9.3 PARAMETRIC EQUATIONS

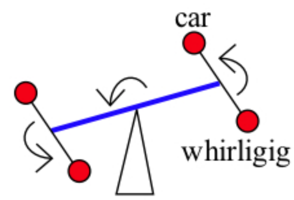


Fig. 1

Some motions and paths are inconvenient, difficult or impossible for us to describe by a single function or formula of the form y = f(x).

- A rider on the "whirligig" (Fig. 1) at the carnival goes in circles at the end of a rotating bar.
- A robot delivering supplies in a factory (Fig. 2) needs to avoid obstacles.
- A fly buzzing around the room (Fig. 3) and a molecule in a solution follow erratic paths.
- A stone caught in the tread of a rolling wheel has a smooth path with some sharp corners (Fig. 4).

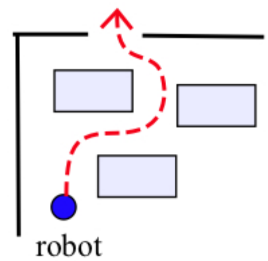


Fig. 2

Parametric equations provide a way to describe all of these motions and paths. And parametric equations generalize easily to describe paths and motions in 3 dimensions.

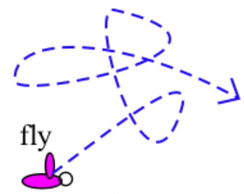


Fig. 3

Parametric equations were used briefly in earlier sections (2.5: Applications of the Chain Rule and 5.2: Arc Length). In those sections the equations were always given. In this section we look at functions given parametrically by data, graphs, and formulas and examine how to build formulas to describe some motions parametrically. The last curve in this section is the cycloid, one of the most famous curves in mathematics. The next section considers calculus with parametric equations: slopes of tangent lines, arc lengths, and areas.

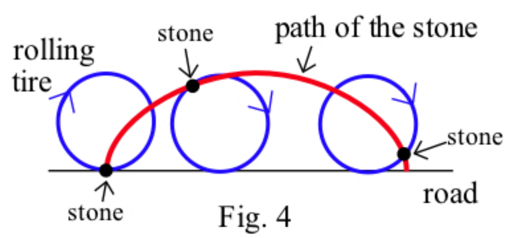


Fig. 4

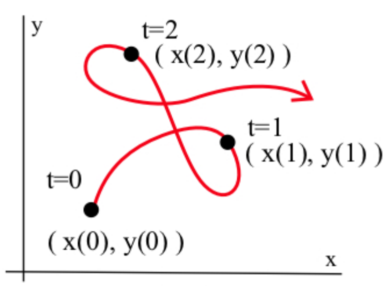


Fig. 5

Parametric equations describe the location of a point (x,y) on a graph or path as a function of a single independent variable t, a "parameter" often representing time. In 2 dimensions, the coordinates x and y are functions of the variable t: $x = x(t)$ and $y = y(t)$ (Fig. 5). In 3 dimensions, the z coordinate is also a function of t: $z = z(t)$. With parametric equations we can also analyze the forces acting on an object separately in each coordinate direction and then combine the results to see the overall behavior of the object. Parametric equations often provide an easier way to understand and build equations for complicated motions.

Graphing Parametric Equations

The data for creating a parametric equation graph can be given as a table of values, as graphs of (t, x(t)) and (t, y(t)), or as formulas for x and y as functions of t.

Example 1: Table 1 is a record of the location of a roller coaster car relative to its starting location. Use the data to sketch a graph of the car's path for the first 7 seconds.

t	x(t)	y(t)	t	x(t)	y(t)
0	0	70	7	90	55
1	30	20	8	105	85
2	70	50	9	125	100
3	60	75	10	130	80
4	30	70	11	150	65
5	32	35	12	180	75
6	60	15	13	200	30

Table 1

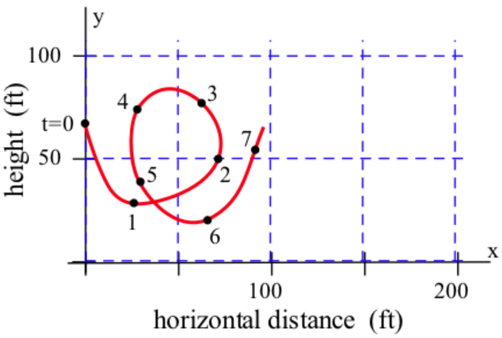


Fig. 6: Roller coaster for $0 \le t \le 7$ seconds

Solution: Figure 6 is a plot of the (x, y) locations of the car for $t = 0$ to 7 seconds. The points are connected by a smooth curve to show a possible path of the car.

