9.4 CALCULUS AND PARAMETRIC EQUATIONS

The previous section discussed parametric equations, their graphs, and some of their uses for visualizing and analyzing information. This section examines some of the ideas and techniques of calculus as they apply to parametric equations: slope of a tangent line, speed, arc length, and area. Slope, speed, and arc length were considered earlier (in optional parts of sections 2.5 and 5.2), and the presentation here is brief. The material on area is new and is a variation on the Riemann sum development of the integral. This section ends with a presentation of some of the properties of the cycloid.

Slope (also see section 2.5)

If x(t) and y(t) are differentiable functions of t, then the derivatives dx/dt and dy/dt measure the rates of change of x and y with respect to t: dx/dt and dy/dt tell how fast each variable is changing. The derivative dy/dx measures the slope of the line tangent to the parametric graph (x(t), y(t)). To calculate dy/dx we need to use the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \,.$$

Dividing each side of the Chain Rule by $\frac{dx}{dt}$, we have $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

Slope with Parametric Equations

If x(t) and y(t) are differentiable functions of t and $\frac{dx}{dt} \neq 0$,

then the **slope** of the line tangent to the parametric graph is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

Example 1: The location of an object is given by the parametric equations $x(t) = t^3 + 1$ feet and $y(t) = t^2 + t$ feet at time t seconds.

(a) Evaluate x(t) and y(t) at $t = -2, -1, 0, 1$, and 2, and then graph the path of the object for $-2 \le t \le 2$.

(b) Evaluate dy/dx for $t = -2, -1, 0, 1$, and 2. Do your calculated values for dy/dx agree with the shape of your graph in part (a)?

t	x	y	dy/dx
–2	–7	2	–3/12= –1/4
–1	0	0	–1/3
0	1	0	undefined
1	2	2	3/3 = 1
2	9	6	5/12

Table 1

Solution: (a) When $t = -2$, $x = (-2)^3 + 1 = -7$ and $y = (-2)^2 + (-2) = 2$. The other values for x and y are given in Table 1.

The graph of (x, y) is shown in Fig. 1.

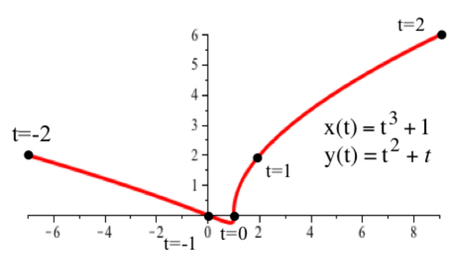


Fig. 1

(b) $dy/dt = 2t + 1$ and $dx/dt = 3t^2$ so $\frac{dy}{dx} = \frac{2t+1}{3t^2}$. When $t = -2$, $\frac{dy}{dx} = \frac{-3}{12}$. The other values for dy/dx are given in Table 1.

Practice 1: Find the equation of the line tangent to the graph of the parametric equations in Example 1 when $t = 3$.

An object can "visit" the same location more than once, and a parametric graph can go through the same point more than once.

Example 2: Fig. 2 shows the x and y coordinates of an object at time t.

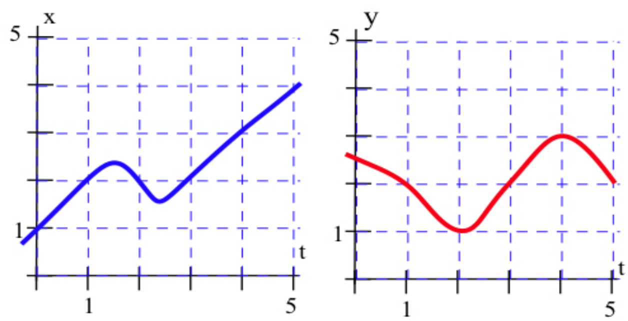


Fig. 2

(a) Sketch the parametric graph $(x(t), y(t))$, the position of the object at time t.

(b) Give the coordinates of the object when $t = 1$ and $t = 3$.

(c) Find the slopes of the tangent lines to the parametric graph when $t = 1$ and $t = 3$.

Solution: (a) By reading the x and y values on the graphs in Fig. 2, we can plot points on the parametric graph. The parametric graph is shown in Fig. 3.

(b) When $t = 1$, $x = 2$ and $y = 2$ so the parametric graph goes through the point (2,2). When $t = 3$, the parametric graph goes through the same point (2,2).

