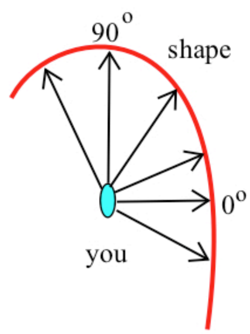
9.1 POLAR COORDINATES

The rectangular coordinate system is immensely useful, but it is not the only way to assign an address to a point in the plane and sometimes it is not the most useful. In many experimental situations, our location is fixed and we or our instruments, such as radar, take readings in different directions (Fig. 1); this information can be graphed using rectangular coordinates (e.g., with the angle on the horizontal axis and the measurement on the vertical axis). Sometimes, however, it is more useful to plot the information in a way similar to the way in which it was collected, as magnitudes along radial lines (Fig. 2). This system is called the Polar Coordinate System.



Taking Measurements

angle	distance
0^o	120 ft
30^o	140
60^o	230
90^o	270
120^o	200

Numerical Data

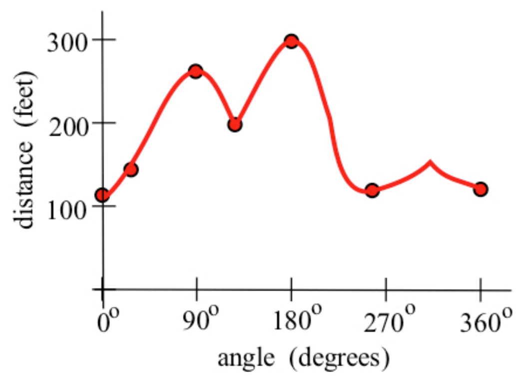
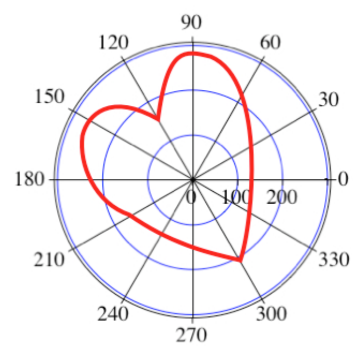


Fig. 1

Rectangular Coordinate Graph of Data

In this section we introduce polar coordinates and examine some of their uses. We start with graphing points and functions in polar coordinates, consider how to change back and forth between the rectangular and polar coordinate systems, and see how to find the slopes of lines tangent to polar graphs. Our primary reasons for considering polar coordinates, however, are that they appear in applications, and that they provide a "natural" and easy way to represent some kinds of information.



Polar coordinate graph of data

Fig. 2

Example 1: SOS! You've just received a distress signal from a ship located at A on your radar screen (Fig. 3). Describe its location to your captain so your vessel can speed to the rescue.

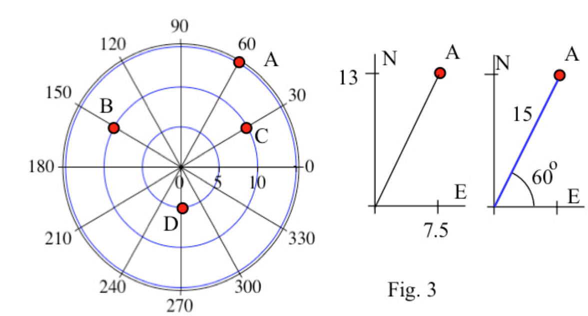


Fig. 3

Solution: You could convert the relative location of the other ship to rectangular coordinates and then tell your captain to go due east for 7.5 miles and north for 13 miles, but that certainly is not the quickest way to reach the other ship. It is better to tell the captain to sail for 15 miles in the direction of 60°. If the distressed ship was at B on the radar screen, your vessel should sail for 10 miles in the direction 150°. (Real radar screens have 0° at the top of the screen, but the convention in mathematics is to put 0° in the direction of the positive x–axis and to measure positive angles counterclockwise from there. And a real sailor speaks of "bearing" and "range" instead of direction and magnitude.)

Practice 1: Describe the locations of the ships at C and D in Fig. 3 by giving a distance and a direction to those ships from your current position at the center of the radar screen.

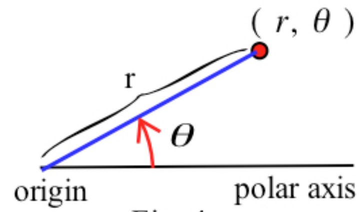


Fig. 4

Points in Polar Coordinates

To construct a polar coordinate system we need a starting point (called the **origin** or **pole**) for the magnitude measurements and a starting direction (called the **polar axis**) for the angle measurements (Fig. 4).

A polar coordinate pair for a point P in the plane is an ordered pair (r,θ) where r is the directed distance along a radial line from O to P, and θ is the angle formed by the polar axis and the segment OP (Fig. 4).

The angle θ is positive when the angle of the radial line OP is measured counterclockwise from the polar axis, and θ is negative when measured clockwise.

Degree or Radian Measure for θ? Either degree or radian measure can be used for the angle in the polar coordinate system, but when we differentiate and integrate trigonometric functions of θ we will want all of the angles to be given in radian measure. From now on, we will primarily use radian measure. You should assume that all angles are given in radian measure unless the units " ° " ("degrees") are shown.

Example 2: Plot the points with the given polar coordinates: A(2, 30°), B(3, $\pi/2$), C(–2, $\pi/6$), and D(–3, 270°).

Solution: To find the location of A, we look along the ray that makes an angle of 30° with the polar axis, and then take two steps in that direction (assuming 1 step = 1 unit). The locations of A and B are shown in Fig. 5.

To find the location of C, we look along the ray which makes an angle of $\pi/6$ with the polar axis, and then we take two steps backwards since r = –2 is negative. Fig. 6 shows the locations of C and D.

Notice that the points B and D have different addresses, (3, $\pi/2$) and (–3, 270°), but the same location.

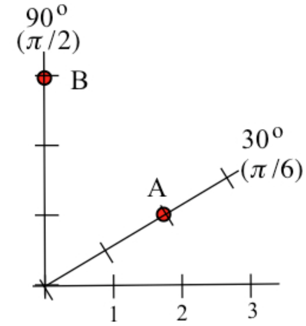


Fig. 5

Practice 2: Plot the points with the given polar coordinates: A(2, π/2), B(2, –120°), C(–2, π/3), D(–2, –135°), and E(2, 135°). Which two points coincide?

