9.2 CALCULUS IN THE POLAR COORDINATE SYSTEM

The previous section introduced the polar coordinate system and discussed how to plot points, how to create graphs of functions (from data, a rectangular graph, or a formula), and how to convert back and forth between the polar and rectangular coordinate systems. This section examines calculus in polar coordinates: rates of changes, slopes of tangent lines, areas, and lengths of curves. The results we obtain may look different, but they all follow from the approaches used in the rectangular coordinate system.

Polar Coordinates and Derivatives

In the rectangular coordinate system, the derivative dy/dx measured both the rate of change of y with respect to x and the slope of the tangent line. In the polar coordinate system two different derivatives commonly appear, and it is important to distinguish between them.

$\dfrac{dr}{d\theta}$	measures the **rate of change** of r with respect to θ.
	The sign of $\dfrac{dr}{d\theta}$ tells us whether r is increasing or decreasing as θ increases.
$\dfrac{dy}{dx}$	measures the **slope** $\dfrac{\Delta y}{\Delta x}$ **of the tangent line** to the polar graph of r.

We can use our usual rules for derivatives to calculate the derivative of a polar coordinate equation r with respect to θ, and dr/dθ tells us how r is changing with respect to (increasing) θ. For example, if dr/dθ > 0 then the directed distance r is increasing as θ increases (Fig. 1). However, dr/dθ is **NOT** the slope of the line tangent to the polar graph of r. For the simple spiral r = θ (Fig. 2), $\dfrac{dr}{d\theta} = 1 > 0$ for all values of θ; but the slope of the tangent line, $\dfrac{dy}{dx}$, may be positive (at A and C) or negative (at B and D).

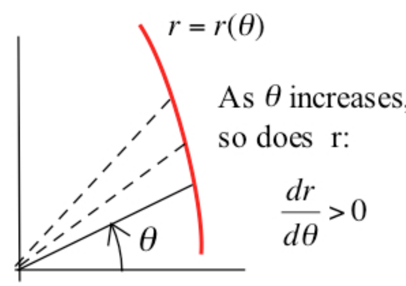

Fig. 1

Similarly, $\dfrac{dx}{d\theta}$ is the rate of change of the x-coordinate of the graph with respect to (increasing) θ, and $\dfrac{dy}{d\theta}$ is the rate of change of the y-coordinate of the graph with respect to (increasing) θ. The values of the derivatives dy/dθ and dx/dθ depend on the location on the graph. They will also be used to calculate the slope $\dfrac{dy}{dx}$ of the tangent line, and also to express the formula for arc length in polar coordinates.

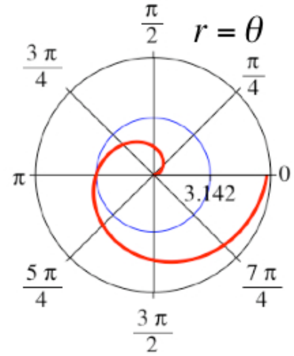

Fig. 2

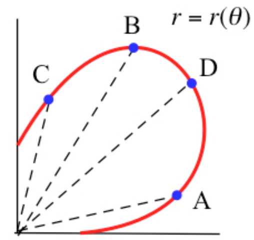

Fig. 3

Example 1: State whether the values of $dr/d\theta$, $dx/d\theta$, $dy/d\theta$, and dy/dx are + (positive), – (negative), 0 (zero), or U (undefined) at the points A and B on the graph in Fig. 3.

Solution: The values of the derivatives at A and B are given in Table 1.

Practice 1: Fill in the rest of Table 1 for points labeled C and D.

Point	$\frac{dx}{d\theta}$	$\frac{dy}{d\theta}$	$\frac{dr}{d\theta}$	$\frac{dy}{dx}$
A	+	+	+	+
B	–	0	–	0
C				
D				

Table 1

When r is given by a formula we can calculate dy/dx , the slope of the tangent line, by using the polar–rectangular conversion formulas and the Chain Rule. By the Chain Rule $\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta}$, so we can solve for $\frac{dy}{dx}$ by dividing each side of the equation by $\frac{dx}{d\theta}$.

