9.5 CONIC SECTIONS

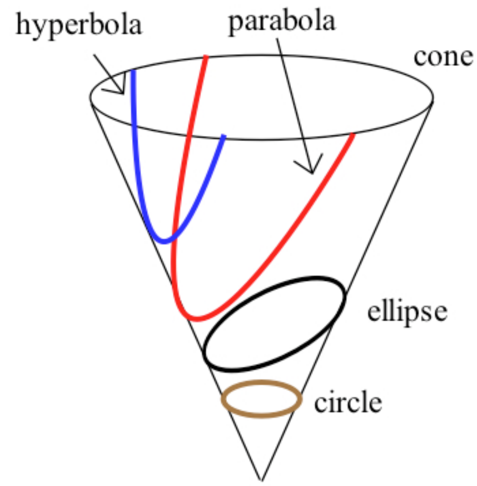


Fig. 1

The conic sections are the curves obtained when a cone is cut by a plane (Fig. 1). They have attracted the interest of mathematicians since the time of Plato, and they are still used by scientists and engineers. The early Greeks were interested in these shapes because of their beauty and their representations by sets of points that met certain distance definitions (e.g., the circle is the set of points at a fixed distance from a given point). Mathematicians and scientists since the 1600s have been interested in the conic sections because the planets, moons, and other celestial objects follow paths that are (approximately) conic sections, and the reflective properties of the conic sections are useful for designing telescopes and other instruments. Finally, the conic sections give the **complete** answer to the question, "what is the shape of the graph of the general quadratic equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$?"

This section discusses the "cut cone" and distance definitions of the conic sections and shows their standard equations in rectangular coordinate form. The section ends with a discussion of the discriminant, an easy way to determine the shape of the graph of any standard quadratic equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Section 9.6 examines the polar coordinate definitions of the conic sections, some of the reflective properties of the conic sections, and some of their applications.

Cutting A Cone

When a (right circular double) cone is cut by a plane, only a few shapes are possible, and these are called the conic sections (Fig. 1). If the plane makes an angle of θ with the horizontal, and $\theta < \alpha$, then the set of points is an ellipse (Fig. 2). When $\theta = 0 < \alpha$, we have a circle, a special case of an ellipse (Fig. 3). If $\theta = \alpha$, a parabola is formed (Fig. 4), and if $\theta > \alpha$, a hyperbola is formed (Fig. 5). When the plane goes through the vertex of the cone, degenerate conics are formed: the degenerate ellipse ($\theta < \alpha$) is a point, the degenerate parabola ($\theta = \alpha$) is a line, and a degenerate hyperbola ($\theta > \alpha$) is a pair of intersecting lines.

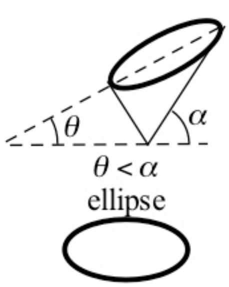


Fig. 2

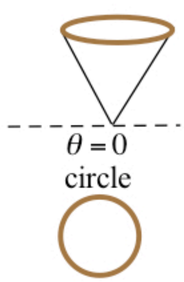


Fig. 3

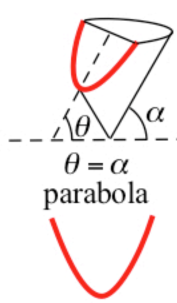


Fig. 4

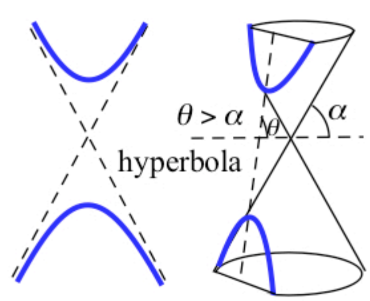


Fig. 5

The conic sections are lovely to look at, but we will not use the conic sections as pieces of a cone because the "cut cone" definition of these shapes does not easily lead to formulas for them. To determine formulas for the conic sections it is easier to use alternate definitions of these shapes in terms of distances of points from fixed points and lines. Then we can use the formula for distance between two points and some algebra to derive formulas for the conic sections.