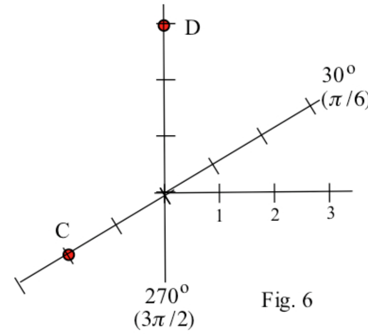


Fig. 6

Each polar coordinate pair (r,θ) gives the location of one point, but each location has lots of different addresses in the polar coordinate system: the polar coordinates of a point are not unique. This nonuniqueness of addresses comes about in two ways. First, the angles θ, $\theta \pm 360°$, $\theta \pm 2{\cdot}360°$, . . . all describe the same radial line (Fig. 7), so the polar coordinates (r, θ), (r, $\theta \pm 360°$), (r, $\theta \pm 2{\cdot}360°$) , . . . all locate the same point.

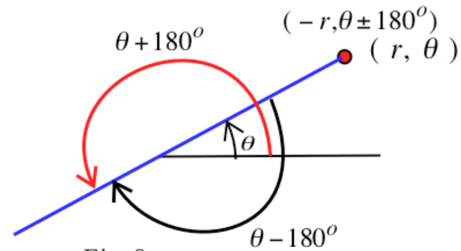


Fig. 8

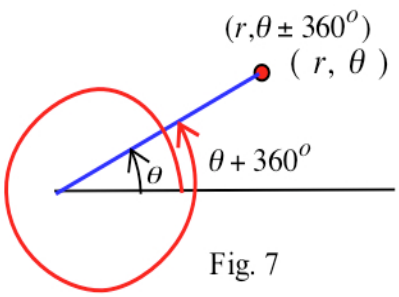


Fig. 7

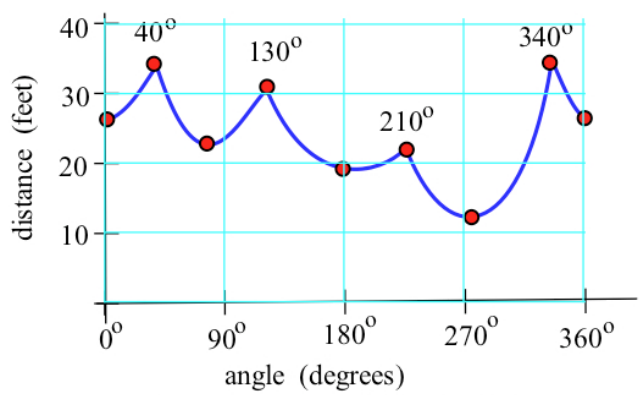
Secondly, the angle $\theta \pm 180°$ describes the radial line pointing in exactly the opposite direction from the radial line described by the angle θ (Fig. 8), so the polar coordinates (r, θ) and (–r, $\theta \pm 180°$) locate the same point. A polar coordinate pair gives the location of exactly one point, but the location of one point is described by many (an infinite number) different polar coordinate pairs.

Note: In the rectangular coordinate system we use (x, y) and y = f(x): first variable independent and second variable dependent. In the polar coordinate system we use (r, θ) and r = f(θ): first variable dependent and second variable independent, a reversal from the rectangular coordinate usage.

Practice 3: Table 1 contains measurements to the edge of a plateau taken by a remote sensor which crashed on the plateau. Fig. 9 shows the data plotted in rectangular coordinates. Plot the data in polar coordinates and determine the shape of the top of the plateau.

angle	distance	angle	distance
0°	28 feet	150°	22 feet
20°	30	180°	18
40°	36	210°	21
60°	27	230°	13
80°	24	270°	10
100°	24	330°	18
130°	30	340°	30

Table 1



Rectangular Coordinate Graph of Data

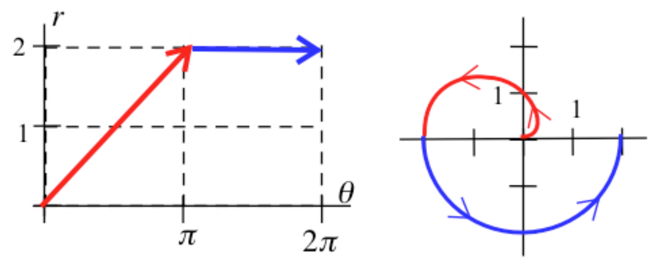
Fig. 9

Graphing Functions in the Polar Coordinate System

In the rectangular coordinate system, we have worked with functions given by tables of data, by graphs, and by formulas. Functions can be represented in the same ways in polar coordinates.

- If a function is given by a table of data, we can graph the function in polar coordinates by plotting individual points in a polar coordinate system and connecting the plotted points to see the shape of the graph. By hand, this is a tedious process; by calculator or computer, it is quick and easy.

- If the function is given by a rectangular coordinate graph of magnitude as a function of angle, we can read coordinates of points on the rectangular graph and replot them in polar coodinates. In essence, as we go from the rectangular coordinate graph to the polar coordinate graph we "wrap" the rectangular graph around the "pole" at the origin of the polar coordinate system. (Fig. 10)



(a) rectangular coordinates (b) polar

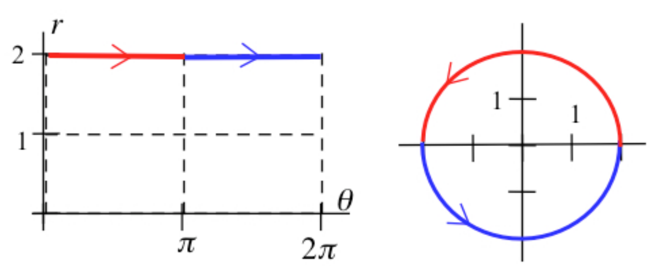
Fig. 10

- If the function is given by a formula, we (or our calculator) can graph the function to help us obtain information about its behavior. Typically, a graph is created by evaluating the function at a lot of points and then plotting the points in the polar coordinate system. Some of the following examples illustrate that functions given by simple formulas may have rather exotic graphs in the polar coordinate system.

If a function is already given by a polar coordinate graph, we can use the graph to answer questions about the behavior of the function. It is usually easy to locate the maximum value(s) of r on a polar coordinate graph, and, by moving counterclockwise around the graph, we can observe where r is increasing, constant, or decreasing.