Then the slope $\frac{dy}{dx}$ of the line tangent to the polar coordinate graph of $r(\theta)$ is

(1) $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d\,(r\cdot\sin(\theta))}{d\theta}}{\frac{d\,(r\cdot\cos(\theta))}{d\theta}} .$$

Since r is a function of θ, $r = r(\theta)$, we may use the product rule and the Chain Rule for derivatives to calculate each derivative and to obtain

(2) $$\frac{dy}{dx} = \frac{r\cdot\cos(\theta) + r'\cdot\sin(\theta)}{-r\cdot\sin(\theta) + r'\cdot\cos(\theta)} \quad (\text{with } r' = dr/d\theta) .$$

The result in (2) is difficult to remember, but the starting point (1) and derivation are straightforward.

Example 2: Find the slopes of the lines tangent to the spiral $r = \theta$ (shown in Fig. 2) at the points $P(\pi/2, \pi/2)$ and $Q(\pi, \pi)$.

Solution: $y = r\cdot\sin(\theta) = \theta\cdot\sin(\theta)$ and $x = r\cdot\cos(\theta) = \theta\cdot\cos(\theta)$ so

$$\frac{dy}{dx} = \frac{\frac{d\,(r\cdot\sin(\theta))}{d\theta}}{\frac{d\,(r\cdot\cos(\theta))}{d\theta}} = \frac{\frac{d\,(\theta\cdot\sin(\theta))}{d\theta}}{\frac{d\,(\theta\cdot\cos(\theta))}{d\theta}} = \frac{\theta\cdot\cos(\theta) + 1\cdot\sin(\theta)}{-\theta\cdot\sin(\theta) + 1\cdot\cos(\theta)} .$$

At the point P, $\theta = \pi/2$ and $r = \pi/2$ so $\frac{dy}{dx} = \frac{\frac{\pi}{2}\cdot 0 + 1\cdot(1)}{-\frac{\pi}{2}\cdot(1) + 1\cdot(0)} = -\frac{2}{\pi} \approx -0.637$.

At the point Q, $\theta = \pi$ and $r = \pi$ so $\frac{dy}{dx} = \frac{\pi\cdot(-1) + 1\cdot(0)}{-\pi\cdot(0) + 1\cdot(-1)} = \frac{-\pi}{-1} = \pi \approx 3.142$.

The function $r = \theta$ is steadily increasing, but the slope of the line tangent to the polar graph can negative or positive or zero or even undefined (where?).

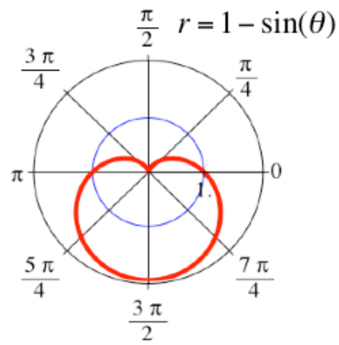

Fig. 4

Practice 2: Find the slopes of the lines tangent to the cardioid $r = 1 - \sin(\theta)$ (Fig. 4) when $\theta = 0, \pi/4$, and $\pi/2$.

Areas in Polar Coordinates

The patterns for calculating areas in rectangular and polar coordinates look different, but they are derived in the same way: partition the area into pieces, calculate areas of the pieces, add the small areas together to get a Riemann sum, and take the limit of the Riemann sum to get a definite integral. The major difference is the shape of the pieces: we use thin rectangular pieces in the rectangular system and thin sectors (pieces of pie) in the polar system. The formula we need for the area of a sector can be found by using proportions (Fig. 5):

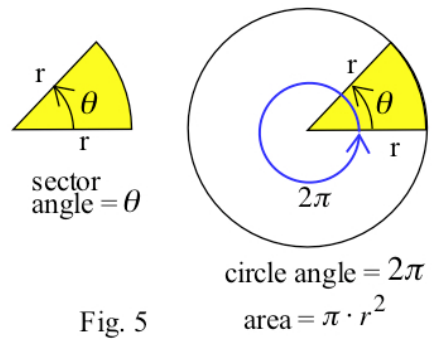

Fig. 5

$$\frac{\text{area of sector}}{\text{area of whole circle}} = \frac{\text{sector angle}}{\text{angle of whole circle}} = \frac{\theta}{2\pi}$$

$$\text{so (area of sector)} = \frac{\theta}{2\pi}(\text{area of whole circle}) = \frac{\theta}{2\pi}(\pi r^2) = \frac{1}{2} r^2 \theta .$$

Figures 6 and 7 refer to the area discussion after the figures.

