,dx + f_y(a,b)\,dy = \{ 6xy^3 \}\,dx + \{ 9x^2y^2 \}\,dy$

(b) at $(2, 1)$, $\mathbf{dz} = \{ 6(2)(1)^3 \}\,dx + \{ 9(2)^2(1)^2 \}\,dy = (12)\,dx + (36)\,dy$.

Example 4: For $z = 5 + 3x^2y^3$ and the point $(2,1)$, use the result of Example 3 that $dz = (12)\,dx + (36)\,dy$ to approximate $f(2.02, 1.01)$ and $f(2.01, 0.97)$. Compare these approximate values with the exact values of $f(2.02, 1.01)$ and $f(2.01, 0.97)$.

Solution: For f(2.02, 1.01) , $dx = 0.02$ and $dy = 0.01$ so $dz = (12)(0.02) + (36)(0.01) = 0.6$.

Then $f(2.02, 1.01) \approx f(2, 1) + dz = 17 + 0.6 = 17.6$.

Actually, $f(2.02, 1.01) = 17.6121206012$ so the approximation "error" using the differential is 0.012 .

For f(2.01, 0.97) , $dx = 0.01$ and $dy = -0.03$, so $dz = (12)(0.01) + (36)(-0.03) = -0.96$.

Then $f(2.01, 0.97) \approx f(2,1) + dz = 17 + (-0.96) = 16.04$.

Actually, $f(2.01, 0.97) = 16.0618705619$ so the approximation "error" using the differential is 0.022 .

Practice 2: Find the differential of $z = 3 + x \cdot \sin(2xy)$ (a) in general, and (b) at the point $(x,y) = (1, \pi/2)$.

(c) Use the result of part (b) to approximate $f(1.3, \frac{\pi}{2} - 0.1)$ and $f(0.99, \frac{\pi}{2} + 0.2)$.

(d) Compare the results of (c) with the exact values of $f(1.3, \frac{\pi}{2} - 0.1)$ and $f(0.99, \frac{\pi}{2} + 0.2)$.

Examples 4 and Practice 2(c and d) compare the value of f found by moving **along the tangent plane** to the actual value of f found **on the surface**. When the sideways movement is "small" (when dx and dy are both small), then the "along the tangent plane" value of z is close to the "on the surface" value of z, the actual value of f.

PROBLEMS

In problems 1 – 8, find an equation for the tangent plane to the given surface at the given point.

1. $z = x^2 + 4y^2$ at (2, 1, 8)
2. $z = x^2 - y^2$ at (3, –2, 5)
3. $z = 5 + (x - 1)^2 + (y + 2)^2$ at (2, 0, 10)
4. $z = \sin(x + y)$ at (1, –1, 0)
5. $z = \ln(2x + y)$ at (–1, 3, 0)
6. $z = e^x \cdot \ln(y)$ at (3, 1, 0)
7. $z = xy$ at (–1, 2, –2)
8. $z = \sqrt{x - y}$ at (5, 1, 2)

In problems 9 – 18, find the differential of the given function.

9. $z = x^2y^3$
10. $z = x^4 - 5x^2y + 6xy^3 + 10$
11. $z = \dfrac{1}{x^2 + y^2}$
12. $z = y \cdot e^{xy}$
13. $u = e^x \cdot \cos(xy)$
14. $v = \ln(2x - 3y)$
15. $w = x^2y + y^2z$
16. $w = x \sin(yz)$
17. $w = \ln(\sqrt{x^2 + y^2 + z^2})$
18. $w = \dfrac{x + y}{y + z}$
19. If $z = 5x^2 + y^2$ and (x, y) changes from (1, 2) to (1.05, 2.1), compare the values of Δz and dz.
20. If $z = x^2 - xy + 3y^2$ and (x, y) changes from (3, –1) to (2.96, –0.95), compare the values of Δz and dz.

In problems 21 – 24, use differentials to approximate the value of f at the given point.

21. $f(x, y) = \sqrt{x^2 - y^2}$ at (5.01, 4.02)

22. $f(x, y) = \sqrt{20 - x^2 - 7y^2}$ at (1.95, 1.08)

23. $f(x, y) = \ln(x - 3y)$ at (6.9, 2.06)

24. $f(x, y) = y \cdot e^{xy}$ at (0.2, 1.96)

25. The length and width of a rectangle are measured as 30 cm and 24 cm, respectively, with an error in measurement of at most 0.1 cm in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

26. The dimensions of a closed rectangular box are measured as 80 cm, 60 cm, and 50 cm, respectively, with a possible error of 0.2 cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.