The Ellipse

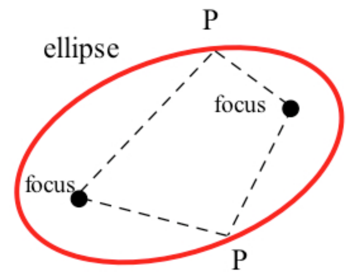


Fig. 6

Ellipse: An ellipse is the set of all points P for which the **sum** of the distances from P to two fixed points (called foci) **is a constant**: dist(P, one focus) + dist(P, other focus) = constant. (Fig. 6)

Example 1: Find the set of points whose distances from the foci $F_1 = (4,0)$ and $F_2 = (-4, 0)$ add up to 10.

Solution: If the point $P = (x, y)$ is on the ellipse, then the distances $PF_1 = \sqrt{(x-4)^2 + y^2}$ and $PF_2 = \sqrt{(x+4)^2 + y^2}$ must total 10 so we have the equation

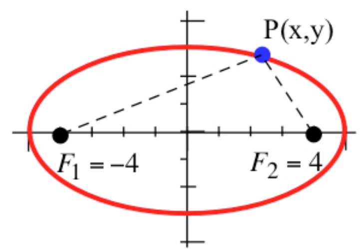
$$PF_1 + PF_2 = \sqrt{(x-4)^2 + y^2} + \sqrt{(x+4)^2 + y^2} = 10 \quad \text{(Fig. 7)}$$

Moving the second radical to the right side of the equation, squaring both sides, and simplifying, we get

$$4x + 25 = 5\sqrt{(x+4)^2 + y^2} \quad .$$

Squaring each side again and simplifying, we have $225 = 9x^2 + 25y^2$ so, after dividing each side by 225,

$$\frac{x^2}{25} + \frac{y^2}{9} = 1 \; .$$



ellipse: $\overline{PF_1} + \overline{PF_2} = 10$
Fig. 7

Practice 1: Find the set of points whose distances from the foci $F_1 = (3,0)$ and $F_2 = (-3, 0)$ add up to 10.

Using the same algebraic steps as in Example 1, it can be shown (see the Appendix at the end of the problems) that the set of points $P = (x,y)$ whose distances from the foci $F_1 = (c,0)$ and $F_2 = (-c, 0)$ add up to $2a$ $(a > c)$ is described by the formula

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ where } b^2 = a^2 - c^2 \; .$$

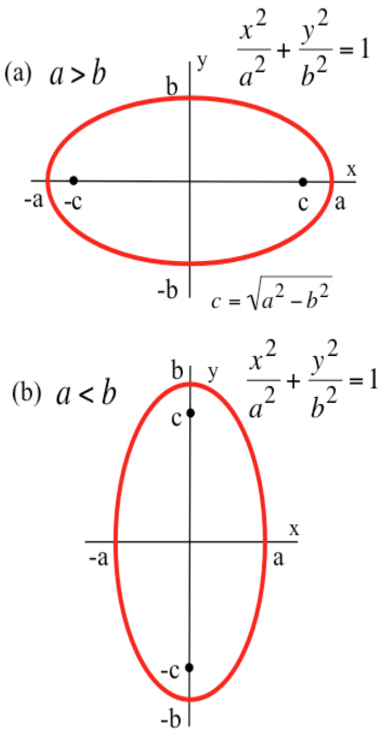


Fig. 8

Ellipse

The standard formula for an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

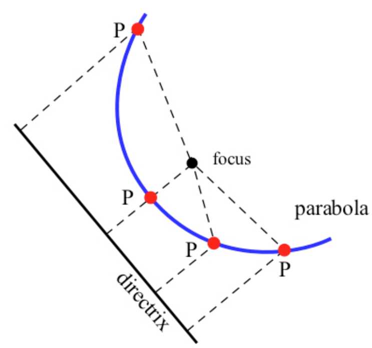
a = b: The ellipse is a circle.