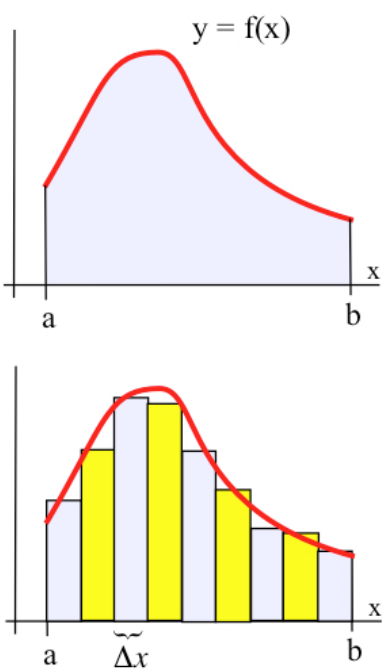

Fig. 6 Area with rectangular coordinates

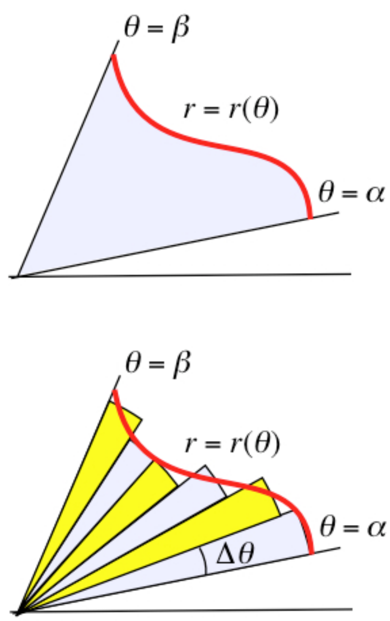

Fig. 7: Area with polar coordinates

Area in Rectangular Coordinates (Fig. 6)	**Area in Polar Coordinates** (Fig. 7)
Partition the domain x of the rectangular coordinate function into small pieces of width Δx .	Partition the domain θ of the polar coordinate function into small pieces of angular width $\Delta\theta$.
Build rectangles on each piece of the domain.	Build "nice" shapes (pieces of pie shaped sectors) along each piece of the domain.
Calculate the area of each piece (rectangle): $\text{area}_i = (\text{base}_i)\cdot(\text{height}_i) = f(x_i)\cdot\Delta x_i$.	Calculate the area of each piece (sector): $\text{area}_i = \frac{1}{2}(\text{radius}_i)^2(\text{angle}_i) = \frac{1}{2} r_i^2 \Delta\theta_i$.
Approximate the total area by adding the small areas together, a Riemann sum: $\text{total area} \approx \sum \text{area}_i = \sum f(x_i)\cdot\Delta x_i$.	Approximate the total area by adding the small areas together, a Riemann sum: $\text{total area} \approx \sum \text{area}_i = \sum \frac{1}{2} r_i^2 \Delta\theta_i$.
The limit of the Riemann sum is a definite integral: $\text{Area} = \int_{x=a}^{b} f(x)\,dx$.	The limit of the Riemann sum is a definite integral: $\text{Area} = \int_{\theta=\alpha}^{\beta} \frac{1}{2} r^2(\theta)\,d\theta$.

If r is a continuous function of θ , then the limit of the Riemann sums is a finite number, and we have a formula for the area of a region in polar coordinates.

Area In Polar Coordinates

The area of the region bounded by a continuous function $r(\theta)$ and radial lines at angles $\theta = \alpha$ and $\theta = \beta$ is

$$\text{area} = \int_{\theta=\alpha}^{\beta} \frac{1}{2} r^2(\theta)\,d\theta \ .$$

Example 3: Find the area inside the cardioid $r = 1 + \cos(\theta)$. (Fig. 8)

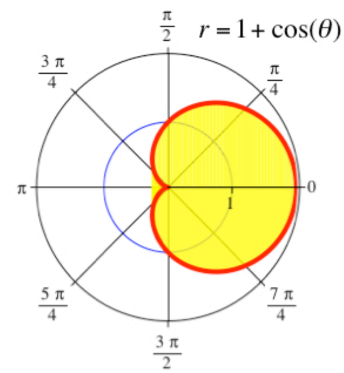

Fig. 8

Solution: This is a straightforward application of the area formula.