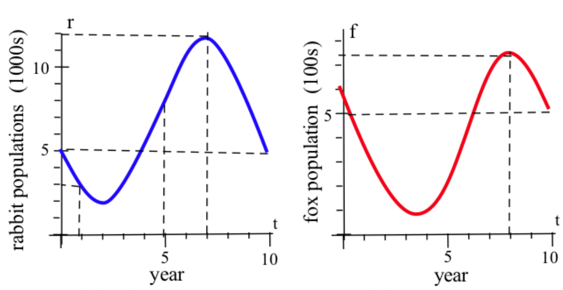
Practice 1: Use the data in Table 1 to sketch the path of the roller coaster for the next 6 seconds.

Note: Clearly the graph in Fig. 6 is not the graph of a function y = f(x). But every y = f(x) function has an easy parametric representation by setting $x(t) = t$ and $y(t) = f(t)$.

Sometimes a parametric graph can show patterns that are not clearly visible in individual graphs.

Example 2: Figures 7a and 7b are graphs of the populations of rabbits and foxes on an island. Use these graphs to sketch a parametric graph of rabbits (x–axis) versus foxes (y–axis) for $0 \le t \le 10$ years.

Solution: The separate rabbit and fox population graphs give us information about each population separately, but the parametric graph helps us see the effects of the interaction between the rabbits and the foxes more clearly.



(a) rabbit population (b) fox population

Fig. 7

For each time t we can read the rabbit and fox populations from the separate graphs (e.g., when $t = 1$, there are approximately 3000 rabbits and 400 foxes so $x \approx 3000$ and $y \approx 400$) and then combine this information to plot a single point on the parametric graph. If we repeat this process for a large number of values of t, we get a graph (Fig. 8) of the "motion" of the rabbit and fox populations over a period of time, and we can ask questions about why the populations might show this behavior.

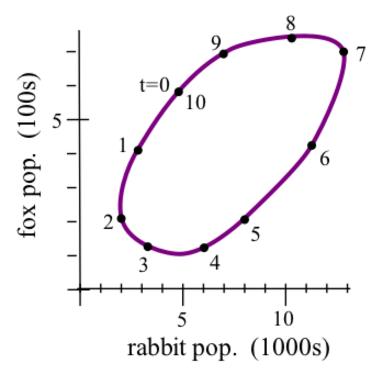


Fig. 8

The type of graph in Fig. 8 is very common for "predator–prey" interactions. Some two–species populations tend to approach a "steady state" or "fixed point" (Fig. 9). However, many two–species population graphs tend to cycle over a period of time as in Fig. 9.

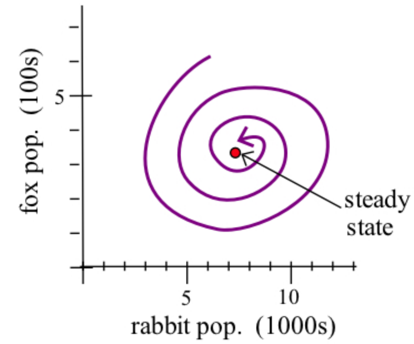


Fig. 9

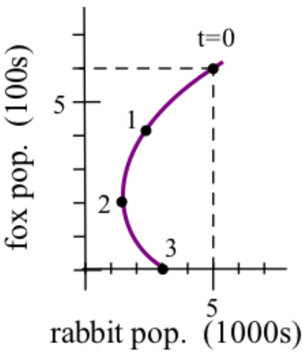


Fig. 10

Practice 2: What would it mean if the rabbit–fox parametric equation graph hit the horizontal axis as in Fig. 10?

Example 3: Graph the pair of parametric equations $x(t) = 2t - 2$ and $y(t) = 3t + 1$.

Solution: Table 2 shows the values of x and y for several values of t. These points are plotted in Fig. 11, and the graph appears to be a straight line.

t	x(t)	y(t)
0	–2	1
1	0	4
2	2	7
–1	–4	–2

Table 2

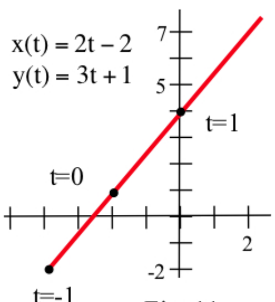


Fig. 11

Usually it is not possible to write y as a simple function of x, but in this case we can do so. By solving $x = 2t - 2$ for $t = \frac{1}{2}x + 1$ and then replacing the t in the equation $y = 3t + 1$,

we get $y = 3t + 1 = 3\{\frac{1}{2}x + 1\} + 1 = \frac{3}{2}x + 4$, a linear function of x.

Practice 3: Graph the pair of parametric equations $x(t) = 3 - t$ and $y(t) = t^2 + 1$. Write y as a function of x alone and identify the shape of the graph.

Example 4: Graph the pair of parametric equations $x(t) = 3\cdot\cos(t)$ and $y(t) = 2\cdot\sin(t)$ for $0 \le t \le 2\pi$,

and show that these equations satisfy the relation $\frac{x^2}{9} + \frac{y^2}{4} = 1$ for all values of t.