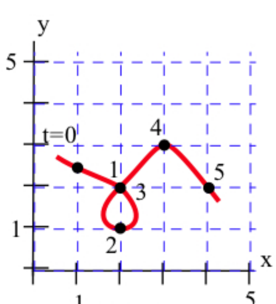


Fig. 3

(c) When $t = 1$, $dy/dt \approx -1$ and $dx/dt \approx +1$ so $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \approx \frac{-1}{+1} = -1$.

When $t = 3$, $dy/dt \approx +1$ and $dx/dt \approx +1$ so $\frac{dy}{dx} \approx \frac{+1}{+1} = +1$.

These values agree with the appearance of the parametric graph in Fig. 3.

The object goes through the point (2,2) twice (when t=1 and t=3), but it is traveling in a different direction each time.

Practice 2: (a) Estimate the slopes of the lines tangent to the parametric graph when $t = 2$ and $t = 5$.

(b) When does $y\,'(t) = 0$ in Fig. 2?

(c) When does the parametric graph in Fig. 3 have a maximum? A minimum?

(d) How are the maximum and minimum points on a parametric graph related to the derivatives of $x(t)$ and $y(t)$?

Speed

If we know how fast an object is moving in the x direction (dx/dt) and how fast in the y direction (dy/dt), it is straightforward to determine the speed of the object, how fast it is moving in the xy –plane.

If, during a short interval of time Δt, the object's position changes Δx in the x direction and Δy in the y direction (Fig. 4), then the object has moved $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ in Δt time. Then

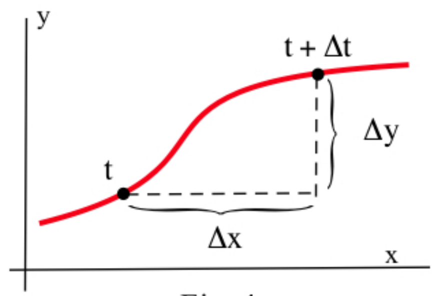


Fig. 4

$$\text{average speed} = \frac{\text{distance moved}}{\text{time change}} = \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta t}$$

$$= \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \ .$$

If x(t) and y(t) are differentiable functions of t, and if we take the limit of the average speed as Δt approaches 0, then

$$\text{speed} = \lim_{\Delta t \to 0} \{\text{average speed}\} = \lim_{\Delta t \to 0} \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ .$$

Speed with Parametric Equations

If an object is located at (x(t), y(t)) at time t, and x(t) and y(t) are differentiable functions of t ,

then the **speed** of the object is $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.

Example 3: At time t seconds an object is located at (cos(t) feet, sin(t) feet) in the plane. Sketch the path of the object and show that it is travelling at a constant speed.

Solution: The object is moving in a circular path (Fig. 5). dx/dt = –sin(t) feet/second and dy/dt = cos(t) feet/second so at all times the speed of the object is

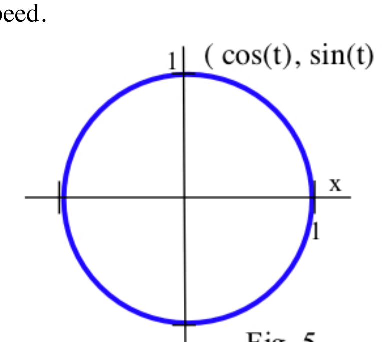


Fig. 5

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(\,-\sin(t)\,)^2 + (\,\cos(t)\,)^2}$$

$$= \sqrt{\sin^2(t) + \cos^2(t)} = \sqrt{1} = 1 \text{ foot per second.}$$

Practice 3: Is the object in Example 2 traveling faster when $t = 1$ or when $t = 3$? When $t = 1$ or when $t = 2$?

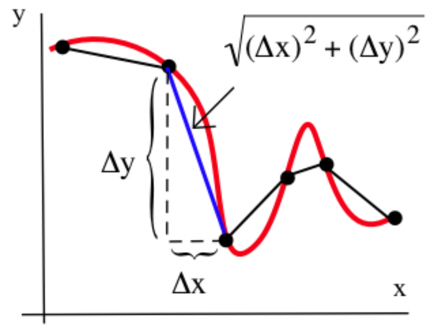


Fig. 6

Arc Length (also see section 5.2)

In section 5.2 we approximated the total length L of a curve by partitioning the curve into small pieces (Fig. 6), approximating the length of each piece using the distance formula, and then adding the lengths of the pieces together to get

$$L \approx \sum \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sum \sqrt{\left(\frac{\Delta x}{\Delta x}\right)^2 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x \text{ , a Riemann sum.}$$