$$\text{Area} = \int_{\theta=0}^{2\pi} \frac{1}{2}(1+\cos(\theta))^2 \, d\theta = \frac{1}{2}\int_{\theta=0}^{2\pi} \{ 1 + 2\cdot\cos(\theta) + \cos^2(\theta) \} \, d\theta$$

$$= \frac{1}{2} \left\{ \theta + 2\cdot\sin(\theta) + \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta) \right] \right\} \Big|_0^{2\pi}$$

$$= \frac{1}{2} \left\{ \left[2\pi + 0 + \frac{1}{2}(2\pi + 0) \right] - [0 + 0 + 0] \right\} = \mathbf{\frac{3}{2}\pi} .$$

We could also have used the symmetry of the region and determined this area by integrating from 0 to π (Fig. 9) and multiplying the result by 2.

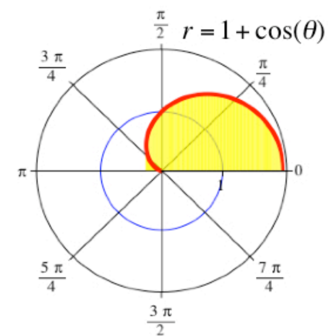

Fig. 9

Practice 3: Find the area inside one "petal" of the rose $r = \sin(3\theta)$. (Fig. 10)

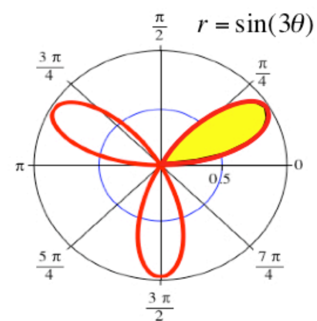

Fig. 10

We can also calculate the area between curves in polar coordinates.

The area of the region (Fig. 11) between the continuous curves $r_1(\theta) \le r_2(\theta)$ for $\alpha \le \theta \le \beta$ is

$$\int_{\theta=\alpha}^{\beta} \frac{1}{2} r_2^2(\theta) \, d\theta - \int_{\theta=\alpha}^{\beta} \frac{1}{2} r_1^2(\theta) \, d\theta$$

$$= \int_{\theta=\alpha}^{\beta} \frac{1}{2} \{ r_2^2(\theta) - r_1^2(\theta) \} \, d\theta \; .$$

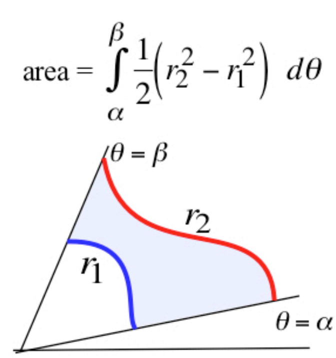

Fig. 11

It is a good idea to sketch the graphs of the curves to help determine the endpoints of integration.

Example 4: Find the area of the shaded region in Fig. 12.

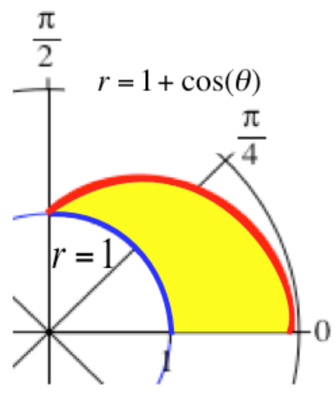

Fig. 12

Solution: A_1 = area between the circle and the origin = $\displaystyle\int_{\theta=0}^{\pi/2} \frac{1}{2}\, 1^2\, d\theta$

$$= \frac{1}{2}\,\theta \Big|_0^{\pi/2} = \frac{\pi}{4} \approx 0.785 .$$

A_2 = area between the cardioid and the origin = $\displaystyle\int_{\theta=0}^{\pi/2} \frac{1}{2}(\,1 + \cos(\theta)\,)^2\, d\theta$

$$= \frac{3}{4}\,\theta + \sin(\theta) + \frac{1}{8}\sin(2\theta) \Big|_0^{\pi/2} = \{\, \frac{3\pi}{8} + 1 + 0 \,\} - \{\, 0 + 0 + 0 \,\} \approx$$

2.178 .