Solution: The graph, an ellipse, is shown in Fig. 12.

$$\frac{x^2}{9} + \frac{y^2}{4} = \frac{3^2\cdot\cos^2(t)}{9} + \frac{2^2\cdot\sin^2(t)}{4} = \cos^2(t) + \sin^2(t) = 1.$$

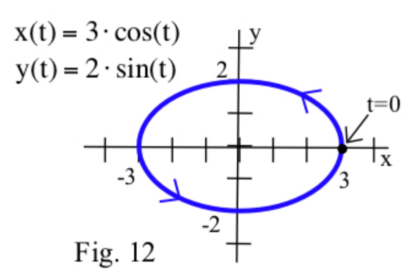


Fig. 12

Practice 4: Graph the pair of parametric equations $x(t) = \sin(t)$ and $y(t) = 5\cdot\cos(t)$ for $0 \le t \le 2\pi$, and show that these equations satisfy the relation $\frac{x^2}{1} + \frac{y^2}{25} = 1$ for all values of t.

Example 5: Describe the motion of a point whose position is
$x(t) = -R \cdot \sin(t)$ and $y(t) = -R \cdot \cos(t)$.

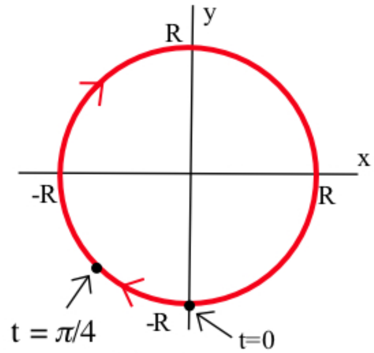


Fig. 13

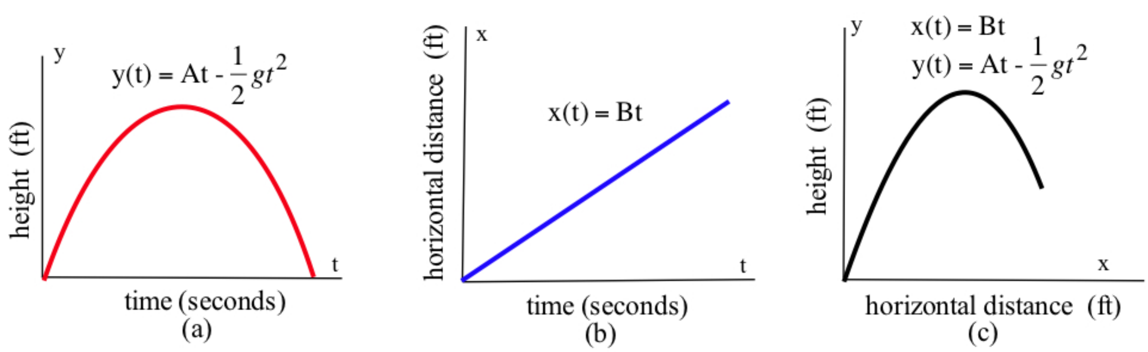
Solution: The point starts at $x(0) = -R \cdot \sin(0) = 0$ and $y(0) = -R \cdot \cos(0) = -R$. By plotting $x(t)$ and $y(t)$ for several other values of t (Fig. 13), we can see that the point is rotating clockwise around the origin. Since $x^2(t) + y^2(t) = R^2 \sin^2(t) + R^2 \cos^2(t) = R^2$, we know the point is always on the circle of radius R which is centered at the origin.

Practice 5: The path of each parametric equation given below is a circle with radius 1 and center at the origin. If an object is located at the point (x, y) at time t seconds:
(a) Where is the object at $t = 0$? (b) Is the object traveling clockwise or counterclockwise around the circle? (c) How long does it take the object to make 1 revolution?
A: $x = \cos(2t), y = \sin(2t)$ B: $x = -\cos(3t), y = \sin(3t)$ C: $x = \sin(4t), y = -\cos(4t)$

Putting Motions Together

If we know how an object moves horizontally and how it moves vertically, then we can put these motions together to see how it moves in the plane.

If an object is thrown straight upward with an initial velocity of A feet per second, then its height after t seconds is $y(t) = A \cdot t - \frac{1}{2} g \cdot t^2$ feet where $g = 32$ feet/second^2 is the downward acceleration of gravity (Fig. 14a). If an object is thrown horizontally with an initial velocity of B feet per second, then its horizontal distance from the starting place after t seconds is $x(t) = B \cdot t$ feet (Fig. 14b).



Fig, 14

Example 6: Write an equation for the position at time t (Fig. 14c) of an object thrown at an angle of 30° with the ground (horizontal) with an initial velocity 100 feet per second.