Example 3: Graph $r = 2$ and $r = \pi - \theta$ in the polar coordinate system for $0 \le \theta \le 2\pi$.

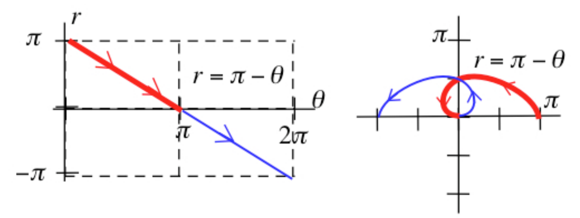
Solution: $r = 2$: In every direction θ, we simply move 2 units along the radial line and plot a point. The resulting polar graph (Fig. 11b) is a circle centered at the origin with a radius of 2. In the rectangular coordinate system, the graph of a constant $y = k$ is a horizontal line. In the polar coordinate system, the graph of a constant $r = k$ is a circle with radius $|k|$.



(a) rectangular coordinates (b) polar

Fig. 11

$r = \pi - \theta$: The rectangular coordinate graph of $r = \pi - \theta$ is shown in Fig. 12a. If we read the values of r and θ from the rectangular coordinate graph and plot them in polar coordinates, the result is the shape in Fig. 12b. The different line thicknesses are used in the figures to help you see which values from the rectangular graph became which parts of the loop in the polar graph.



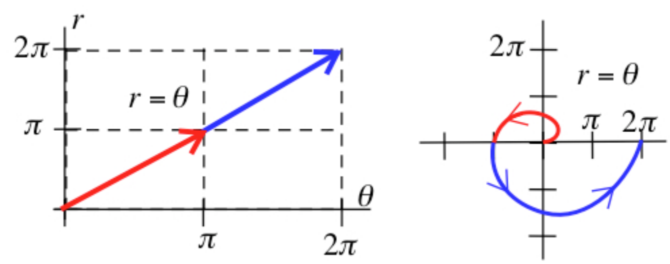
(a) rectangular coordinates (b) polar

Fig. 12

Practice 4: Graph $r = -2$ and $r = \cos(\theta)$ in the polar coordinate system.

Example 4: Graph $r = \theta$ and $r = 1 + \sin(\theta)$ in the polar coordinate system.

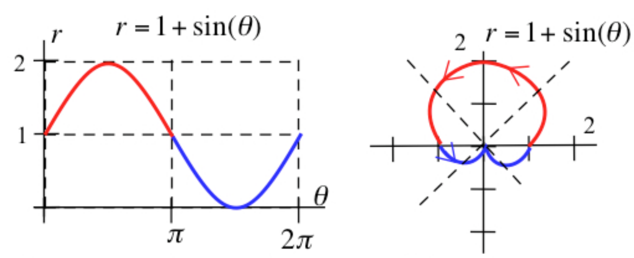
Solution: $r = \theta$: The rectangular coordinate graph of $r = \theta$ is a straight line (Fig. 13a). If we read the values of r and θ from the rectangular coordinate graph and plot them in polar coordinates, the result is the spiral, called an Archimedean spiral, in Fig. 13b.



(a) rectangular coordinates (b) polar

Fig. 13

$r = 1 + \sin(\theta)$: The rectangular coordinate graph of $r = 1 + \sin(\theta)$ is shown in Fig. 14a, and it is the graph of the sine curve shifted up 1 unit. In polar coordinates, the result of adding 1 to sine is much less obvious and is shown in Fig. 14b.



(a) rectangular coordinates (b) polar

Fig. 14

Practice 5: Plot the points in Table 2 in the polar coordinate system and connect them with a smooth curve. Describe the shape of the graph in words.

angle (radians)	distance (meters)
0	3.0
$\pi/6$	1.9
$\pi/4$	1.7
$\pi/3$	1.6
$\pi/2$	2.0

Table 2

Fig. 15 shows the effects of adding various constants to the rectangular and polar graphs of r = sin(θ). In rectangular coordinates the result is a graph shifted up or down by k units. In polar coordinates, the result **may** be a graph with an entirely different shape (Fig. 16).

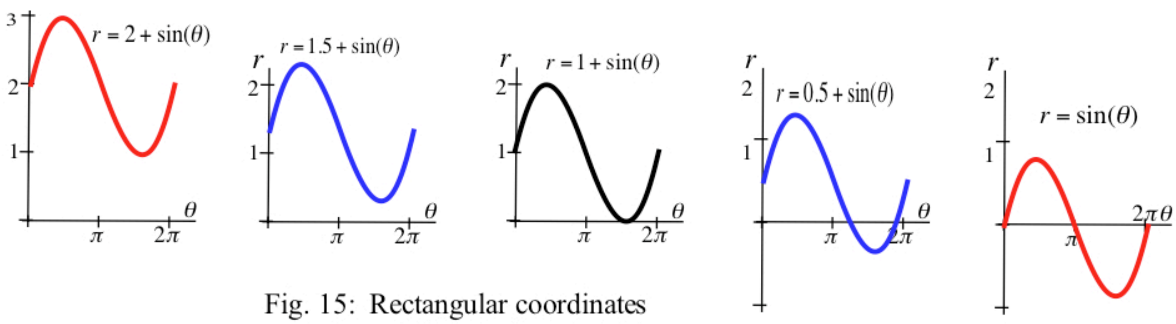


Fig. 15: Rectangular coordinates

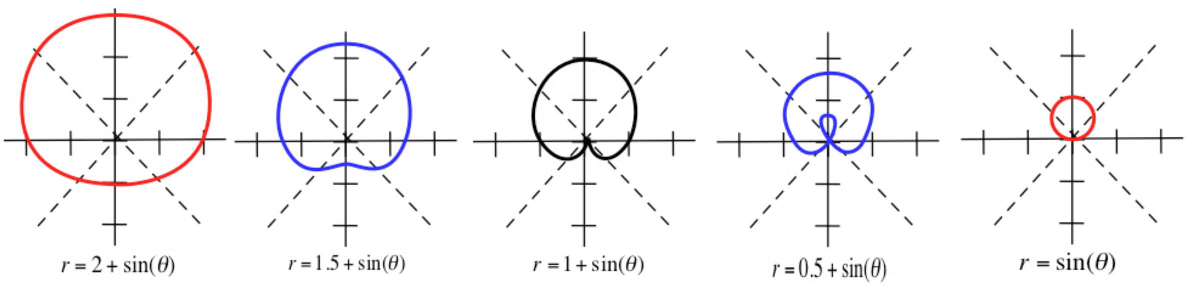


Fig. 16: Polar coordinates

Fig. 17 shows the effects of adding a constant to the independent variable in rectangular coordinates, and the result is a horizontal shift of the original graph. In polar coordinates, Fig. 18, the result is a rotation of the original graph. Generally it is difficult to find formulas for rotated figures in rectangular coordinates, but rotations are easy in polar coordinates.