27. Use differentials to estimate the amount of tin in a closed tin can with diameter 8 cm and height 12 cm if the tin is 0.04 cm thick.

28. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the wall is 0.05 cm thick and the metal in the top and bottom is 0.1 cm thick.

29. A boundary stripe 3 in. wide is painted around a rectangle whose dimensions are 100 ft. by 200 ft. Use differentials to approximate the number of square feet of paint in the stripe.

30. The pressure, volume, and temperature of a mole of an ideal gas are related by the equation $PV = 8.31T$, where P is measured in kilopascals, V in liters, and T in ^{o}K (= ^{o}C + 273). Use differentials to find the approximate change in pressure if the volume increases from 12 L to 12.3 L and the temperature decreases from $310^{o}K$ to $305^{o}K$.

PRACTICE ANSWERS

Practice 1: $z = f(a,b) + f_x(a,b)(x - a) + f_y(a,b)(y - b)$

with $f(x,y) = 5xy^2 + 7y + \sin(xy) - 2$, $a = 0$, $b = 1$, and $f(0,1) = 5$.

$f_x(x,y) = 5y^2 + y \cdot \cos(xy)$ so $f_x(0,1) = 5 + 1 = 6$.

$f_y(x,y) = 10xy + 7 + x \cdot \cos(xy)$ so $f_y(0,1) = 7$.

Then the equation of the tangent plane to f at (0,1,5) is $z = 5 + 6(x - 0) + 7(y - 1) = 6x + 7y - 2$.

Practice 2: (a) $\mathbf{dz} = f_x(a, b)\,dx + f_y(a, b)\,dy = \{ x\bullet\cos(2xy)\bullet 2y + \sin(2xy)\}\,dx + \{ x\bullet\cos(2xy)\bullet 2x \}\,dy$

(b) at $(1, \pi/2)$,

$\mathbf{dz} = \{1\bullet\cos(2\bullet 1\bullet\pi/2)\bullet 2\bullet\pi/2 + \sin(2\bullet 1\bullet\pi/2)\}\,dx + \{1\bullet\cos(2\bullet 1\bullet\pi/2)\bullet 2\bullet 1\}\,dy$ so

$\mathbf{dz} = (-\pi)\,dx + (-2)\,dy$.

(c)&(d) For $f(1.3, \frac{\pi}{2} - 0.1)$, $dx = 0.3$ and $dy = -0.1$ so $dz = (-\pi)(0.3) + (-2)(-0.1) \approx -0.742$.

Then $f(1.3, \frac{\pi}{2} - 0.1) \approx f(1, \frac{\pi}{2}) + dz = 3 + (-0.742) = 2.258$.

Actually, $f(1.3, \frac{\pi}{2} - 0.1) = 2.18006691401$ so the approximation "error" is 0.078,

an "error" of less than 4%.

For $f(0.99, \frac{\pi}{2} + 0.2)$, $dx = -0.01$ and $dy = 0.2$ so $dz = (-\pi)(-0.01) + (-2)(0.2) = -0.369$.

Then $f(0.99, \frac{\pi}{2} + 0.2) \approx f(1, \frac{\pi}{2}) + dz = 3 + (-0.369) = 2.631$.

Actually, $f(0.99, \frac{\pi}{2} + 0.2) = 2.64700487061$ so the approximation "error" is 0.016,

an "error" of less than 1%.

Selected Answers

1. $4x + 8y - z = 8$

3. $2x + 4y - z + 6 = 0$

5. $2x + y - z = 1$

7. $2x - y - z + 2 = 0$

9. $dz = 2xy^3\,dx + 3x^2y^2\,dy$

11. $dz = -2x(x^2 + y^2)^{-2}\,dx - 2y(x^2 + y^2)^{-2}\,dy$

13. $du = e^x\bullet(\cos(xy) - y\bullet\sin(xy))\,dx - x\bullet e^x\bullet\sin(xy)\,dy$