Solution: If the object travels 100 feet along a line at an angle of 30° to the horizontal ground (Fig. 15), then it travels $100 \cdot \sin(30°) = 50$ feet upward and $100 \cdot \cos(30°) \approx 86.6$ feet sideways, so $A = 50$ and $B = 86.6$. The position of the object at time t is

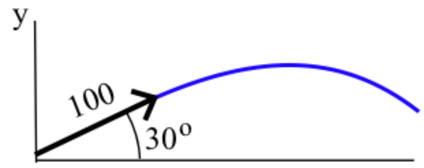


Fig. 15

$$y(t) = 50 \cdot t - \frac{1}{2}\, g\, t^2 \quad \text{and} \quad x(t) = 86.6 \cdot t .$$

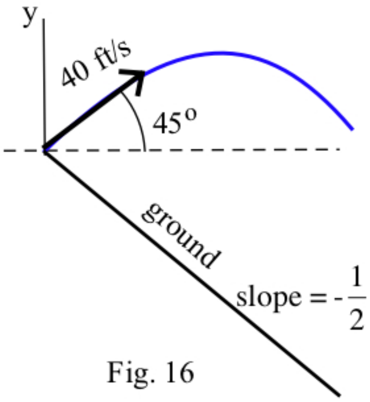


Fig. 16

Practice 6: A ball is thrown upward at an angle of 45° (Fig. 16) with an initial velocity of 40 ft/sec.

(a) Write the parametric equations for the position of the ball as a function of time.

(b) Use the parametric equations to find when and then where the ball will hit the sloped ground. (Suggestion: set $y(t) = -0.5x(t)$ from part (a) and solve for t. Then use that value of t to evaluate $x(t)$ and $y(t)$.)

Sometimes the location or motion of an object is measured by an instrument which is in motion itself (e.g., tracking a pod of migrating whales from a moving ship), and we want to determine the path of the object independent of the location of the instrument. In that case, the "absolute" location of the object with respect to the origin is the sum of the relative location of the object (pod of whales) with respect to the instrument (ship) and the location of the instrument (ship) with respect to the origin. The same approach works for describing the motion of linked objects such as connected gears.

Example 7: **Carnival Ride** The car (Fig. 17) makes one counterclockwise revolution ($r = 8$ feet) about the pivot point A every 2 seconds and the long arm ($R = 20$ feet) makes one counterclockwise revolution about its pivot point (the origin) every 5 seconds. Assume that the ride begins with the two arms along the positive x–axis and sketch the path you think the car will follow. Find a pair of parametric equations to describe the position of the car at time t.

Solution: The position of the car relative to its pivot point A is

$$x_c(t) = 8 \cdot \cos\left(\frac{2\pi}{2}\, t\right) \quad \text{and} \quad y_c(t) = 8 \cdot \sin\left(\frac{2\pi}{2}\, t\right).$$

The position of the pivot point A relative to the origin is

$$x_p(t) = 20 \cdot \cos\left(\frac{2\pi}{5}\, t\right) \quad \text{and} \quad y_p(t) = 20 \cdot \sin\left(\frac{2\pi}{5}\, t\right),$$ so the

location of the car, relative to the origin, is

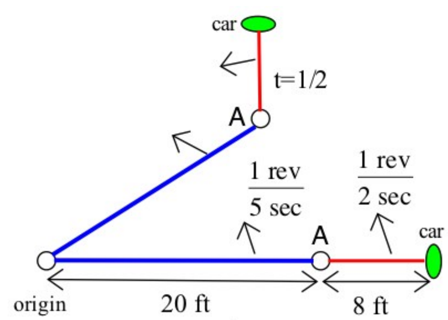


Fig. 17

$$x(t) = x_p(t) + x_c(t) = 20 \cdot \cos(\frac{2\pi}{5} t) + 8 \cdot \cos(\frac{2\pi}{2} t) \quad \text{and}$$
$$y(t) = y_p(t) + y_c(t) = 20 \cdot \sin(\frac{2\pi}{5} t) + 8 \cdot \sin(\frac{2\pi}{2} t) .$$

Use a graphing calculator to graph the path of the car for 5 seconds.

Example 8: **Cycloid** A light is attached to the edge of a wheel of radius R which is rolling along a level road (Fig. 18). Find parametric equations to describe the location of the light.

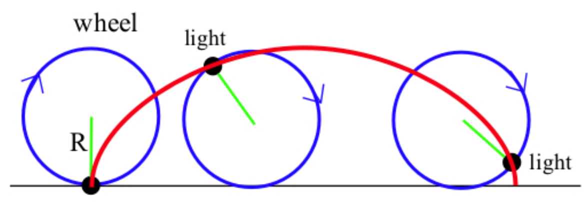


Fig. 18: Cycloid

Solution: We can describe the location of the axle of the wheel, the location of the light relative to the axle, and then put the results together to get the location of the light.

The axle of the wheel is always R inches off the ground, so the y coordinate of the axle is $y_a(t) = R$ (Fig. 19). When the wheel has rotated t radians about its axle, the wheel has rolled a distance of $R \cdot t$ along the road, and the x coordinate of the axle is $x_a(t) = R \cdot t$.