The area we want is $A_2 - A_1 = 1 + \frac{3\pi}{8} - \frac{\pi}{4} = 1 + \frac{\pi}{8} \approx 1.393$.

Practice 4: Find the area of the region outside the cardioid $1 + \cos(\theta)$ and inside the circle $r = 2$. (Fig. 13)

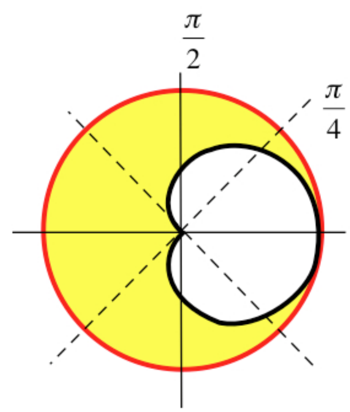

Fig. 13

Arc Length in Polar Coordinates

The patterns for calculating the lengths of curves in rectangular and polar coordinates look different, but they are derived from the Pythagorean Theorem and the same sum we used in Section 5.2 (Fig. 14):

$$\text{length} \approx \sum \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sum \sqrt{\left(\frac{\Delta x}{\Delta\theta}\right)^2 + \left(\frac{\Delta y}{\Delta\theta}\right)^2}\ \Delta\theta \ .$$

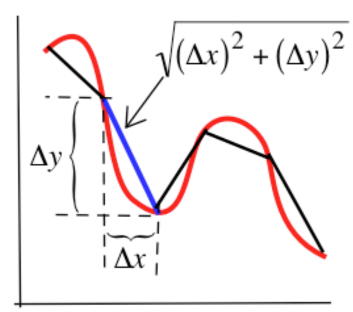

Fig. 14

If x and y are differentiable functions of θ, then as $\Delta\theta$ approaches 0, $\Delta x/\Delta\theta$ approaches $dx/d\theta$, $\Delta y/\Delta\theta$ approaches $dy/d\theta$, and the Riemann sum approaches the definite integral

$$\text{length} = \int_{\theta=\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta \ .$$

Replacing x with $r\cdot\cos(\theta)$ and y with $r\cdot\sin(\theta)$, we have $dx/d\theta = -r\cdot\sin(\theta) + r'\cdot\cos(\theta)$ and $dy/d\theta = r\cdot\cos(\theta) + r'\cdot\sin(\theta)$. Then $(\,dx/d\theta\,)^2 + (\,dy/d\theta\,)^2$ inside the square root simplifies to $r^2 + (\,r'\,)^2$ and we have a more useful form of the integral for arc length in polar coordinates.

Arc Length

If r is a differentiable function of θ for $\alpha \le \theta \le \beta$, then the length of the graph of r is

$$\text{Length} = \int_{\theta=\alpha}^{\beta} \sqrt{(r)^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta \ .$$

Problems

Derivatives

In problems 1–4, fill in the table for each graph with + (positive), – (negative), 0 (zero), or U (undefined) for each derivative at each labeled point.

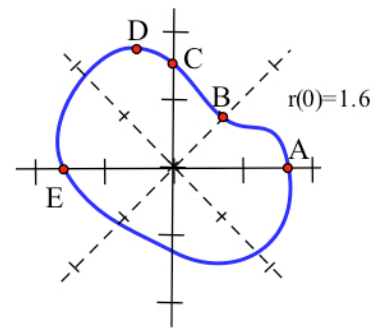

Fig. 15

1. Use Fig. 15.

Point	$\frac{dr}{d\theta}$	$\frac{dx}{d\theta}$	$\frac{dy}{d\theta}$	$\frac{dy}{dx}$
A				
B				
C				
D				
E				

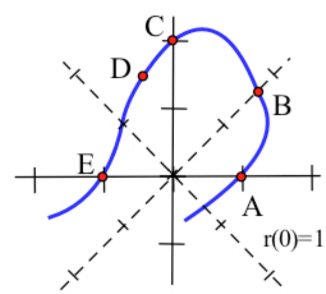

Fig. 16

2. Use Fig. 16.

Point	$\frac{dr}{d\theta}$	$\frac{dx}{d\theta}$	$\frac{dy}{d\theta}$	$\frac{dy}{dx}$
A				
B				
C				
D				
E				