As Δx approaches 0, the Riemann sum approaches the definite integral

$$L = \int_{x=a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx\ .$$

A similar approach also works for parametric equations, but in this case we factor out a Δt from the original summation:

$$L \approx \sum \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\ \Delta t \text{ (a Riemann sum)}$$

$$\longrightarrow \int_{t=a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt \text{ as } \Delta t \to 0.$$

Arc Length with Parametric Equations

If $x(t)$ and $y(t)$ are differentiable functions of t

then the length of the parametric graph from $(x(a), y(a))$ to $(x(b), y(b))$ is

$$L = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Example 4: Find the length of the cycloid $x = R(t - \sin(t))$ $y = R(1 - \cos(t))$ for $0 \le t \le 2\pi$. (Fig. 7)

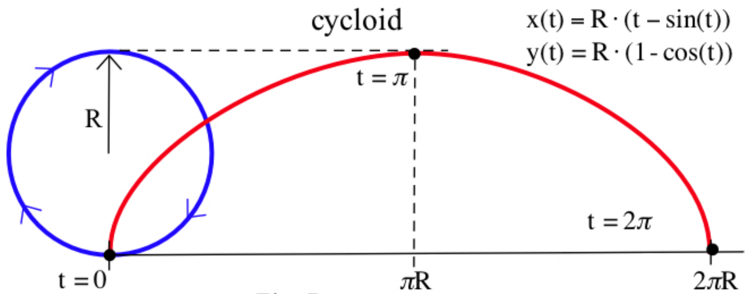


Fig. 7

Solution: Since $dx/dt = R(1 - \cos(t))$ and $dy/dt = R\cdot\sin(t)$,

$$L = \int_{t=a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt =$$

$$\int_{t=0}^{2\pi} \sqrt{(R(1 - \cos(t)))^2 + (R\cdot\sin(t))^2}\, dt$$

$$= R\int_{t=0}^{2\pi} \sqrt{1 - 2\cos(t) + \cos^2(t) + \sin^2(t)}\, dt = R\int_{t=0}^{2\pi} \sqrt{2 - 2\cdot\cos(t)}\, dt .$$

By replacing θ with $t/2$ in the formula $\sin^2(\theta) = \dfrac{1 - \cos(2\theta)}{2}$ we have $\sin^2(t/2) = \dfrac{1 - \cos(t)}{2}$ so $2 - 2\cdot\cos(t) = 4\cdot\sin^2(t/2)$, and the integral becomes

$$L = R\int_{t=0}^{2\pi} 2\cdot\sin(t/2)\, dt = 2R\{-2\cdot\cos(t/2)\}\Big|_0^{2\pi} = 2R\{-2\cdot\cos(\pi) + 2\cdot\cos(0)\} = \mathbf{8R} .$$

The length of a cycloid arch is 8 times the radius of the rolling circle that generated the cycloid.

Practice 4: Represent the length of the ellipse $x = 3\cdot\cos(t)$ $y = 2\cdot\sin(t)$ for for $0 \le t \le 2\pi$ (Fig. 8). as a definite integral. Use a calculator to approximate the value of the integral.

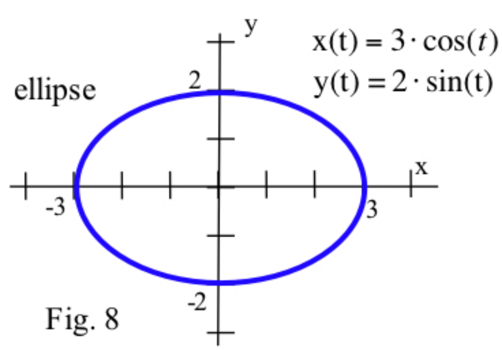


Fig. 8

Area

When we first discussed area and developed the definite integral, we approximated the area of a positive function y (Fig. 9) by partitioning the domain $a \le x \le b$ into pieces of length Δx, finding the areas of the thin rectangles, and approximating the total area by adding the little areas together:

$$A \approx \sum y \, \Delta x \quad \text{(a Riemann sum).}$$

As Δx approached 0, the Riemann sum approached the definite integral $\int_{x=a}^{x=b} y \, dx$.