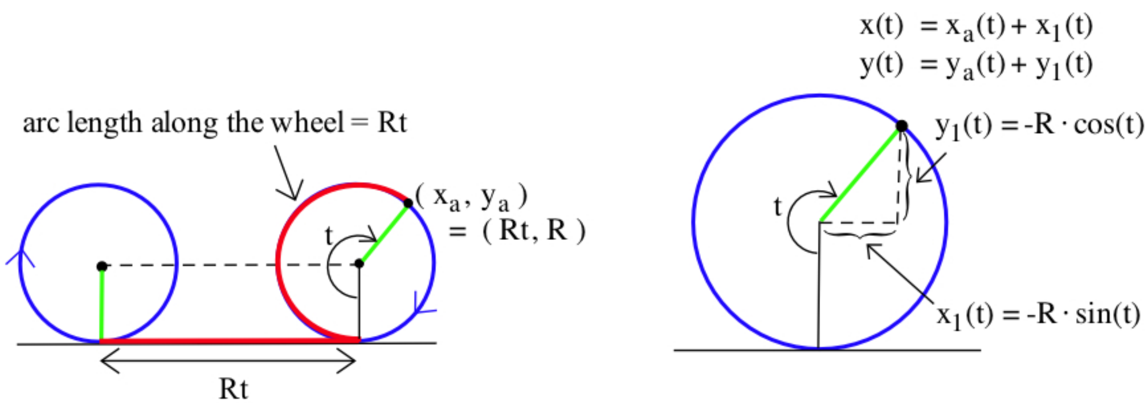


Fig. 19

The position of the light relative to the axle is $x_l(t) = -R \cdot \sin(t)$ and $y_l(t) = -R \cdot \cos(t)$ (see Example 3) so the position of the light is

$$x(t) = x_a(t) + x_l(t) = R \cdot t - R \cdot \sin(t) = R \cdot \{ t - \sin(t) \} \quad \text{and}$$
$$y(t) = y_a t) + y_l(t) = R - R \cdot \cos(t) = R \cdot \{ 1 - \cos(t) \} .$$

This curve is called a **cycloid**, and it is one of the most famous and interesting curves in mathematics. Many great mathematicians and physicists (Mersenne, Galileo, Newton, Bernoulli, Huygens, and others) examined the cycloid, determined its properties, and used it in physical applications.

Practice 7: A light is attached r units from the axle of an R inch radius wheel (r < R) that is rolling along a level road (Fig. 20). Use the approach of the solution to Example 8 to find parametric equations to describe the location of the light. The resulting curve is called an curate cycloid.

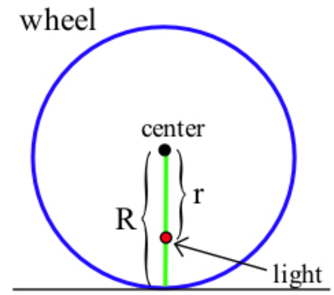


Fig. 20

The cycloid, the path of a point on a rolling circle, was studied in the early 1600's by Mersenne (1588–1648) who thought the path might be part of an ellipse (it isn't). In 1634 Roberval determined the parametric form of the cycloid and found the area under the cycloid as did Descartes and Fermat. This was done before Newton (1642–1727) was even born; they used various specialized geometric approaches to solve the area problem. About the same time Galileo determined the area experimentally by cutting a cycloid region from a sheet of lead and balancing it against a number of circular regions (with the same radius as the circle which generated the cycloid) cut from the same material. How many circles do you think balanced the cycloid region's area (Fig. 21)?

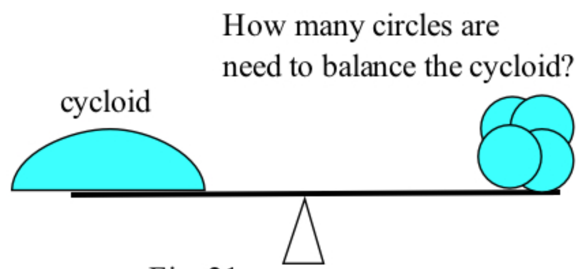


Fig. 21

However, the most amazing properties of the cycloid involve motion along a cycloid–shaped path, and their discovery had to wait for Newton and the calculus. These calculus–based properties are discussed at the end of the next section.

PROBLEMS

For problems 1–4, use the data in each table to create three graphs: (a) $(t, x(t))$, (b) $(t, y(t))$, and (c) the parametric graph $(x(t), y(t))$. (Connect the points with straight line segments to create the graph.)

1. Use Table 3.
2. Use Table 4.
3. Use Table 5.
4. Use Table 6.

t	x(t)	y(t)
0	2	1
1	2	0
2	–1	0
3	1	–1

Table 3

t	x(t)	y(t)
0	0	1
1	1	1
2	1	–1
3	2	0

Table 4

t	x(t)	y(t)
0	1	2
1	–1	–1
2	1	2
3	0	2

Table 5

t	x(t)	y(t)
0	0	1
1	–1	0
2	0	–2
3	3	1

Table 6

For problems 5–8, use the data in the given graphs of $(t, x(t))$ and $(t, y(t))$ to sketch the parametric graph $(x(t), y(t))$.