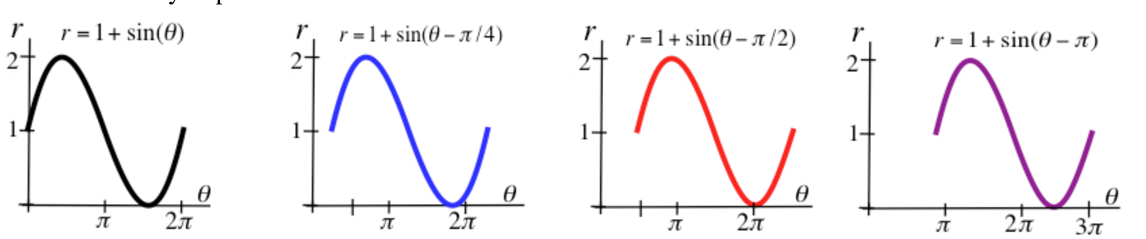


Fig. 17: Rectangular coordinates

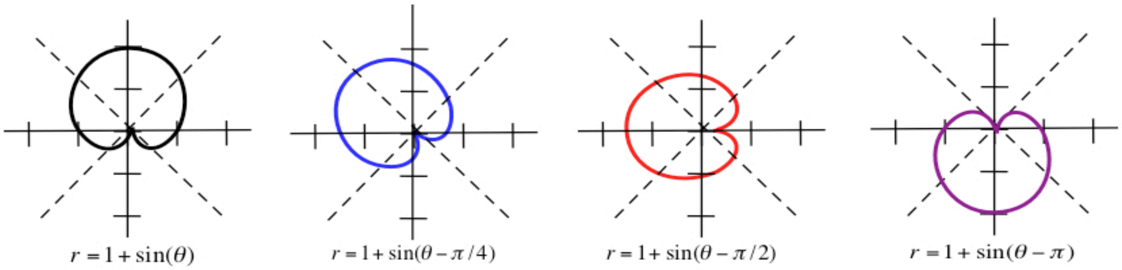


Fig. 18: Polar coordinates

The formulas and names of several functions with exotic shapes in polar coordinates are given in the problems. Many of them are difficult to graph "by hand," but by using a graphing calculator or computer you can enjoy the shapes and easily examine the effects of changing some of the constants in their formulas.

Converting Between Coordinate Systems

Sometimes both rectangular and polar coordinates are needed in the same application, and it is necessary to change back and forth between the systems. In such a case we typically place the two origins together and align the polar axis with the positive x–axis. Then the conversions are straightforward exercises using trigonometry and right triangles (Fig. 19).

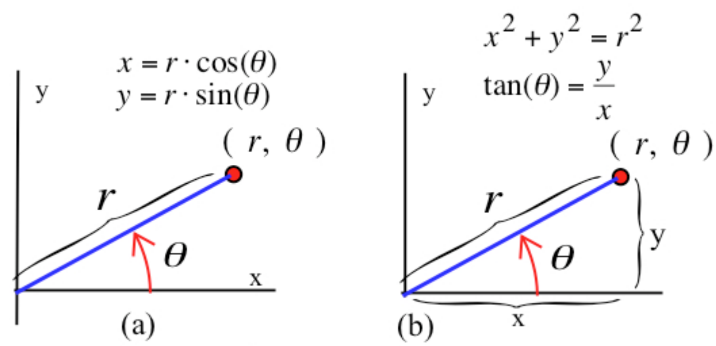


Fig. 19

Polar to Rectangular (Fig. 19a)	Rectangular to Polar (Fig. 19b)
$x = r \cdot \cos(\theta)$	$r^2 = x^2 + y^2$
$y = r \cdot \sin(\theta)$	$\tan(\theta) = \frac{y}{x}$ (if $x \neq 0$)

Example 5: Convert (a) the polar coordinate point P(7, 0.4) to rectangular coordinates, and (b) the rectangular coordinate point R(12, 5) to polar coordinates.

Solution: (a) $r = 7$ and $\theta = 0.4$ (Fig. 20) so $x = r \cdot \cos(\theta) = 7 \cdot \cos(0.4) \approx 7(0.921) = 6.447$ and $y = 7 \cdot \sin(0.4) \approx 7(0.389) = 2.723$.

(b) $x = 12$ and $y = 5$ so $r^2 = x^2 + y^2 = 144 + 25 = 169$ and $\tan(\theta) = y/x = 5/12$ so we can take $r = 13$ and $\theta = \arctan(5/12) \approx 0.395$. The polar coordinate addresses $(13, 0.395 \pm n \cdot 2\pi)$ and $(-13, 0.395 \pm (2n+1) \cdot \pi)$ give the location of the same point.

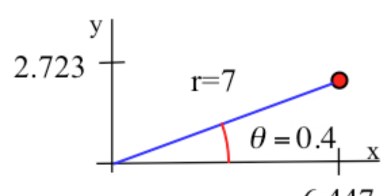


Fig. 20

The conversion formulas can also be used to convert function equations from one system to the other.

Example 6: Convert the rectangular coordinate linear equation $y = 3x + 5$ (Fig. 21) to a polar coordinate equation.

Solution: This simply requires that we replace x with $r \cdot \cos(\theta)$ and y with $r \cdot \sin(\theta)$. Then

$$y = 3x + 5 \text{ becomes } r \cdot \sin(\theta) = 3r \cdot \cos(\theta) + 5$$

so $r \cdot (\sin(\theta) - 3\cos(\theta)) = 5$ and $r = 5/(\sin(\theta) - 3\cos(\theta))$. This final representation is valid only for θ such that $\sin(\theta) - 3\cos(\theta) \neq 0$.

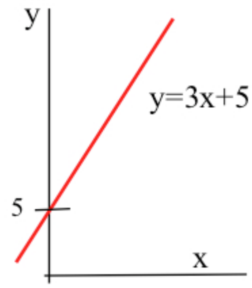


Fig. 21

Practice 6: Convert the polar coordinate equation $r^2 = 4r \cdot \sin(\theta)$ to a rectangular coordinate equation.