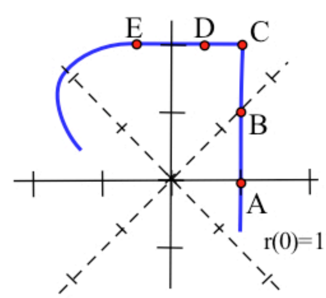

Fig. 17

3. Use Fig. 17.

Point	$\frac{dr}{d\theta}$	$\frac{dx}{d\theta}$	$\frac{dy}{d\theta}$	$\frac{dy}{dx}$
A				
B				
C				
D				
E				

4. Use Fig. 18.

Point	$\frac{dr}{d\theta}$	$\frac{dx}{d\theta}$	$\frac{dy}{d\theta}$	$\frac{dy}{dx}$
A				
B				
C				
D				
E				

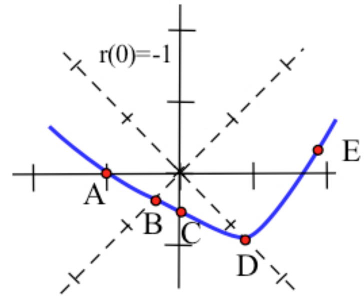

Fig. 18

In problems 5–8, sketch the graph of the polar coordinate function $r = r(\theta)$ for $0 \le \theta \le 2\pi$, label the points with the given polar coordinates on the graph, and calculate the values of $\frac{dr}{d\theta}$ and $\frac{dy}{dx}$ at the points with the given polar coordinates.

5. $r = 5$ at $A(5, \pi/4)$, $B(5, \pi/2)$, and $C(5, \pi)$.

6. $r = 2 + \cos(\theta)$ at $A(2 + \frac{\sqrt{2}}{2}, \pi/4)$, $B(2, \pi/2)$, and $C(1, \pi)$.

7. $r = 1 + \cos^2(\theta)$ at $A(2, 0)$, $B(3/2, \pi/4)$, and $C(1, \pi/2)$.

8. $r = \frac{6}{2 + \cos(\theta)}$ at $A(2, 0)$, $B(3, \pi/2)$, and $C(\frac{24 - 6\sqrt{2}}{7}, \pi/4) \approx (2.216, \pi/4)$.

9. Graph $r = 1 + 2 \cdot \cos(\theta)$ for $0 \le \theta \le 2\pi$, and show that the graph goes through the origin when $\theta = 2\pi/3$ and $\theta = 4\pi/3$. Calculate dy/dx when $\theta = 2\pi/3$ and $\theta = 4\pi/3$. How can a curve have two different tangent lines (and slopes) when it goes through the origin?

10. Graph the cardiod $r = 1 + \sin(\theta)$ for $0 \le \theta \le 2\pi$.
    (a) At what points on the cardioid does $dx/d\theta = 0$? (b) At what points on the cardiod does $dy/d\theta = 0$?
    (c) At what points on the cardioid does $dr/d\theta = 0$? (d) At what points on the cardiod does $dy/dx = 0$?

11. Show that if a polar coordinate graph goes through the origin when the angle is θ_0 (and if $dr/d\theta$ exists and does not equal 0 there), then the slope of the tangent line at the origin is $\tan(\theta_0)$. (Suggestion: Evaluate formula (2) for dy/dx at the point $(0, \theta_0)$.)

Areas

In problems 12–20, represent each area as a definite integral. Then evaluate the integral exactly or using Simpson's rule (with n = 100).