15. $dw = 2xy\,dx + (x^2 + 2yz)\,dy + y^2\,dz$

17. $dw = (x^2 + y^2 + z^2)^{-1}(x\,dx + y\,dy + z\,dz)$

19. $\Delta z = 0.9225$ and $dz = 0.9$

21. 2.9923. –0.28

25. 5.4 cm^2

27. 16 cm^3

29. 150

Appendix: Maple commands to create figures

In the following Maple routines:

xa and ya are the x and y coordinates, respectively, of the point of tangency,
the formula for za is our function (in terms of xa and ya),
fx is f_x (in terms of xa and ya), and
fy is f_y (in terms of xa and ya).
L is the length and width of one of the rectangles making up the plane, and
N is the number of rectangles in the plane to each side of the point.
xmin, xmax, ymin, ymax, zmin, and zmax specify the viewing "rectange in 3D.
orientation= [θdegrees, ϕdegrees] is the viewing position in spherical coordinates.

```
> with(plots);
```

> MAKES FIG. 2

```
xa:=.6: ya:=2:za:=16-2*xa^2-ya^2:
fx:=-4*xa: fy:=-2*ya:
L:=0.3:N:=2:
xmin:=0: xmax:=1.5: ymin:=0: ymax:=3: zmin:=0: zmax:=16:
SURF:=plot3d(16-2*x^2-y^2, x=xmin..xmax, y=ymin..ymax, axes=normal, grid=[9,17], color=red):
PT:=pointplot3d( {[xa,ya,za]}, color=black, symbol=circle):
TANPL:=plot3d( [u,v,za+fx*(u-xa)+fy*(v-ya)], u=xa-N*L..xa+N*L, v=ya-N*L..ya+N*L, color=blue,
grid=[2*N+1,2*N+1], thickness=2):
display3d( {SURF, PT, TANPL }, orientation=[25,80], tickmarks=[2,2,4], view=0..16);
```

MAKES FIG. 4 (these commands also do all of the work for figures 5 and 6)

To modify these commands for your own function at a point you pick, you only need to change the part of the routine in boldface type.

> xa:=1: ya:=2:za:=2*xa^3+ya^2+3:
fx:=6*xa^2: fy:=2*ya:

```
L:=0.3:N:=2:
xmin:=0: xmax:=2: ymin:=0: ymax:=3: zmin:=0: zmax:=25:
```

SURF:=plot3d(**2*x^3+y^2+3**, x=xmin..xmax, y=ymin..ymax, axes=normal, grid=[9,17], color=red):

```
PT:=pointplot3d( {[xa,ya,za]}, color=black, symbol=circle):
TANPL:=plot3d( [u,v,za+fx*(u-xa)+fy*(v-ya)], u=xa-N*L..xa+N*L, v=ya-N*L..ya+N*L, color=blue,
grid=[2*N+1,2*N+1], thickness=2):
DX:=spacecurve( [u,ya,za+fx*(u-xa)],u=xa..xa+3*L, color=black, thickness=2):
DY:=spacecurve( [xa,v,za+fy*(v-ya)],v=ya..ya+3*L, color=black, thickness=2):
display3d( {SURF, PT }, orientation=[25,80], tickmarks=[3,3,4], view=0..zmax);
```

> MAKES FIG. 5

```
display3d( {SURF, PT, DX, DY }, orientation=[25,80], tickmarks=[3,3,4], view=0..zmax);
```

> MAKES FIG. 6 two views

```
display3d( {SURF, PT, DX, DY, TANPL }, orientation=[25,80], tickmarks=[3,3,4], view=0..zmax);
display3d( {SURF, PT, DX, DY, TANPL }, orientation=[70,57], tickmarks=[3,3,4], view=0..zmax);
```

MAKES FIG. 7 two views

```
> xa:=1: ya:=1:za:=2*xa^2*ya^3+ln(xa*ya)+7:
fx:=4*xa*ya^3 +1/xa: fy:=6*xa^2*ya^2+1/ya:
L:=0.3:N:=2:
xmin:=.1: xmax:=2: ymin:=.1: ymax:=2: zmin:=0: zmax:=30:
SURF:=plot3d(2*x^2*y^3+ln(x*y)+7, x=xmin..xmax, y=ymin..ymax, axes=normal, grid=[9,17],
color=red):
PT:=pointplot3d( {[xa,ya,za]}, color=black, symbol=circle):
TANPL:=plot3d( [u,v,za+fx*(u-xa)+fy*(v-ya)], u=xa-N*L..xa+N*L, v=ya-N*L..ya+N*L, color=blue,
grid=[2*N+1,2*N+1], thickness=2):
display3d( {SURF, PT, TANPL}, orientation=[30,80], tickmarks=[3,3,4], view=0..zmax);
display3d( {SURF, PT, TANPL}, orientation=[10,25], tickmarks=[3,3,4], view=0..zmax);
```