Example 7: **Robotic Arm**: A robotic arm has a hand at the end of a 12 inch long forearm which is connected to an 18 inch long upper arm (Fig. 22). Determine the position of the hand, relative to the shoulder, when $\theta = 45°$ ($\pi/4$) and $\phi = 30°$ ($\pi/6$).

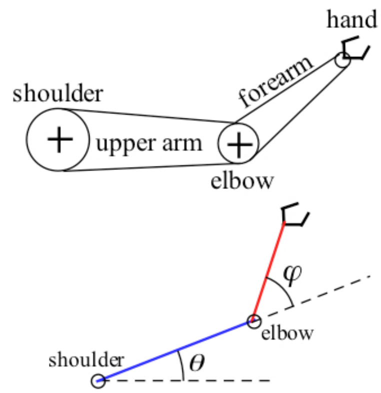


Fig. 22: Robot arm

Solution: The hand is $12 \cdot \cos(\pi/4 + \pi/6) \approx 3.1$ inches to the right of the elbow (Fig. 23) and $12\sin(\pi/4 + \pi/6) \approx 11.6$ inches above the elbow. Similarly, the elbow is $18 \cdot \cos(\pi/4) \approx 12.7$ inches to the right of the shoulder and $18 \cdot \sin(\pi/4) \approx 12.7$ inches above the shoulder. Finally, the hand is approximately $3.1 + 12.7 = 15.8$ inches to the right of the shoulder and approximately $11.6 + 12.7 = 24.3$ inches above the shoulder. In polar coordinates, the hand is approximately 29 inches from the shoulder, at an angle of about 57° (about 0.994 radians) above the horizontal.

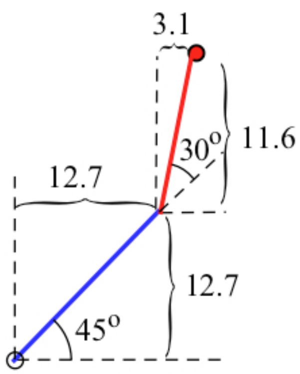


Fig. 23

Practice 7: Determine the position of the hand, relative to the shoulder, when $\theta = 30°$ and $\phi = 45°$.

Graphing Functions in Polar Coordinates on a Calculator or Computer

Some calculators and computers are programmed to graph polar functions simply by keying in the formula for r, either as a function of θ or of t, but others are only designed to display rectangular coordinate graphs. However, we can graph polar functions on most of them as well by using the rectangular to polar conversion formulas, selecting the parametric mode (and the radian mode) on the calculator, and then graphing the resulting parametric equations in the rectangular coordinate system:

To graph $r = r(\theta)$ for θ between 0 and 3π,

define $x(t) = r(t) \cdot \cos(t)$ and $y(t) = r(t) \cdot \sin(t)$

and graph the parametric equations $x(t), y(t)$ for t taking values from 0 to 9.43.

Which Coordinate System Should You Use?

There are no rigid rules. Use whichever coordinate system is easier or more "natural" for the problem or data you have. Sometimes it is not clear which system to use until you have graphed the data both ways, and some problems are easier if you switch back and forth between the systems.

Generally, the polar coordinate system is easier if

- the data consists of measurements in various directions (radar)
- your problem involves locations in relatively featureless locations (deserts, oceans, sky)
- rotations are involved.

Typically, the rectangular coordinate system is easier if

- the data consists of measurements given as functions of time or location (temperature, height)
- your problem involves locations in situations with an established grid (a city, a chess board)
- translations are involved.

PROBLEMS

1. Give the locations in polar coordinates (using radian measure) of the points labeled A, B, and C in Fig. 24.

2. Give the locations in polar coordinates (using radian measure) of the points labeled D, E, and F in Fig. 24.

3. Give the locations in polar coordinates (using radian measure) of the points labeled A, B, and C in Fig. 25.

4. Give the locations in polar coordinates (using radian measure) of the points labeled D, E, and F in Fig. 25.

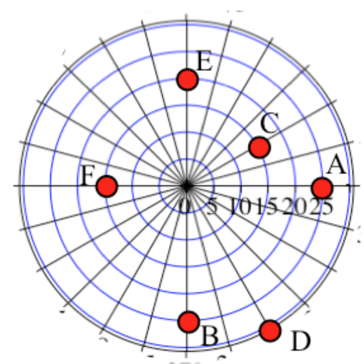


Fig. 24

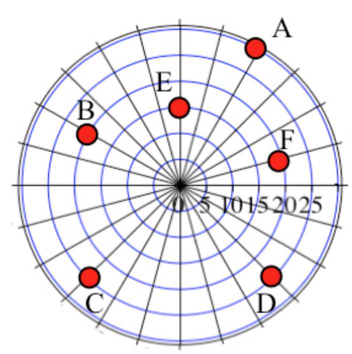


Fig. 25

In problems 5–8, plot the points A – D in polar coordinates, connect the dots by line segments in order (A to B to C to D to A), and name the approximate shape of the resulting figure.

5. A(3, 0°), B(2, 120°), C(2, 200°), and D(2.8, 315°).

6. A(3, 30°), B(2, 130°), C(3, 150°), and D(2, 280°).

7. A(2, 0.175), B(3, 2.269), C(2, 2.618), and D(3, 4.887).

8. A(3, 0.524), B(2, 2.269), C(3, 2.618), and D(2, 4.887).

In problems 9–14, the rectangular coordinate graph of a function $r = r(\theta)$ is shown. Sketch the polar coordinate graph of $r = r(\theta)$.