a > b: (Fig. 8a) The vertices are at (±a, 0) on the x–axis,
the foci are at (±c, 0) with $c = \sqrt{a^2 - b^2}$, and
for any point P on the ellipse,
dist(P, one focus) + dist(P, other focus) = 2a.
The length of the semimajor axis is a.

a < b: (Fig. 8b) The vertices are at (0, ±b) on the y–axis,
the foci are at (0, ±c) with $c = \sqrt{b^2 - a^2}$, and
for any point P on the ellipse,
dist(P, one focus) + dist(P, other focus) = 2b.
The length of the semimajor axis is b.

Practice 2: Use the information in the box to determine the vertices, foci, and length of the semimajor axis of the ellipse $\frac{x^2}{169} + \frac{y^2}{25} = 1$.

The Parabola



dist(P, focus) = dist(P, directrix)
Fig. 9

Parabola: A parabola is the set of all points P for which the distance from P to a fixed point (focus) **is equal to** the distance from P to a fixed line (directrix): dist(P, focus) = dist(P, directrix). (Fig. 9)

Example 2: Find the set of points P = (x,y) whose distance from the focus F = (4,0) equals the distance from the directrix $x = -1$.

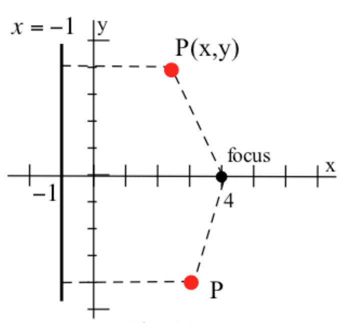


Fig. 10

Solution: The distance $PF = \sqrt{(x-4)^2 + y^2}$, and the distance from P to

to

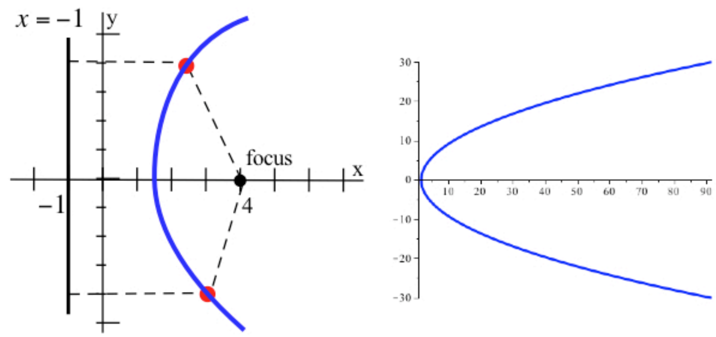
the directrix (Fig. 10) is x+1. If these two distances are equal then we have the equation $\sqrt{(x-4)^2 + y^2} = x + 1$.

Squaring each side,

$$(x-4)^2 + y^2 = (x+1)^2$$

so $x^2 - 8x + 16 + y^2 = x^2 + 2x + 1$.

This simplifies to $x = \frac{1}{10} y^2 + \frac{3}{2}$, the equation of a parabola opening to the right (Fig. 11).



parabola: $x = \frac{3}{2} + \frac{1}{10} y^2$ (two scales)

Fig. 11

Practice 3: Find the set of points P = (x,y) whose distance from the focus F = (0,2) equals the distance from the directrix $y = -2$..