12. The area of the shaded region in Fig. 19.

13. The area of the shaded region in Fig. 20.

14. The area of the shaded region in Fig. 21.

15. The area in the first quadrant outside the circle $r = 1$ and inside the cardiod $r = 1 + \cos(\theta)$.

16. The region in the second quadrant bounded by $r = \theta$ and $r = \theta^2$.

17. The area inside one "petal" of the graph of (a) $r = \sin(3\theta)$ and (b) $r = \sin(5\theta)$.

18. The area (a) inside the "peanut" $r = 1.5 + \cos(2\theta)$ and (b) inside $r = a + \cos(2\theta)$ $(a > 1)$.

19. The area inside the circle $r = 4 \cdot \sin(\theta)$.

20. The area of the shaded region in Fig. 22.

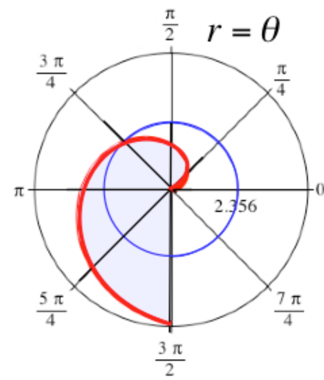

Fig. 19

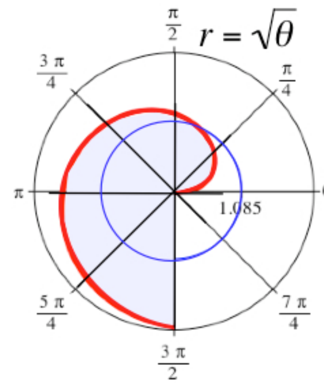

Fig. 20

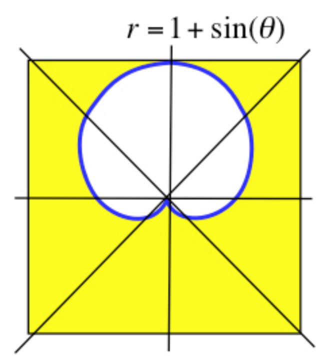

Fig. 21

21. Goat and Square Silo: (This problem does not require calculus.)
One end of a 40 foot long rope is attached to the middle of a wall of a 20 foot square silo, and the other end is tied to a goat (Fig. 23).
   (a) Sketch the region that the goat can reach.
   (b) Find the area of the region that the goat can reach.
   (c) Can the goat reach more area if the rope is tied to the corner of the silo?

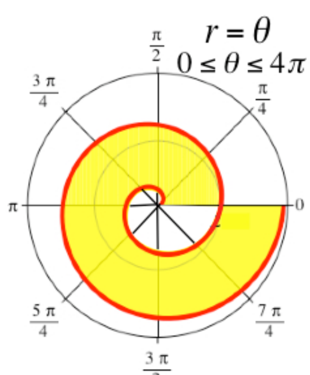

Fig. 22

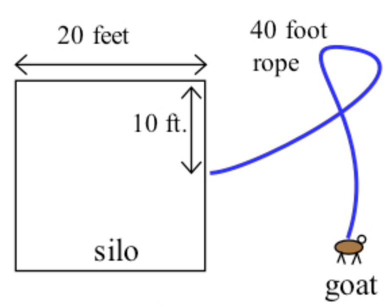

Fig. 23

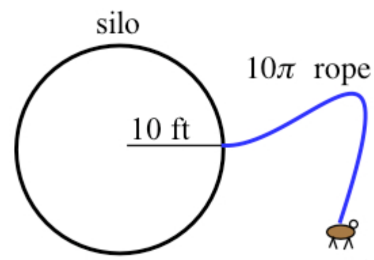

Fig. 24

22. Goat and Round Silo: One end of a 10π foot long rope is attached to the wall of a round silo that has a radius of 10 feet, and the other end is tied to a goat (Fig. 24).
   (a) Sketch the region the goat can reach.
   (b) Justify that the area of the region in Fig. 25 as the goat goes around the silo from having θ feet of rope taut against the silo to having $\theta + \Delta\theta$ feet taut against the silo is approximately $\frac{1}{2}(10\pi - 10\cdot\theta)^2\,\Delta\theta$.
   (c) Use the result from part (b) to help calculate the area of the region that the goat can reach.

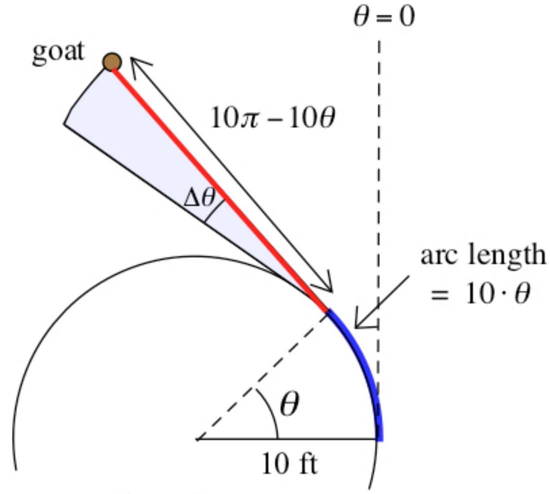

Fig. 25

Arc Lengths

In problems 23–29, represent the length of each curve as a definite integral. Then evaluate the integral exactly or using your calculator.