9. The graph in Fig. 26.

10. The graph in Fig. 27.

11. The graph in Fig. 28.

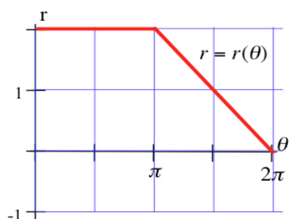


Fig. 26

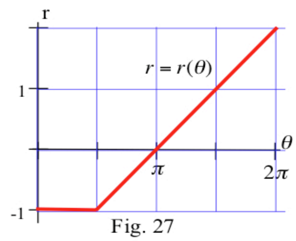


Fig. 27

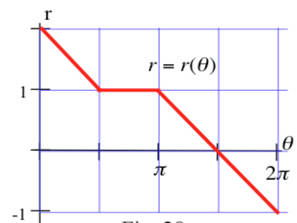


Fig. 28

12. The graph in Fig. 29.

13. The graph in Fig. 30.

14. The graph in Fig. 31.

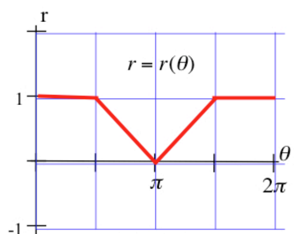


Fig. 29

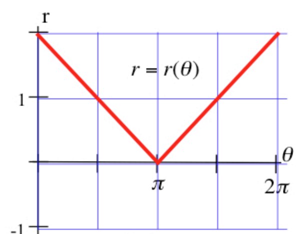


Fig. 30

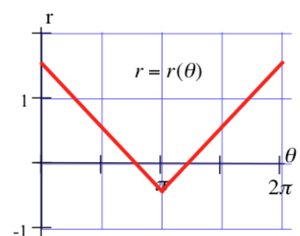


Fig. 31

15. The rectangular coordinate graph of $r = f(\theta)$ is shown in Fig. 32.
   (a) Sketch the rectangular coordinate graphs of $r = 1 + f(\theta)$, $r = 2 + f(\theta)$, and $r = -1 + f(\theta)$.
   (b) Sketch the polar coordinate graphs of $r = f(\theta)$, $r = 1 + f(\theta)$, $r = 2 + f(\theta)$, and $r = -1 + f(\theta)$.

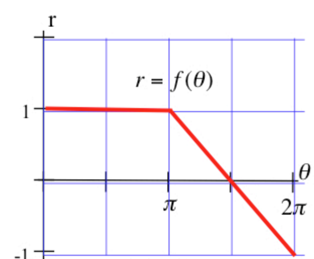


Fig. 32

16. The rectangular coordinate graph of $r = g(\theta)$ is shown in Fig. 33.
   (a) Sketch the rectangular coordinate graphs of $r = 1 + g(\theta)$, $r = 2 + g(\theta)$, and $r = -1 + g(\theta)$.
   (b) Sketch the polar coordinate graphs of $r = g(\theta)$, $r = 1 + g(\theta)$, $r = 2 + g(\theta)$, and $r = -1 + g(\theta)$.

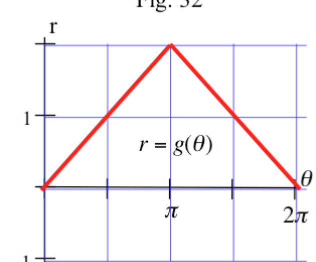


Fig. 33

17. The rectangular coordinate graph of $r = f(\theta)$ is shown in Fig. 34.
   (a) Sketch the rectangular coordinate graphs of $r = 1 + f(\theta)$, $r = 2 + f(\theta)$, and $r = -1 + f(\theta)$.
   (b) Sketch the polar coordinate graphs of $r = f(\theta)$, $r = 1 + f(\theta)$, $r = 2 + f(\theta)$, and $r = -1 + f(\theta)$.

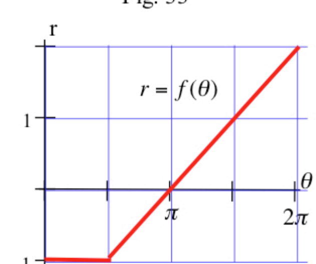


Fig. 34

18. The rectangular coordinate graph of $r = g(\theta)$ is shown in Fig. 35.
    (a) Sketch the rectangular coordinate graphs of $r = 1 + g(\theta)$, $r = 2 + g(\theta)$, and $r = -1 + g(\theta)$.
    (b) Sketch the polar coordinate graphs of $r = g(\theta)$, $r = 1 + g(\theta)$, $r = 2 + g(\theta)$, and $r = -1 + g(\theta)$.

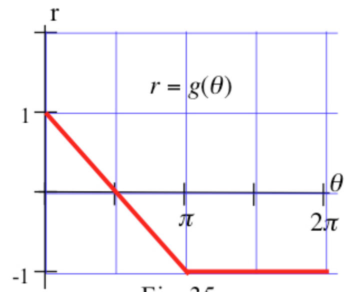


Fig. 35

19. Suppose the rectangular coordinate graph of $r = f(\theta)$ has the horizontal asymptote $r = 3$ as θ grows arbitrarily large. What does that tell us about the polar coordinate graph of $r = f(\theta)$ for large values of θ?

20. Suppose the rectangular coordinate graph of $r = f(\theta)$ has the vertical asymptote $\theta = \pi/6$: $\lim_{\theta \to \pi/6} f(\theta) = +\infty$. What does that tell us about the polar coordinate graph of $r = f(\theta)$ for values of θ near $\pi/6$?

A computer or graphing calculator is recommended for the problems marked with a * .

In problems 21–40, graph the functions in polar coordinates for $0 \le \theta \le 2\pi$.

21. $r = -3$	22. $r = 5$	23. $\theta = \pi/6$	24. $\theta = 5\pi/3$
25. $r = 4 \cdot \sin(\theta)$	26. $r = -2 \cdot \cos(\theta)$	27. $r = 2 + \sin(\theta)$	28. $r = -2 + \sin(\theta)$
29. $r = 2 + 3 \cdot \sin(\theta)$	30. $r = \sin(2\theta)$	*31. $r = \tan(\theta)$	*32. $r = 1 + \tan(\theta)$
*33. $r = \dfrac{3}{\cos(\theta)}$	*34. $r = \dfrac{2}{\sin(\theta)}$	*35. $r = \dfrac{1}{\sin(\theta) + \cos(\theta)}$	36. $r = \dfrac{\theta}{2}$
37. $r = 2 \cdot \theta$	38. $r = \theta^2$	39. $r = \dfrac{1}{\theta}$	40. $r = \sin(2\theta) \cdot \cos(3\theta)$

*41. $r = \sin(m\theta) \cdot \cos(n\theta)$ produces lovely graphs for various small integer values of m and n. Go exploring with a graphic calculator to find values of m and n which result in shapes you like.

*42. Graph $r = \dfrac{1}{1 + 0.5 \cdot \cos(\theta + a)}$, $0 \le \theta \le 2\pi$, for $a = 0, \pi/6, \pi/4$, and $\pi/2$. How are the graphs related?