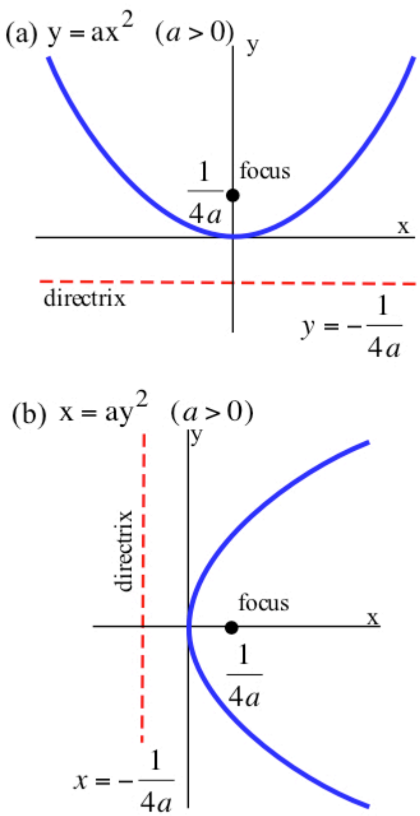


Fig. 12

Parabola

The standard parabola $y = ax^2$ opens around the y–axis (Fig. 12a) with vertex = (0,0), focus = $(0 , \frac{1}{4a})$, and directrix $y = -\frac{1}{4a}$.

The standard parabola $x = ay^2$ opens around the x–axis (Fig. 12b) with vertex = (0,0), focus = $(\frac{1}{4a} , 0)$, and directrix $x = -\frac{1}{4a}$.

Proof for the case $y = ax^2$:

The set of points p = (x,y) that are equally distant from the focus $F = (0 , \frac{1}{4a})$ and the directrix $y = -\frac{1}{4a}$ satisfy the distance equation

PF = PD so

$\sqrt{x^2 + (y - \frac{1}{4a})^2} = (y + \frac{1}{4a})$. Squaring each side, we have

$x^2 + (y - \frac{1}{4a})^2 = (y + \frac{1}{4a})^2$ and $x^2 + y^2 - \frac{2}{4a} y + \frac{1}{16a^2} = y^2 + \frac{2}{4a} y + \frac{1}{16a^2}$.

Then $x^2 = \frac{2}{4a} y + \frac{2}{4a} y = \frac{1}{a} y$ and, finally, $y = ax^2$.

Practice 4: Prove that the set of points $P = (x,y)$ that are equally distant from the focus $F = (\frac{1}{4a}, 0)$, and directrix $x = -\frac{1}{4a}$ satisfy the equation $x = ay^2$.

Hyperbola

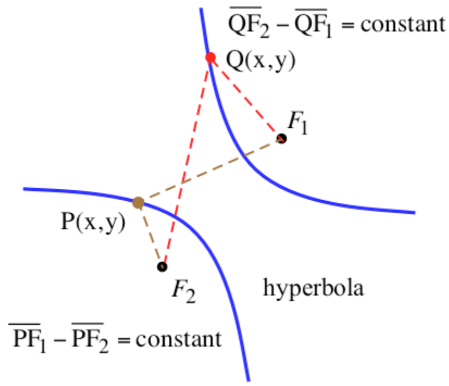


Fig. 13

Hyperbola: A hyperbola is the set of all points P for which the **difference** of the distances from P to two fixed points (foci) **is a constant**:

dist(P, one focus) – dist(P, other focus) = constant. (Fig. 13)

Example 3: Find the set of points for which the **difference** of the distances from the points to the foci $F_1 = (5,0)$ and $F_2 = (-5, 0)$ is always 8.

Solution: If the point $P = (x, y)$ is on the hyperbola, then the difference of the distances $PF_1 = \sqrt{(x-5)^2 + y^2}$ and $PF_2 = \sqrt{(x+5)^2 + y^2}$ is 8 so we have the equation

$$PF_1 - PF_2 = \sqrt{(x-5)^2 + y^2} - \sqrt{(x+5)^2 + y^2} = 8 \quad \text{(Fig. 14)}.$$

Moving the second radical to the right side of the equation, squaring both sides, and simplifying, we get

$$5x + 16 = -4\sqrt{(x+5)^2 + y^2}\ .$$

Squaring each side again and simplifying, we have $9x^2 - 16y^2 = 144$.