23. The length of the spiral $r = \theta$ from $\theta = 0$ to $\theta = 2\pi$.

24. The length of the spiral $r = \theta$ from $\theta = 2\pi$ to $\theta = 4\pi$.

25. The length of the cardiod $r = 1 + \cos(\theta)$.

26. The length of $r = 4\cdot\sin(\theta)$ from $\theta = 0$ to $\theta = \pi$.

27. The length of the circle $r = 5$ from $\theta = 0$ to $\theta = 2\pi$.

28. The length of the "peanut" $r = 1.2 + \cos(2\theta)$.

29. The length (a) of one "petal" of the graph of $r = \sin(3\theta)$ and (b) of one "petal" of $r = \sin(5\theta)$.

30. Assume that r is a differentiable function of θ. Verify that $\left\{\frac{dx}{d\theta}\right\}^2 + \left\{\frac{dy}{d\theta}\right\}^2 = \{r\}^2 + \left\{\frac{dr}{d\theta}\right\}^2$ by replacing x with $r\cdot\cos(\theta)$ and y with $r\cdot\sin(\theta)$ in the left side of the equation, differentiating, and then simplifying the result to obtain the right side of the equation.

Section 9.2 PRACTICE Answers

Practice 1: The values are shown in Fig. 26.

Point	$\frac{dx}{d\theta}$	$\frac{dy}{d\theta}$	$\frac{dr}{d\theta}$	$\frac{dy}{dx}$
A	+	+	+	+
B	–	0	–	0
C	–	–	–	+
D	–	+	+	–

Fig. 26

Practice 2: $r = 1 - \sin(\theta)$ and $r' = -\cos(\theta)$.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r\cdot\cos(\theta) + r'\cdot\sin(\theta)}{-r\cdot\sin(\theta) + r'\cdot\cos(\theta)}$$

$$= \frac{(1 - \sin(\theta))\cdot\cos(\theta) + (-\cos(\theta))\cdot\sin(\theta)}{-(1 - \sin(\theta))\cdot\sin(\theta) + (-\cos(\theta))\cdot\cos(\theta)} = \frac{\cos(\theta) - 2\cdot\sin(\theta)\cdot\cos(\theta)}{-\sin(\theta) + \sin^2(\theta) - \cos^2(\theta)} .$$

When $\theta = 0$, $\frac{dy}{dx} = \frac{1-0}{-0+0-1} = -1$.

When $\theta = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{(1/\sqrt{2}) - 2(1/\sqrt{2})(1/\sqrt{2})}{-(1/\sqrt{2}) + (1/\sqrt{2})^2 - (1/\sqrt{2})^2} = \frac{1/\sqrt{2} - 1}{-1/\sqrt{2} + \frac{1}{2} - \frac{1}{2}} = \sqrt{2} - 1 \approx 0.414$.

When $\theta = \frac{\pi}{2}$, $\frac{dy}{dx} = \frac{0-0}{-1+1-0}$ which is undefined. Why does this result make sense in terms of the graph of the cardioid $r = 1 - \sin(\theta)$?

Practice 3: One "petal" of the rose $r = \sin(3\theta)$ is swept out as θ goes from 0 to $\pi/3$ (see Fig. 10) so the endpoints of the area integral are 0 and $\pi/3$.

$$\text{area} = \int_{\theta=\alpha}^{\beta} \frac{1}{2} r^2(\theta)\, d\theta = \int_{\theta=0}^{\pi/3} \frac{1}{2} \{ \sin(3\theta) \}^2\, d\theta \qquad \text{(then using integral table entry \#13)}$$

$$= \frac{1}{2} \left\{ \frac{1}{2}\theta - \frac{1}{4(3)} \sin(2\cdot 3\theta) \right\} \Big|_0^{\pi/3} = \frac{1}{2} \left\{ \left[\frac{\pi}{6} - \frac{1}{12}(0) \right] - [0 - 0] \right\} = \frac{\pi}{12} \approx 0.262 .$$

Practice 4: The area we want in Fig. 13 is

$$\{\text{area of circle}\} - \{\text{area of cardioid from Example 3}\} = \pi(2)^2 - \frac{3}{2}\pi = \frac{5}{2}\pi \approx 7.85 .$$