*43. Graph $r = \dfrac{1}{1 + 0.5 \cdot \cos(\theta - a)}$, $0 \le \theta \le 2\pi$, for $a = 0, \pi/6, \pi/4$, and $\pi/2$. How are the graphs related?

*44. Graph $r = \sin(n\theta)$, $0 \le \theta \le 2\pi$, for n = 1, 2, 3, and 4 and count the number of "petals" on each graph. Predict the number of "petals" for the graphs of $r = \sin(n\theta)$ for n = 5, 6, and 7, and then test your prediction by creating those graphs.

*45. Repeat the steps in problem 44 but using $r = \cos(n\theta)$.

In problems 46–49, convert the rectangular coordinate locations to polar coordinates.

46. (0, 3), (5, 0), and (1, 2)

47. (–2, 3), (2, –3), and (0, –4).

48. (0, –2), (4, 4), and (3, –3)

49. (3, 4), (–1, –3), and (–7, 12).

In problems 50–53, convert the polar coordinate locations to rectangular coordinates.

50. (3, 0), (5, 90°), and (1, π)

51. (–2,3), (2,–3), and (0,–4).

52. (0,3), (5,0), and (1,2)

53. (2,3), (–2,–3), and (0,4).

Problems 54–60 refer to the robotic arm in Fig. 36.

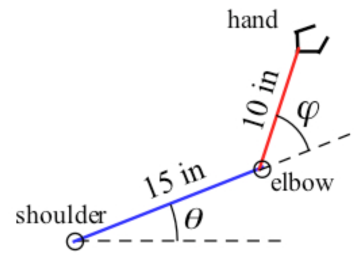


Fig. 36

54. Determine the position of the hand, realtive to the shoulder, when $\theta = 60°$ and $\phi = -45°$.

55. Determine the position of the hand, relative to the shoulder, when $\theta = -30°$ and $\phi = 30°$.

56. Determine the position of the hand, relative to the shoulder, when $\theta = 0.6$ and $\phi = 1.2$.

57. Determine the position of the hand, relative to the shoulder, when $\theta = -0.9$ and $\phi = 0.4$.

58. Suppose the robot's shoulder can pivot so that $-\pi/2 \le \theta \le \pi/2$, but the elbow is broken and ϕ is always $0°$. Sketch the points the hand can reach.

59. Suppose the robot's shoulder can pivot so that $-\pi/2 \le \theta \le \pi/2$, and the elbow can pivot so that $-\pi/2 \le \phi \le \pi/2$. Sketch the points the hand can reach.

60. Suppose the robot's shoulder can pivot so that $-\pi/2 \le \theta \le \pi/2$, and the elbow can pivot completely so $-\pi \le \phi \le \pi$. Sketch the points the hand can reach.

*61. Graph $r = \dfrac{1}{1 + a\cdot\cos(\theta)}$ for $0 \le \theta \le 2\pi$ and $a = 0.5, 0.8, 1, 1.5,$ and 2. What shapes do the various values of a produce?

*62. Repeat problem 61 with $r = \dfrac{1}{1 + a\cdot\sin(\theta)}$.

Some Exotic Curves (and Names)

Many of the following curves were discovered and named even before polar coordinates were invented. In most cases the path of a point moving on or around some object is described. You may enjoy using your calculator to graph some of these curves or you can invent your own exotic shapes. (An inexpensive source for these shapes and names is A Catalog Of Special Plane Curves by J. Dennis Lawrence, Dover Publications, 1972, and the page references below are to that book)

Some Classics:

Cissoid ("like ivy") of Diocles (about 200 B.C.):	$r = a\sin(\theta)\cdot\tan(\theta)$	p. 98	
Right Strophoid ("twisting") of Barrow (1670):	$r = a(\sec(\theta) - 2\cos(\theta))$	p. 101	
Trisectrix of Maclaurin (1742):	$r = a\sec(\theta) - 4a\cos(\theta)$	p. 105	
Lemniscate ("ribbon") of Bernoulli (1694):	$r^2 = a^2\cos(2\theta)$	p. 122	
Conchoid ("shell") of Nicomedes (225 B.C.):	$r = a + b\cdot\sec(\theta)$	p. 137	
Hippopede ("horse fetter") of Proclus (about 75 B.C.):	$r^2 = 4b(a - b\sin^2(\theta))$	p. 144	b = 3, a = 1, 2, 3, 4
Devil's Curve of Cramer (1750):	$r^2(\sin^2(\theta) - \cos^2(\theta)) = a^2\sin^2(\theta) - b^2\cos^2(\theta)$	p. 151	a= 2, b=3
Nephroid ("kidney") of Freeth:	$r = a\cdot(1 + 2\sin(\frac{\theta}{2}))$	p. 175	a = 3

Some of our own: (Based on their names, what shapes do you expect for the following curves?)

Piscatoid of Pat (1992): $r = \frac{1}{\cos(\theta)} - 3\cos(\theta)$ for $-1.1 \le \theta \le 1.1$ Window x: (–2, 1) and y: (–1, 1)

Kermitoid of Kelcey (1992) :

$$r = 2.5\cdot\sin(2\theta)\cdot(\theta - 4.71)\cdot\mathrm{INT}(\theta/\pi) + \{5\cdot\sin^3(\theta) - 3\cdot\sin^9(\theta)\}\cdot\{1 - \mathrm{INT}(\theta/\pi)\} \quad \text{for } 0 \le \theta \le 2\pi$$

Window x: (–3, 3) and y: (–1, 4)

Bovine Oculoid: $r = 1 + \mathrm{INT}(\theta/(2\pi))$ for $0 \le \theta \le 6\pi$ (≈ 18.85) Window x: (–5, 5) and y: (–4, 4)

A Few Reference Facts

The polar form of the linear equation $Ax + By + C = 0$ is $r\cdot(A\cdot\cos(\theta) + B\cdot\sin(\theta)) + C = 0$

The equation of the line through the polar coordinate points (r_1, θ_1) and (r_2, θ_2) is

$$r\cdot\{r_1\cdot\sin(\theta - \theta_1) + r_2\cdot\sin(\theta_2 - \theta)\} = r_1\cdot r_2\cdot\sin(\theta_2 - \theta_1)$$

The graph of $r = a\cdot\sin(\theta) + b\cdot\cos(\theta)$ is a circle through the origin with center $(b/2, a/2)$ and radius $\frac{1}{2}\sqrt{a^2 + b^2}$. (Hint: multiply each side by r, and then convert to rectangular coordinates.)