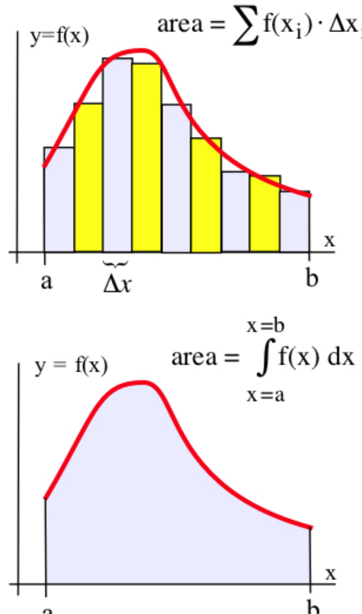


Fig. 9

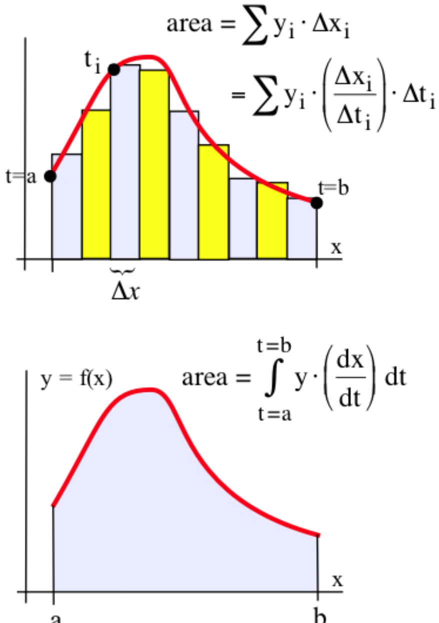


Fig. 10

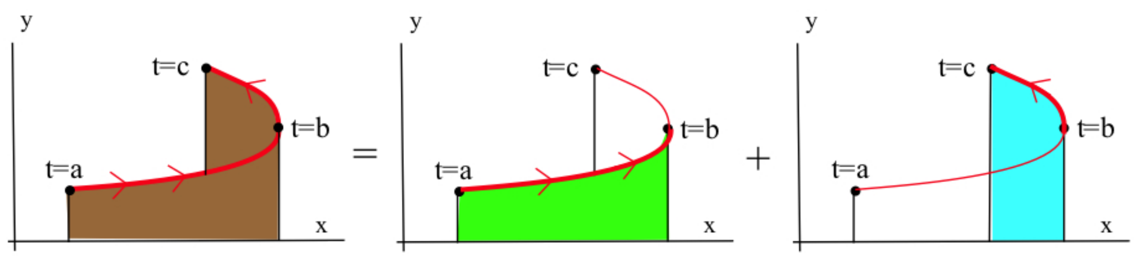
For parametric equations, the independent variable is t and the domain is an interval [a, b].

If x is an increasing function of t, then a partition of the t–interval [a, b] into pieces of length Δt induces a partition along the x–axis (Fig. 10), and we can use the induced partition of the x–axis to approximate the total area by

$$A \approx \sum y \, \Delta x = \sum y \, \frac{\Delta x}{\Delta t} \, \Delta t$$ which approaches the definite integral $A = \int_{t=a}^{t=b} y \cdot \left(\frac{dx}{dt} \right) dt$ as Δt approaches 0.

> **Area** with Parametric Equations
>
> If y and dx/dt do not change sign for $a \le t \le b$,
>
> then the **area** between the graph (x , y) and the x–axis is $A = \left| \int_{t=a}^{t=b} y \cdot \left(\frac{dx}{dt} \right) dt \right|$.

The requirement that y not change sign for $a \le t \le b$ is to prevent the parametric graph from being above the x–axis sometimes and below the x–axis sometimes. The requirement that dx/dt not change sign for $a \le t \le b$ is to prevent the graph from "turning around" (Fig. 11). If either of those situations occurs, some of the area is evaluated as positive and some of the area is evaluated as negative.



$$\text{area} = \int_{t=a}^{t=c} y \cdot \left(\frac{dx}{dt}\right) dt$$

may be positive or negative even though y is always positive

from t=a to t=b

$$\frac{dx}{dt} > 0$$

$$\int_{t=a}^{t=b} y \cdot \left(\frac{dx}{dt}\right) dt \quad \text{is } \textbf{positive}$$

from t=b to t=c

$$\frac{dx}{dt} < 0$$

$$\int_{t=b}^{t=c} y \cdot \left(\frac{dx}{dt}\right) dt \quad \text{is } \textbf{negative}$$

Fig. 11

Example 5: Find the area of the ellipse $x = a \cdot \cos(t)$, $y = b \cdot \sin(t)$ $(a,b > 0)$ in the first quadrant (Fig. 12).