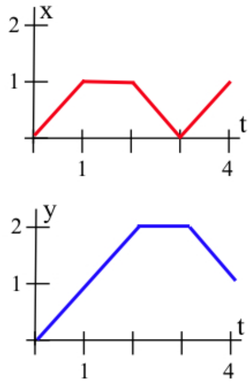


Fig. 22

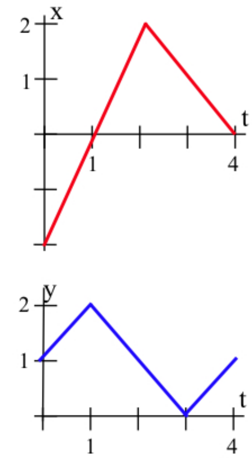


Fig. 23

5. Use x and y from Fig. 22.

6. Use x and y from Fig. 23.

7. Use x and y from Fig. 24.

8. Use x and y from Fig. 25.

9. Graph $x(t) = 3t - 2$, $y(t) = 1 - 2t$. What shape is this graph?

10. Graph $x(t) = 2 - 3t$, $y(t) = 3 + 2t$.
    What shape is this graph?

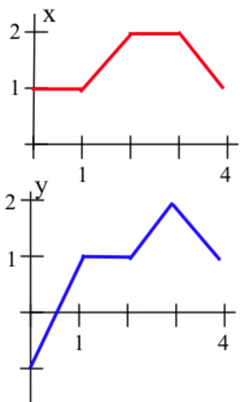


Fig. 24

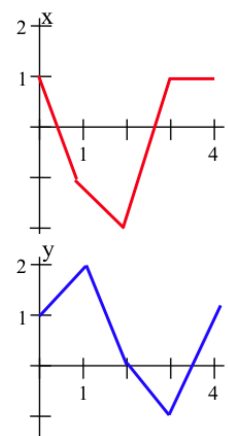


Fig. 25

11. Calculate the slope of the line through the points
    $P = (x(0), y(0))$ and $Q = (x(1), y(1))$ for $x(t) = at + b$
    and $y(t) = ct + d$.

12. Graph $x(t) = 3 + 2\cdot\cos(t)$, $y(t) = -1 + 3\cdot\sin(t)$ for
    $0 \le t \le 2\pi$. Describe the shape of the graph.

13. $x(t) = -2 + 3\cdot\cos(t)$, $y(t) = 1 - 4\cdot\sin(t)$ for $0 \le t \le 2\pi$.
    Describe the shape of the graph.

14. Graph (a) $x(t) = t^2$, $y(t) = t$, (b) $x(t) = \sin^2(t)$, $y(t) = \sin(t)$, and
    (c) $x(t) = t$, $y(t) = \sqrt{t}$. Describe the similarities and the differences among these graphs.

15. Graph (a) $x(t) = t$, $y(t) = t$, (b) $x(t) = \sin(t)$, $y(t) = \sin(t)$, and (c) $x(t) = t^2$, $y(t) = t^2$.
    Describe the similarities and the differences among these graphs.

16. Graph $x(t) = (4 - \frac{1}{t})\cos(t)$, $y(t) = (4 - \frac{1}{t})\sin(t)$ for $t \ge 1$. Describe the behavior of the graph.

17. Graph $x(t) = \frac{1}{t}\cdot\cos(t)$, $y(t) = \frac{1}{t}\cdot\sin(t)$ for $t \ge \pi/4$. Describe the behavior of the graph.

18. Graph $x(t) = t + \sin(t)$, $y(t) = t^2 + \cos(t)$ for $0 \le t \le 2\pi$. Describe the behavior of the graph.

Problems 19–22 refer to the rabbit–fox population graph shown in Fig. 26 which shows several different population cycles depending on the various numbers of rabbits and foxes. Wildlife biologists sometimes try to control animal populations by "harvesting" some of the animals, but it needs to be done with care. The thick dot on the graph is the fixed point for this two–species population.

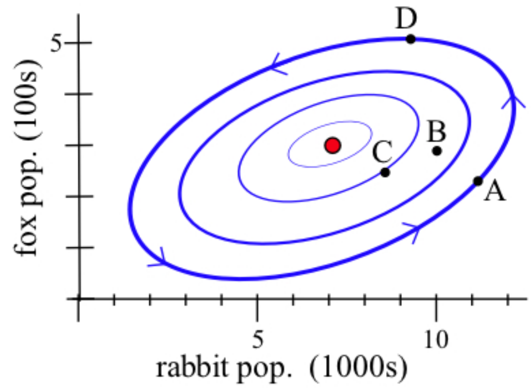


Fig. 26

19. Suppose there are currently 11,000 rabbits and 200 foxes (point A on the graph), and 1,000 rabbits are "harvested" (removed from the population). Does the harvest shift the populations onto a cycle closer to or farther from the fixed point?

