13.7 LAGRANGE MULTIPLIER METHOD

Suppose we go on a walk on a hillside, but we have to stay on a path. Where along this path are we at the highest elevation? That is the basic problem we consider in this section: how to find a maximum or minimum subject to a constraint (staying on a path). Our method, Lagrange Multipliers, is very algebraic, but it also has a geometric interpretation.

Example 1: Fig. 1 shows the level curves for a hill and several paths. The dots on path A and B are at the highest elevations along those two paths.

Practice 1: Mark the locations of maximum elevation along paths C and D.

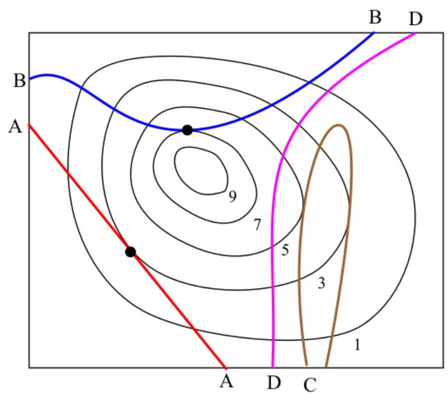

Fig. 1: A hillside and several paths

You might have noticed that at each maximum along a path the path was tangent to a level curve. This is the basis of the Lagrange Multiplier method – find the points along a path (constraint) where the path is tangent to the level curve of the function. But finding the tangent to level curves can be difficult so instead we use the fact that if two curves have parallel tangent vectors then they have parallel gradient vectors. And it is easy to calculate the gradient vector for a function.

Lagrange Multiplier Method to Maximize/Minimize f(x,y) along a path:

To find the maximum and minimum values of $f(x,y)$ subject to the constraint (condition) that $g(x,y) = k$

(a) Find all values of x, y, and λ such that $\nabla f(x,y) = \lambda \nabla g(x,y)$ and $g(x,y) = k$.

(b) Evaluate f at the points (x,y) found in step (a). The largest value of $f(x,y)$ at these points is the maximum value of f. The smallest value of $f(x,y)$ is the minimum.

Example 2: Find the maximum and minimum values of $f(x,y) = y^2 - x^2$ on the ellipse $x^2 + 4y^2 = 4$.

Solution: $f(x,y) = y^2 - x^2$ and $g(x,y) = x^2 + 4y^2 = 4$.

Then $\nabla f(x,y) = -2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g(x,y) = 2x\mathbf{i} + 8y\mathbf{j}$. The Lagrange condition that $\nabla f(x,y) = \lambda \nabla g(x,y)$ means $-2x\mathbf{i} + 2y\mathbf{j} = \lambda(2x\mathbf{i} + 8y\mathbf{j}) = 2\lambda x\mathbf{i} + 8\lambda y\mathbf{j}$ so

i: $-2x = 2\lambda x$

j: $2y = 8\lambda y$ (3 equations in 3 unknowns x, y, and λ)

constraint: $x^2 + 4y^2 = 4$

If $x = 0$, then $y^2 = 1$ so $y = +1$ or $y = -1$ (from constraint) and $\lambda = 1/4$ (from $\mathbf{j}$ condition).

Then we have the points $(x,y) = (0, 1)$ and $(0, -1)$

If $x \neq 0$, then $-2 = 2\lambda$ (from $\mathbf{i}$ condition) so $\lambda = -1$. Then $2y = -8y$ (from $\mathbf{j}$ condition) so $y = 0$ and $x^2 = 4$ so $x = +2$ or $x = -2$. Then we have the points $(x,y) = (2, 0)$ and $(-2, 0)$.

Finally, $f(0, 1) = 1$ $\quad f(0, -1) = 1$ $\quad$ (maximum value of f is 1)

$f(2, 0) = -4$ $\quad f(-2, 0) = -4$ $\quad$ (minimum value of f is -4).

Fig. 2 shows our constraint, the surface and the path along the surface.

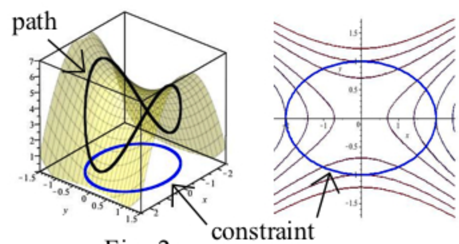

Fig. 2

The Lagrange Multiplier method allows us to trade a calculus problem for the algebra problem of solving a system of equations. But that algebra can be difficult.