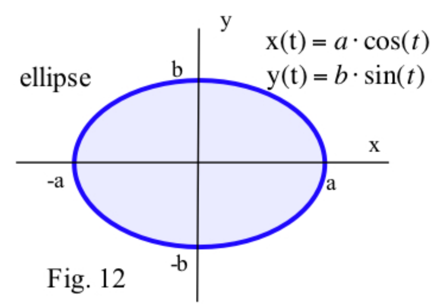


Fig. 12

Solution: The derivative $dx/dt = -a \cdot \sin(t)$, and in the first quadrant $0 \le t \le \pi/2$. Then the

$$\text{area of the ellipse in first quadrant} = \left| \int_{t=a}^{b} y \cdot \left(\frac{dx}{dt}\right) dt \right|$$

$$= \left| \int_{t=0}^{\pi/2} \{ b \cdot \sin(t) \} \cdot \left(-a \cdot \sin(t) \right) dt \right|$$

$$= \left| -ab \int_{t=0}^{\pi/2} \sin^2(t)\ dt \right| = ab \int_{t=0}^{\pi/2} \sin^2(t)\ dt \qquad \left(\text{replace } \sin^2(t) \text{ with } \frac{1-\cos(2t)}{2}\right)$$

$$= \frac{1}{2}\, ab \int_{t=0}^{\pi/2} 1 - \cos(2t)\ dt = \frac{1}{2}\, ab \left\{ t - \frac{1}{2} \cdot \sin(2t) \right\} \Big|_0^{\pi/2} = \frac{1}{4}\, ab\pi .$$

The area of the whole ellipse is $4\{ \frac{1}{4}\, ab\pi \} = \boldsymbol{\pi ab}$.

If $a = b$, the ellipse is a circle with radius $r = a = b$, and its area is πr^2 as expected.

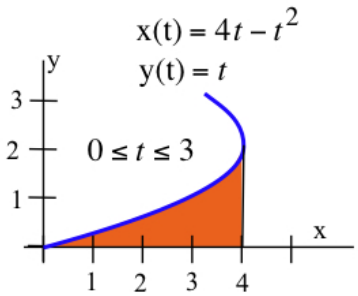


Fig. 13

Practice 5: Let $x(t) = 4t - t^2$ and $y(t) = t$ (Fig. 13).

(a) Represent the shaded area in Fig. 13 as an integral and evaluate the integral.

(b) Evaluate $\displaystyle\int_{t=0}^{t=3} t\cdot(4 - 2t)\, dt$. Does this value represent an area?

Area under a Cycloid: (Fig. 7) For all $t \ge 0$, $x = R(t - \sin(t)) \ge 0$, $y = R(1 - \cos(t)) \ge 0$, and $dx/dt = R(1 - \cos(t)) \ge 0$ so we can use the area formula. Then

$$\text{area} = \left| \int_{t=a}^{b} y\cdot\left(\frac{dx}{dt}\right) dt \right| = \left| \int_{t=0}^{2\pi} \{R(1-\cos(t))\}\cdot\big(R(1-\cos(t))\big)\, dt \right|$$

$$= R^2 \int_{t=0}^{2\pi} 1 - 2\cdot\cos(t) + \cos^2(t)\; dt \qquad (\text{ replace } \cos^2(t) \text{ with } \frac{1+\cos(2t)}{2} \text{ and integrate })$$

$$= R^2 \left\{ t - 2\cdot\sin(t) + \frac{1}{2}t + \frac{1}{4}\sin(2t) \right\} \Bigg|_0^{2\pi} = R^2\{2\pi + \pi\} = \mathbf{3\pi R^2}.$$

The area under one arch of a cycloid is 3 times the area of the circle that generates the cycloid.

Properties of the Cycloid

Suppose you and a friend decide to have a contest to see who can build a slide that gets a person from point A to point B (Fig. 14) in the shortest time. What shape should you make your slide — a straight line, part of a circle, or something else? Assuming that the slide is frictionless and that the only acceleration is due to gravity, John Bernoulli showed that the **shortest time** ("brachistochrone" for "brachi" = short and "chrone" = time) path is a cycloid that starts at A that also goes through the point B. Fig. 15 shows the cycloid paths for A and B as well as the cycloid paths for two other "finish" points, C and D.