20. Suppose there are currently 10,000 rabbits and 300 foxes (point B on the graph), and 100 foxes are "harvested." Does the harvest shift the populations onto a cycle closer to or farther from the fixed point?

21. Suppose there are currently 8,000 rabbits and 250 foxes (point C on the graph), and 1,000 rabbits die during a hard winter. Does the wildlife biologist need to take action to maintain the population balance? Justify your response.

22. Suppose there are currently 9,000 rabbits and 500 foxes (point D on the graph), and 2,000 rabbits die during a hard winter. Does the wildlife biologists need to take action to maintain the population balance? Justify your response.

23. Suppose x and y are functions of the form $x(t) = a \cdot t + b$ and $y(t) = c \cdot t + d$ with $a \neq 0$ and $c \neq 0$. Write y as a function of x alone and show that the parametric graph (x, y) is a straight line. What is the slope of the resulting line?

24. The parametric equations given in (a) – (e) all satisfy $x^2 + y^2 = 1$, and, for $0 \le t \le 2\pi$, the path of each object is a circle with radius 1 and center at the origin. Explain how the motions of the objects **differ**.
(a) $x(t) = \cos(t)$, $y(t) = \sin(t)$, (b) $x(t) = \cos(-t)$, $y(t) = \sin(-t)$, (c) $x(t) = \cos(2t)$, $y(t) = \sin(2t)$,
(d) $x(t) = \sin(t)$, $y(t) = \cos(t)$, and (e) $x(t) = \cos(t + \pi/2)$, $y(t) = \sin(t + \pi/2)$

25. From a tall building you observe a person is walking along a straight path while twirling a light (parallel to the ground) at the end of a string. (a) If the person is walking slowly, sketch the path of the light. (b) How would the graph change if the person was running? (c) Sketch the path for a person walking (running) along a parabolic path.

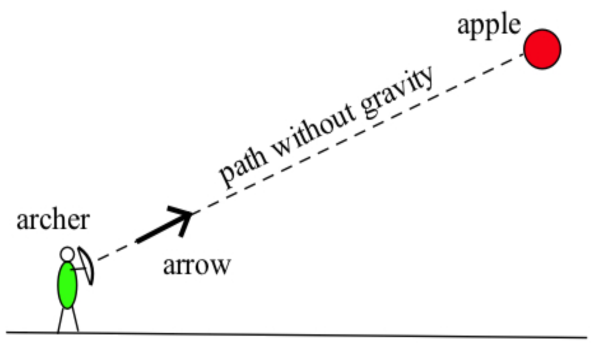


Fig. 27

26. William Tell and the Falling Apple: William Tell is aiming directly at an apple, and releases the arrow at exactly the same instant that the apple stem breaks. In a world without gravity (or air resistance), the apple remains in place after the stem breaks, and the arrow flies in a straight line to hit the apple (Fig. 27). Sketch the path of the apple and the arrow in a world with gravity (but still no air). Does the arrow still hit the apple? Why or why not?

27. Find the radius R of a circle which generates a cycloid starting at the point (0,0) and

(a) passing through the point $(10\pi, 0)$ on its first complete revolution $(0 \le t \le 2\pi)$..

(b) passing through the point $(5, 2)$ on its first complete revolution. (A calculator is helpful here.)

(c) passing through the point $(2, 3)$ on its first complete revolution. (A calculator is helpful here.)

(d) passing through the point $(4\pi, 8)$ on its first complete revolution.

The Ferris Wheel and the Apple (problems 28 – 30).

28. Your friends are on the Ferris wheel illustrated in Fig. 28, and at time t seconds, their location is given parametrically as

$$\left(-20 \sin\left(\frac{2\pi}{15} t\right), 30 - 20 \cos\left(\frac{2\pi}{15} t\right) \right).$$

(a) Is the Ferris wheel turning clockwise of counterclockwise?

(b) How many seconds does it take the Ferris wheel to make a revolution?

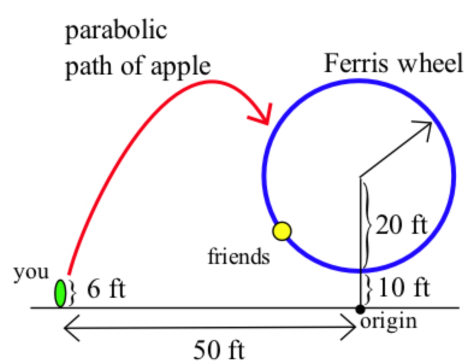


Fig. 28

29. You are 50 feet to the left of the Ferris wheel in problem 28, and you toss an apple from a height of 6 feet above the ground at an angle of 45°. Write parametric equations for the location of the apple (relative to the origin in Fig. 29) at time t if

(a) its initial velocity is 30 feet per second, and (b) its initial velocity is V feet per second.

30. Help — the Ferris wheel won't stop! To keep your friends on the Ferris wheel in problem 28 from getting too hungry, you toss an apple to them (at time $t = 0$). Write an equation for the distance between the apple and your friends at time t. Somehow, find a value for the initial velocity V of the apple so that it comes close enough for your friend to catch it, within 2 feet. (Note: A calculator or computer is probably required for this problem.)