Practice 2: Use the Lagrange Multiplier method to find the maximum and minimum values of $f(x,y) = 7x + 3y + 25$ on the circle $x^2 + y^2 = 9$.

Example 3: Find the maximum and minimum values of $f(x,y) = x^2 + y + 2$ on the circle $x^2 + y^2 = 1$.

Solution: $f(x,y) = x^2 + y + 2$ and $g(x,y) = x^2 + y^2 = 1$.

Then $\nabla f(x,y) = 2x\mathbf{i} + 1\mathbf{j}$ and $\nabla g(x,y) = 2x\mathbf{i} + 2y\mathbf{j}$. The Lagrange condition that $\nabla f(x,y) = \lambda \nabla g(x,y)$ means $2x\mathbf{i} + 1\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) = 2\lambda x\mathbf{i} + 2\lambda y\mathbf{j}$ so

$\mathbf{i}$: $\quad 2x = 2\lambda x$

$\mathbf{j}$: $\quad 1 = 2\lambda y$ $\qquad$ (3 equations in 3 unknowns x, y, and λ)

constraint: $\quad x^2 + y^2 = 1$

If $x = 0$, then $y^2 = 1$ so $y = +1$ or $y = -1$ (from the constraint $x^2 + y^2 = 1$).

If $y = +1$, then $\lambda = 1/2$ (from $\mathbf{j}$ condition) so one solution is $x = 0$, $y = 1$, and $\lambda = 1/2$.

If $y = -1$, then $\lambda = -1/2$ (from $\mathbf{j}$ condition) so one solution is $x = 0$, $y = -1$, and $\lambda = -1/2$.

Then we have the points $(x,y) = (0, 1)$ and $(0, -1)$

If $x \neq 0$, then $2x = 2\lambda x$ (from $\mathbf{i}$ condition) so $\lambda = 1$. Then $1 = 2y$ (from $\mathbf{j}$ condition) so $y = 1/2$ and $x^2 + (1/2)^2 = 1$ so $x = +\sqrt{3}/2$ or $x = -\sqrt{3}/2$. Then we have the points $(x,y) = (+\sqrt{3}/2 , 1/2)$ and $(-\sqrt{3}/2 , 1/2)$.

Finally, $f(0, 1) = 3$

$f(0, -1) = 1$ $\quad$ (minimum of f is 1)

$f(+\sqrt{3}/2 , 1/2) = f(-\sqrt{3}/2 , 1/2) = 13/4$

(maximum value of f is $13/4$).

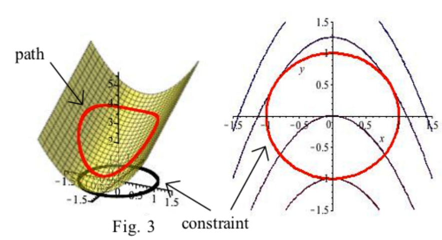

Fig. 3

Fig. 3 shows the constraint, the surface and the path.

The same idea also works for functions and constraints with three (or more) variables:

Find all values of x, y, z and λ such that $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ and $g(x,y,z) = k$,

but now we need to solve four equations in four unknowns.

Example 4: Find the volume of the largest rectangular box with a divider but no top (see Fig. 4) that can be constructed from 288 square inches of material.

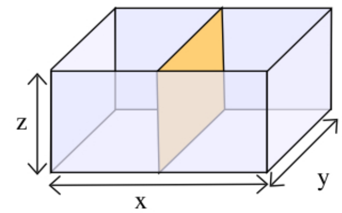

Fig. 4: Rectangular box with one divider and no top

Solution: $V(x,y,z) = xyz$ and $g(x,y,z) = xy + 2xz + 3yz = 288$.

Then $\nabla V(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and

$\nabla g(x,y,z) = (y + 2z)\mathbf{i} + (x + 3z)\mathbf{j} + (2x + 3y)\mathbf{k}$.