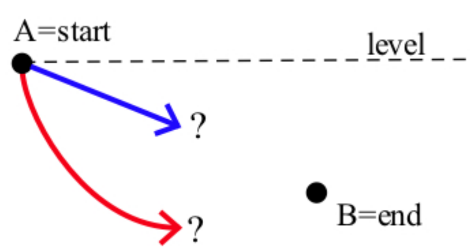


Fig. 14

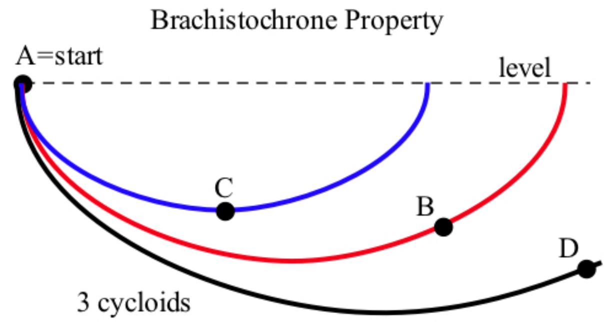


Fig. 15

Even before Bernoulli solved the brachistochrone problem, the astronomer (physicist, mathematician) Huygens was trying to design an accurate pendulum clock. On a standard pendulum clock (Fig. 16), the path of the bob is part of a circle, and the period of the swing depends on the displacement angle of the bob. As friction slows the bob, the displacement angle gets smaller and the clock slows down. Huygens designed a clock (Fig. 17) whose bob swung in a curve so that the period of the swing did not depend on the displacement angle. The curve Huygens found to solve the **same time** ("tautochrone" for "tauto" = same and "chrone" = time) problem was the cycloid. Beads strung on a wire in the shape of a cycloid (Fig. 18) reach the bottom in the same amount of time, no matter where along the wire (except the bottom point) they are released.

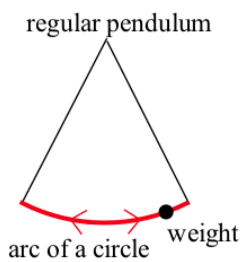


Fig. 16

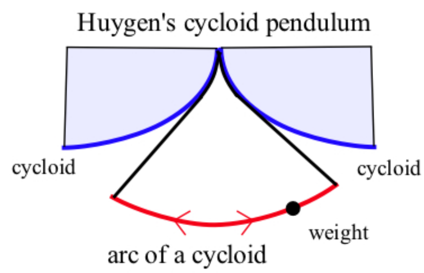


Fig. 17

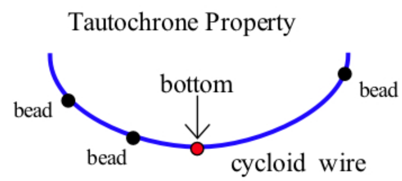


Fig. 18

The brachistochane and tautochrone problems are examples from a field of mathematics called the Calculus of Variations. Typical optimization problems in calculus involve finding a point or number that maximizes or minimizes some quantity. Typical optimization problems in the Calculus of Variations involve finding the curve or function that maximizes or minimizes some quantity. For example, what curve or shape with a given length encloses the greatest area? (Answer: a circle) Modern applications of Calculus of Variations include finding routes for airliners and ships to minimize travel time or fuel consumption depending on prevailing winds or currents.

PROBLEMS

Slope

For problems 1–8, (a) sketch the parametric graph (x,y),

(b) find the slope of the line tangent to the parametric graph at the given values of t, and

(c) find the points (x,y) at which dy/dx is either 0 or undefined.

1. $x(t) = t - t^2$, $y(t) = 2t + 1$ at $t = 0, 1$, and 2.

2. $x(t) = t^3 + t$, $y(t) = t^2$ at $t = 0, 1$, and 2.

3. $x(t) = 1 + \cos(t)$, $y(t) = 2 + \sin(t)$ at $t = 0, \pi/4$, and $\pi/2$.

4. $x(t) = 1 + 3\cdot\cos(t)$, $y(t) = 2 + 2\cdot\sin(t)$ at $t = 0, \pi/4, \pi/2$, and π.

5. $x(t) = \sin(t)$, $y(t) = \cos(t)$ at $t = 0, \pi/4, \pi/2$, and 17.3.

6. $x(t) = 3 + \sin(t)$, $y(t) = 2 + \sin(t)$ at $t = 0, \pi/4, \pi/2$, and 17.3.

7. $x(t) = \ln(t)$, $y(t) = 1 - t^2$ at $t = 1, 2$, and e.