Section 9.3 **PRACTICE Answers**

Practice 1: A possible path for the car is shown in Fig. 29.

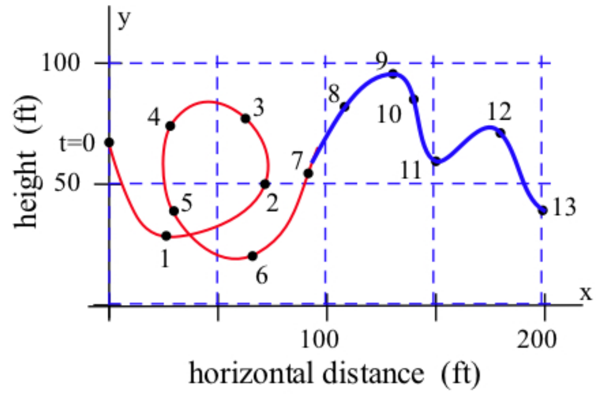


Fig. 29

Practice 2: If the (rabbit, fox) parametric graph touches the horizontal axis, then there are 0 foxes: the foxes are extinct.

Practice 3: $x = 3 - t$ and $y = t^2 + 1$.

Then $t = 3 - x$ and $y = (3 - x)^2 + 1 = x^2 - 6x + 10$.

The graph in Fig. 30 is parabola, opening upward, with vertex at (3,1).

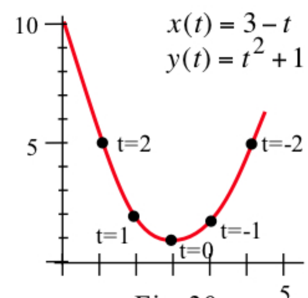


Fig. 30

Practice 4: The parametric graph of $x(t) = \sin(t)$ and $y(t) = 5\cos(t)$ is shown in Fig. 31. For all t,

$$\frac{x^2}{1} + \frac{y^2}{25} = \frac{\sin^2(t)}{1} + \frac{25\cos^2(t)}{25} = \sin^2(t) + \cos^2(t) = 1.$$

Practice 5: A: Starts at (1,0), travels counterclockwise, and takes $2\pi/2 = \pi$ seconds to make one revolution.

B: Starts at (–1,0), travels clockwise, and takes $2\pi/3$ seconds to make one revolution.

C: Starts at (0,–1), travels counterclockwise, and takes $2\pi/4 = \pi/2$ seconds to make one revolution.

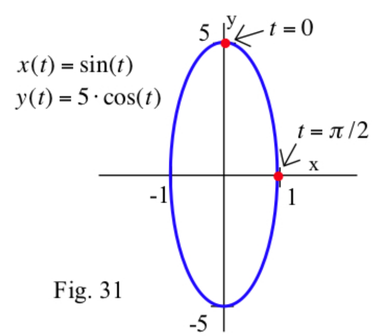


Fig. 31

Practice 6: (a) $x(t) = 40 \cdot \cos(45^\circ) \cdot t$, $y(t) = 40 \cdot \sin(45^\circ) \cdot t - 16t^2$.

(b) Let $A = 40 \cdot \sin(45^\circ) = 40 \cdot \cos(45^\circ) \approx 28.284$.

Then the ball is at $x(t) = At$ and $y(t) = At - 16t^2$.

Along the ground line, $y = -\frac{1}{2}x$ so the ball intersects the ground when $y(t) = -\frac{1}{2}x(t)$: $At - 16t^2 = -\frac{1}{2}At$.

When $t \neq 0$, we can solve $At - 16t^2 = -\frac{1}{2}At$ for $t = \frac{3}{32}A$.

Putting $t = \frac{3}{32}A$ into the equations for the location of the ball, we have

$$x\left(\frac{3}{32}A\right) = A \cdot \left(\frac{3}{32}A\right) = \frac{3}{32}A^2 \quad \text{and} \quad y\left(\frac{3}{32}A\right) = A \cdot \left(\frac{3}{32}A\right) - 16\left(\frac{3}{32}A\right)^2 = -\frac{3}{64}A^2.$$

The ball hits the ground after $t = \frac{3}{32}A = \frac{3}{32} \cdot 40 \cdot \sin(45^\circ)$ **≈ 2.652 seconds**.

The ball hits the ground at the location **x =** $\frac{3}{32}A^2$ **= 75 feet** and **y =** $-\frac{3}{64}A^2$ **= – 37.5 feet**.

Practice 7: Axle: $x_a = R \cdot t$ and $y_a = R$. Light relative to the axle: $x_l = -r \cdot \sin(t)$ and $y_l = -r \cdot \cos(t)$.

Then $x(t) = x_a + x_l = R \cdot t - r \cdot \sin(t)$ and $y(t) = y_a + y_l = R - r \cdot \cos(t)$.