The Lagrange condition that $\nabla f(x,y) = \lambda \nabla g(x,y)$ means

$yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(y + 2z)\mathbf{i} + \lambda(x + 3z)\mathbf{j} + \lambda(2x + 3y)\mathbf{k}$ so

$\mathbf{i}$: $yz = \lambda y + \lambda 2z$

$\mathbf{j}$: $xz = \lambda x + \lambda 3z$ (4 equations in 4 unknowns x, y, z and λ)

$\mathbf{k}$: $xy = \lambda 2x + \lambda 3y$

constraint: $xy + 2xz + 3yz = 288$

There are a variety of ways to solve this system, but this algebraic way is relatively easy. Multiply the first equation by x, the second by y, and the third by z to get

$\mathbf{i}$: $xyz = \lambda xy + \lambda 2xz$

$\mathbf{j}$: $xyz = \lambda xy + \lambda 3yz$

$\mathbf{k}$: $xyz = \lambda 2xz + \lambda 3yz$

Then all 3 equations are equal to xyz so $\lambda xy + \lambda 2xz = \lambda xy + \lambda 3yz = \lambda 2xz + \lambda 3yz$.

Since $\lambda xy + \lambda 2xz = \lambda xy + \lambda 3yz$ then $y = \frac{2}{3}x$. Since $\lambda xy + \lambda 2xz = \lambda 2xz + \lambda 3yz$ then $z = \frac{1}{3}x$.

Putting those values for y and z into the constraint we get

$$288 = x\left(\frac{2}{3}x\right) + 2x\left(\frac{1}{3}x\right) + 3\left(\frac{2}{3}x\right)\left(\frac{1}{3}x\right) = 2x^2$$ so x = 12 inches, y = 8 inches and z = 4 inches.

The maximum volume is $V(12, 8, 4) = (12)(8)(4) = 384$ cubic inches.

Practice 3: Find the dimensions of the largest volume rectangular box with two dividers but no top (see Fig. 5) that can be constructed from 384 square centimeters of material.

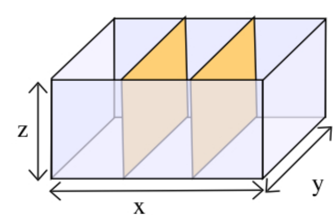

Fig. 5: Rectangular box with two dividers and no top

PROBLEMS

1. In Fig. 6 locate the maximum and minimum values of z along each path and estimate their values.

2. In Fig. 7 locate the maximum and minimum values of z along each path and estimate their values.

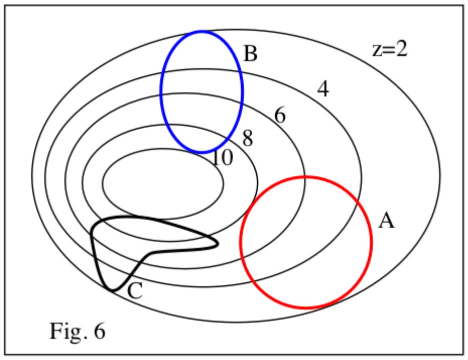

Fig. 6

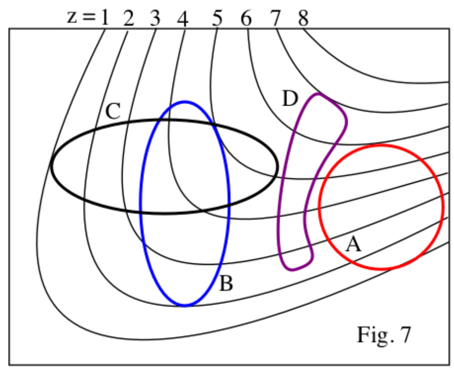

Fig. 7

In problems 3 – 11, use the Lagrange Multiplier method to find the maximum and minimum values of the given function subject to the given constraint.

3. $f(x,y) = x^2 - y^2$; $x^2 + y^2 = 1$

4. $f(x,y) = 2x + y$; $x^2 + 4y^2 = 1$

5. $f(x,y) = xy$; $9x^2 + y^2 = 4$

6. $f(x,y) = x^2 + y^2$; $x^4 + y^4 = 1$

7. $f(x,y,z) = x + 3y + 5z$; $x^2 + y^2 + z^2 = 1$

8. $f(x,y,z) = x - y + 3z$; $x^2 + y^2 + 4z^2 = 4$

9. $f(x,y,z) = xyz$; $x^2 + 2y^2 + 3z^2 = 6$

10. $f(x,y,z) = x^2y^2z^2$; $x^2 + y^2 + z^2 = 1$

11. $f(x,y,z) = x^2 + y^2 + z^2$; $x^4 + y^4 + z^4 = 1$

12. Find the maximum volume of a rectangular box with no top that has a surface area of 48 square inches.

13. Find the maximum volume of a rectangular box with no top that has a surface area of A square inches.

14.. Using your result from problem 11*, show that the area of the bottom is A/3, the total area of the front and back sides is A/3, and the total area of the two end sides is A/3.