8. $x(t) = \arctan(t)$, $y(t) = e^t$ at $t = 0, 1$, and 2.

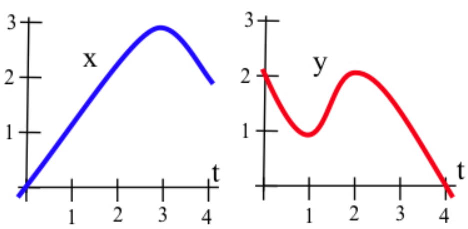


Fig. 19

In problems 9–12, the graphs of $x(t)$ and $y(t)$ are given.
Use this graphical information to estimate

(a) the slope of the line tangent to the parametric graph at $t = 0, 1, 2$, and 3, and

(b) the points (x,y) at which dy/dx is either 0 or undefined.

9. $x(t)$ and $y(t)$ in Fig. 19.

10. $x(t)$ and $y(t)$ in Fig. 20.

11. $x(t)$ and $y(t)$ in Fig. 21.

12. $x(t)$ and $y(t)$ in Fig. 22.

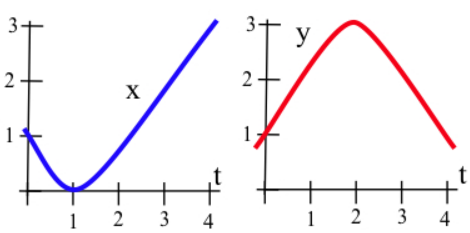


Fig. 20

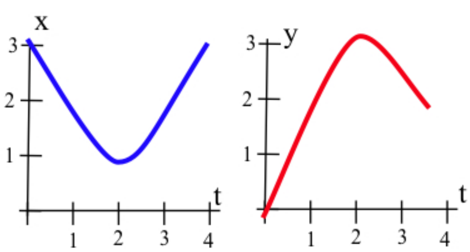


Fig. 21

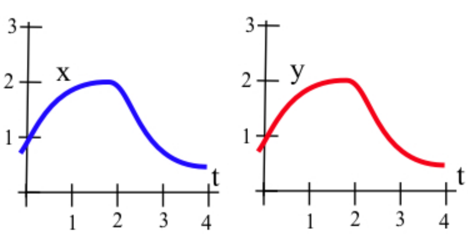


Fig. 22

Speed

For problems 13–20, the locations $x(t)$ and $y(t)$ (in feet) of an object are given at time t seconds. Find the speed of the object at the given times.

13. $x(t) = t - t^2$, $y(t) = 2t + 1$ at $t = 0, 1$, and 2.

14. $x(t) = t^3 + t$, $y(t) = t^2$ at $t = 0, 1$, and 2.

15. $x(t) = 1 + \cos(t)$, $y(t) = 2 + \sin(t)$ at $t = 0, \pi/4, \pi/2$, and π .

16. $x(t) = 1 + 3\cdot\cos(t)$, $y(t) = 2 + 2\cdot\sin(t)$ at $t = 0, \pi/4, \pi/2$, and π.

17. x and y in Fig. 19 at $t = 0, 1, 2, 3$, and 4.

18. x and y in Fig. 20 at $t = 0, 1, 2$, and 3.

19. x and y in Fig. 21 at $t = 0, 1, 2$, and 3.

20. x and y in Fig. 22 at $t = 0, 1, 2$, and 3.

21. At time t seconds an object is located at the point $x(t) = R\cdot(t - \sin(t))$, $y(t) = R\cdot(1 - \cos(t))$ (in feet).
    (a) Find the speed of the object at time t. (b) At what time is the object traveling fastest?
    (c) Where is the object on the cycloid when it is traveling fastest?

22. At time t seconds an object is located at the point $x(t) = 5\cdot\cos(t)$, $y(t) = 2\cdot\sin(t)$ (in feet).
    (a) Find the speed of the object at time t. (b) At what time is the object traveling fastest?
    (c) Where is the object on the ellipse when it is traveling fastest?

Arc Length

For problems 23–28, (a) represent the arc length of each parametric function as a definite integral, and (b) evaluate the integral (if necessary, use your calculator's **fnInt()** feature to evaluate the integral).

23. $x(t) = t - t^2$, $y(t) = 2t + 1$ for $t = 0$ to 2 .

24. $x(t) = t^3 + t$, $y(t) = t^2$ for $t = 0$ to 2 .