The equations $r = \dfrac{1}{1 \pm a\cdot\cos(\theta)}$ and $r = \dfrac{1}{1 \pm a\cdot\sin(\theta)}$ are conic sections with one focus at the origin.

If $a < 1$, the denominator is **never** 0 for $0 \le \theta < 2\pi$ and the graph is an **ellipse**.

If $a = 1$, the denominator is 0 for **one** value of θ, $0 \le \theta < 2\pi$, and the graph is a **parabola**.

If $a > 1$, the denominator is 0 for **two** values of θ, $0 \le \theta < 2\pi$, and the graph is a **hyperbola**.

Section 9.1 **PRACTICE Answers**

Practice 1: Point C is at a distance of 10 miles in the direction 30^{o}. D is 5 miles at 270^{o}.

Practice 2: The points are plotted in Fig. 37.

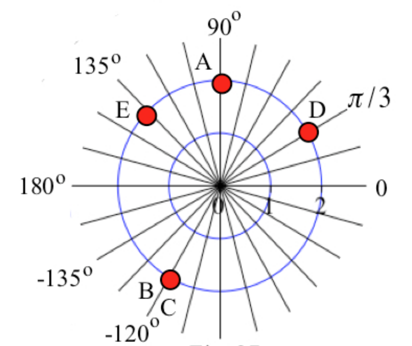


Fig. 37

Practice 3: See Fig. 38.

The top of the plateau is roughly rectangular.

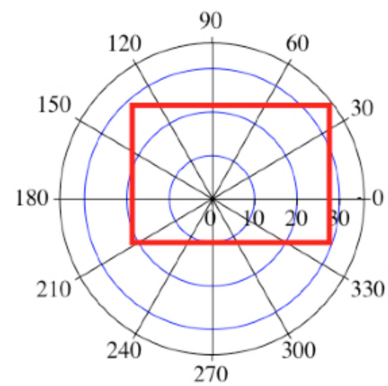
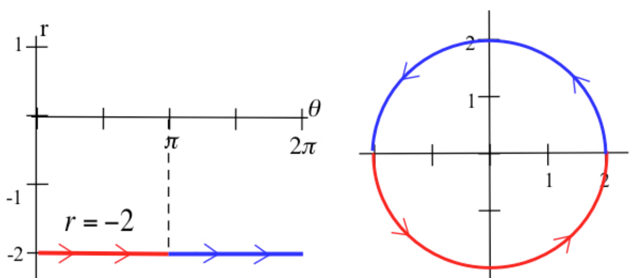


Fig. 38

Practice 4: The graphs are shown in Figs. 39 and 40.

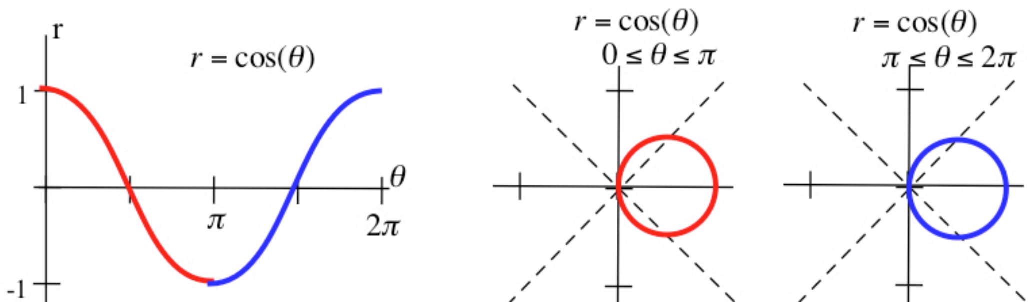
Note that the graph of r = cos(θ) traces out a circle **twice**; once as θ goes from 0 to π, and a second time as θ goes from π to 2π.



(a) rectangular coordinates

(b) polar coordinates

Fig. 39



(a) rectangular coordinates

(b) polar coordinates

Fig. 40

Practice 5: The points are plotted in Fig. 41.

The points (almost) lie on a straight line.

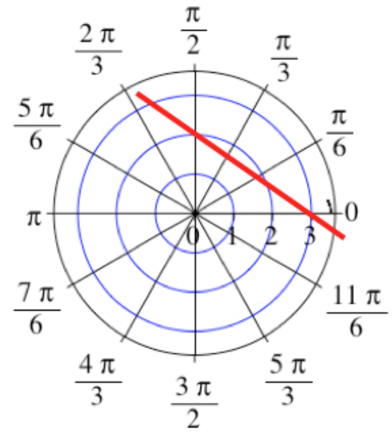


Fig. 41

Practice 6: $r^2 = x^2 + y^2$ and $r \cdot \sin(\theta) = y$ so $r^2 = 4r \cdot \sin(\theta)$ becomes $x^2 + y^2 = 4y$.

Putting this last equation into the standard form for a circle (by completing the square) we have $x^2 + (y-2)^2 = 4$, the equation of a circle with center at $(0, 2)$ and radius 2.

Practice 7: See Fig. 42.

For point A, the "elbow," relative to O, the "shoulder:"

$$x = 18 \cdot \cos(30^o) \approx 15.6 \text{ inches and } y = 18 \cdot \sin(30^o) = 9 \text{ inches.}$$

For point B, the "hand," relative to A:

$$x = 12 \cdot \cos(75^o) \approx 3.1 \text{ inches and } y = 12 \cdot \sin(75^o) \approx 11.6 \text{ inches.}$$

Then the retangular coordinate location of the B relative to O is

$$x \approx 15.6 + 3.1 = 18.7 \text{ inches and } y \approx 9 + 11.6 = 20.6 \text{ inches.}$$

The polar coordinate location of B relative to O is

$$r = \sqrt{x^2 + y^2} \approx 27.8 \text{ inches and } \theta \approx 47.7^o \text{ (or 0.83 radians)}$$

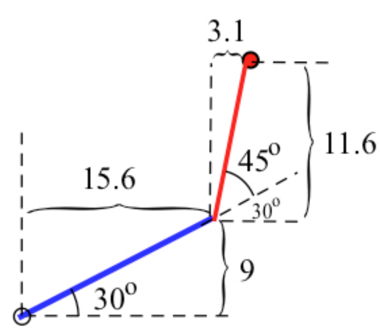


Fig. 42