15. Find the maximum volume of a rectangular box with no top that be built at a cost of \$15.00 if the bottom material costs \$0.05/in^2 and the materials for the sides costs \$0.01/in^2 .

16.. Find the maximum volume of a rectangular box with no top that be built at a cost of \$T if the bottom material costs \$B/in^2 and the materials for the 4 sides costs \$S/in^2 .

17. Find the maximum volume of a cylinder with no top that has a surface area of 48 square inches.

18. Find the maximum volume of a cylinder with a top that has a surface area of 48 square inches.

Practice Answers

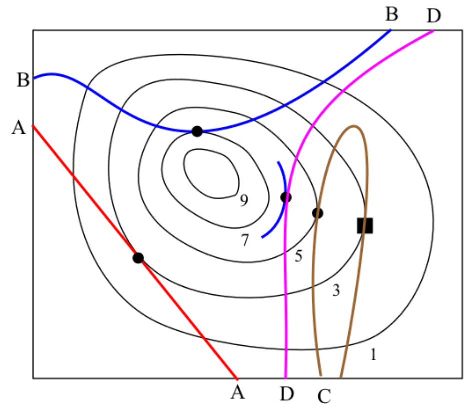

Fig. P1: A hillside and several paths

Practice 1: The dots in Fig. P1 show the locations of the maximum elevations along paths C and D. The little square on path C is a local maximum along that path. The figure also includes part of the level curve that goes through the dot on path D.

Practice 2: f(x,y)= 7x + 3y + 25 and g(x,y) = $x^2 + y^2$ so $\nabla g = \langle 2x,\ 2y \rangle$ and $\nabla f = \langle\ 7,\ 3 \rangle$. Putting these into the Lagrange equation $\nabla f = \lambda \cdot \nabla g$ we have the algebraic system

i: $7 = 2\lambda x$ (so $x \neq 0$)

i: $3 = 2\lambda y$ (so $y \neq 0$)

constraint: $x^2 + y^2 = 9$

Then (from **i**) $x = \dfrac{7}{2\lambda}$ and (from **j**) $y = \dfrac{3}{2\lambda}$. Putting these into the constraint $\left(\dfrac{7}{2\lambda}\right)^2 + \left(\dfrac{3}{2\lambda}\right)^2 = 9$

Then $49 + 9 = 36\lambda^2$ and $\lambda = \pm\dfrac{\sqrt{58}}{6} \approx \pm 1.269$. Putting these back into x and y equations, for $\lambda = \dfrac{\sqrt{58}}{6}$ we have $x = \dfrac{21}{\sqrt{58}} \approx 2.757$, $y = \dfrac{9}{\sqrt{58}} \approx 1.182$ and $f\left(\dfrac{21}{\sqrt{58}}, \dfrac{9}{\sqrt{58}}\right) = \dfrac{169}{\sqrt{58}} + 25 \approx 57.19$ the maximum value of f on the elliptical path. For $\lambda = -\dfrac{\sqrt{58}}{6}$, we have $x = \dfrac{-21}{\sqrt{58}}$, $y = \dfrac{-9}{\sqrt{58}}$ and

$f\left(\dfrac{-21}{\sqrt{58}}, \dfrac{-9}{\sqrt{58}}\right) = \dfrac{-169}{\sqrt{58}} + 25 \approx 2.81$.

Figure P2 shows the surface f, the contours for f, and the constraint g.

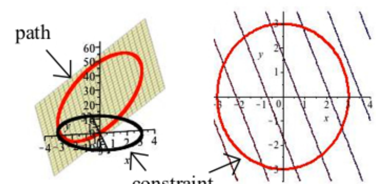

Fig. P2

Practice 3: V(x,y,z) = xyz and g(x,y,z) = xy + 2xz + 4yz = 384 cm^2.