25. $x(t) = 1 + \cos(t)$, $y(t) = 2 + \sin(t)$ for $t = 0$ to π .

26. $x(t) = 1 + 3\cdot\cos(t)$, $y(t) = 2 + 2\cdot\sin(t)$ for $t = 0$ to π .

27. x and y in Fig. 23 for $t = 1$ to 3.

28. x and y in Fig. 22 for $t = 0$ to 2.

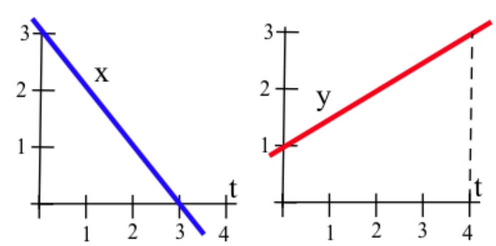


Fig. 23

Area

For problems 29–34, (a) represent the area of each region as a definite integral, and (b) evaluate the integral (if necessary, use your calculator's **fnInt()** feature to evaluate the integral).

29. $x(t) = t^2$, $y(t) = 4t^2 - t^4$ for $0 \le t \le 2$.

30. $x(t) = 1 + \sin(t)$, $y(t) = 2 + \sin(t)$ for $0 \le t \le \pi$.

31. $x(t) = t^2$, $y(t) = 1 + \cos(t)$ for $0 \le t \le 2$.

32. $x(t) = \cos(t)$, $y(t) = 2 - \sin(t)$ for $0 \le t \le \pi/2$.

33. "Cycloid" with a square wheel: Find the area under one "arch" of the path of a point on the corner of a "rolling" square that has sides of length R. (This problem does not require calculus.)

34. The region bounded between the x–axis and the curate cycloid $x(t) = R \cdot t - r \cdot \sin(t)$, $y(t) = R - r \cdot \cos(t)$ for $0 \le t \le 2\pi$.

Section 9.4 **PRACTICE Answers**

Practice 1: $\frac{dy}{dt} = 2t + 1$, so when $t = 3$, $\frac{dy}{dt} = 7$. $\frac{dx}{dt} = 3t^2$, so when $t = 3$, $\frac{dx}{dt} = 27$.

Finally, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ so when $t = 3$, $\frac{dy}{dx} = \frac{7}{27}$.

When $t = 3$, $x = 28$ and $y = 12$ so the equation of the tangent line is $y - 12 = \frac{7}{27}(x - 28)$.

Practice 2: (a) When $t = 2$, $dy/dx \approx 0$. When $t = 5$, $dy/dx \approx -1$.

(b) In Fig. 2, $\frac{dy}{dt} = 0$ when $t \approx 2$ and $t \approx 4$.

(c) In Fig. 3, a minimum occurs when $t \approx 2$ and a maximum when $t \approx 4$.

(d) If the parametric graph has a maximum or minimum at $t = t^*$, then dy/dt is either 0 or undefined when $t = t^*$.

Practice 3: When $t = 1$, speed $= \sqrt{(dx/dt)^2 + (dy/dt)^2} \approx \sqrt{(1)^2 + (-1)^2} = \sqrt{2} \approx 1.4$ ft/sec.
When $t = 2$, speed $= \sqrt{(dx/dt)^2 + (dy/dt)^2} \approx \sqrt{(-1)^2 + (0)^2} = \sqrt{1} = 1$ ft/sec.
When $t = 3$, speed $= \sqrt{(dx/dt)^2 + (dy/dt)^2} \approx \sqrt{(1)^2 + (1)^2} = \sqrt{2} \approx 1.4$ ft/sec.

Practice 4: Length $= \int_{t=0}^{2\pi} \sqrt{(-3\sin(t))^2 + (2\cos(t))^2}\, dt$

≈ 15.87 (using my calculator's **fnInt()** feature)

Practice 5: (a) $A = \int_{t=0}^{2} t \cdot (4 - 2t)\, dt = 2 \cdot t^2 - \frac{2}{3} t^3 \Big|_0^2 = \{ 8 - \frac{16}{3} \} - \{ 0 - 0 \} = \frac{8}{3}$.

(b) $\int_{t=0}^{3} t \cdot (4 - 2t)\, dt = 2 \cdot t^2 - \frac{2}{3} t^3 \Big|_0^3 = \{ 18 - 18 \} - \{ 0 - 0 \} = 0$.

This integral represents {shaded area in Fig. 13 } – { area from $t = 2$ to $t = 3$ }.