Then ∇V(x,y,z) = yz**i** + xz**j** + xy**k** and

∇g(x,y,z) = (y + 2z)**i** + (x + 4z)**j** + (2x + 4y)**k** so

yz**i** + xz**j** + xy**k** = λ(y + 2z)**i** + λ(x + 4z)**j** + λ(2x + 4y)**k** so

i: $yz = \lambda y + \lambda 2z$

j: $xz = \lambda x + \lambda 4z$ (4 equations in 4 unknowns x, y, z and λ)

k: $xy = \lambda 2x + \lambda 4y$

constraint: $xy + 2xz + 4yz = 384$

Multiply the first equation by x, the second by y, and the third by z,

Then all 3 equations are equal to xyz so λxy + λ2xz = λxy + λ4yz = λ2xz + λ4yz. Then

$384 = x\left(\dfrac{1}{2}x\right) + 2x\left(\dfrac{1}{4}x\right) + 4\left(\dfrac{1}{2}x\right)\left(\dfrac{1}{4}x\right) = \dfrac{3}{2}x^2$ so x=16 cm, y=8 cm and z=2 cm

and he maximum volume is 512 cm^3 .

13.7 Selected Answers

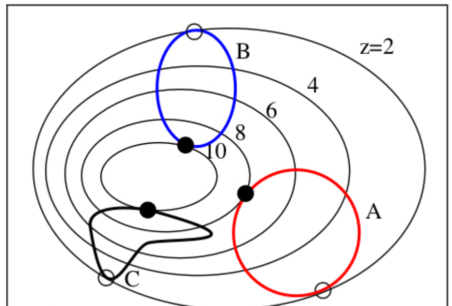

Fig. 8

1. See Fig. 8. The solid circles marks the locations of the maximum z values along each path, and the open circles mark the locations of the minimums, A: max z=8, min z=2. B: max z=10, min z=2, C: max z=10, min z=2.

3. maximum $f(\pm 1,0) = 1$, minimum $f(0,\pm 1) = -1$

4. maximum $f(\sqrt{2}/3, \sqrt{2}) = f(-\sqrt{2}/3, -\sqrt{2}) = 2/3$, minimum $f(\sqrt{2}/3, -\sqrt{2}) = f(-\sqrt{2}/3, \sqrt{2}) = -2/3$

7. maximum $f(1/\sqrt{35}, 3/\sqrt{35}, 5/\sqrt{35}) = \sqrt{35}$, minimum $f(-1/\sqrt{35}, -3/\sqrt{35}, -5/\sqrt{35}) = -\sqrt{35}$

9. $x = \pm\sqrt{2}$, $y = \pm 1$, $z = \pm\sqrt{\frac{2}{3}}$. maximum f is $2/\sqrt{3}$ (when all are positive or one is positive and two are negative), minimum f is $-2/\sqrt{3}$.

11. maximum is $\sqrt{3} = f\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$, minimum is $1 = f(\pm 1,0,0) = f(0,\pm 1,0) = f(0,0,\pm 1)$

13. V=xyz with xy+2xz+2yz=A. Maximum V is $\frac{1}{2}\left(\frac{A}{3}\right)^{3/2}$ and that occurs when $x = y = \sqrt{\frac{A}{3}}$ and $z = \frac{1}{2}\sqrt{\frac{A}{3}}$.

15. V=xyz with $5xy + 1(2xz + 2yz) = 1500$ (working in cents). Maximum volume is 2500 in^3 when x = y = 10 inches and z = 25 inches. Note that the cost of the bottom is \$5.00, the total cost of the two ends is \$5.00, and the total costs of the other two sides is \$5.00 .

16. $V = xyz$ with $Bxy + S2xz + 2Syz = T$, $x = y = \sqrt{\frac{T}{3B}}$, $z = \frac{1}{2S}\sqrt{\frac{BT}{3}}$ and maximum volume is $V = \frac{T}{3B} \cdot \frac{1}{2S} \cdot \sqrt{\frac{BT}{3}}$. The cost of the bottom is T/3.

17. maximum volume is $64/\sqrt{\pi}$ when $r = 4/\sqrt{\pi}$ and $h = 4/\sqrt{\pi}$.