After dividing each side by 144, $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

If we start with the difference $PF_2 - PF_1 = 8$, we have the equation

$$\sqrt{(x+5)^2 + y^2} - \sqrt{(x-5)^2 + y^2} = 8\ .$$

Solving this equation, we again get $9x^2 - 16y^2 = 144$ and

$$\frac{x^2}{16} - \frac{y^2}{9} = 1\ .$$

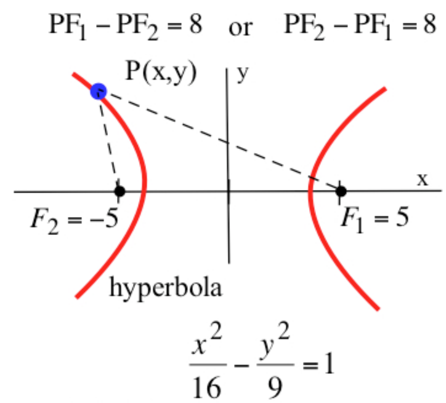


Fig. 14

Using the same algebraic steps as in the Example 3, it can be shown (see the Appendix at the end of the problems) that the set of points $P = (x,y)$ whose distances from the foci $F_1 = (c,0)$ and $F_2 = (-c, 0)$ differ by $2a$ $(a < c)$ is described by the formula

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{where } b^2 = c^2 - a^2 .$$

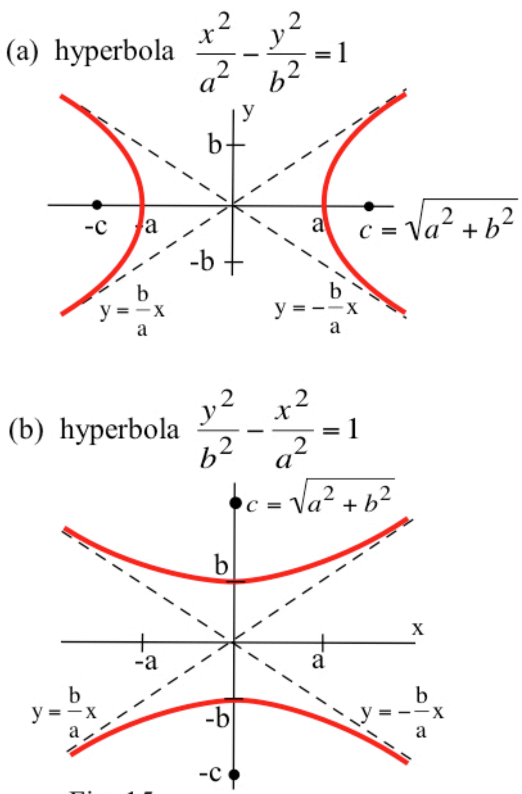


Fig. 15

Hyperbola

The standard hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

opens around the x–axis (Fig. 15a)

with vertices at $(\pm a, 0)$, foci at $(\pm\sqrt{a^2 + b^2}, 0)$,

and linear asymptotes $y = \pm \frac{b}{a} x$.

The standard hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

opens around the y–axis (Fig. 15b)

with vertices at $(0, \pm b)$, foci at $(0, \pm\sqrt{a^2 + b^2})$,

and linear asymptotes $y = \pm \frac{b}{a} x$.

Practice 5: Graph the hyperbolas $\frac{x^2}{25} - \frac{y^2}{16} = 1$ and $\frac{y^2}{25} - \frac{x^2}{16} = 1$ and find the linear asymptotes for each hyperbola.

Visually distinguishing the conic sections

If you only observe a small part of the graph of a conic section, it may be impossible to determine which conic section it is, and you may need to look at more of its graph. Near a vertex or in small pieces, all of the conic sections can be quite similar in appearance, but on a larger graph the ellipse is easy to distinguish from the other two. On a large graph, the hyperbola and parabola can be distinguished by noting that the hyperbola has two linear asymptotes and the parabola has no linear asymptotes.

The General Quadratic Equation and the Discriminant

Every equation that is quadratic in the variables x or y or both can be written in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \text{ where } A \text{ through } F \text{ are constants.}$$

The form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is called the **general quadratic equation**.

In particular, each of the conic sections can be written in the form of a general quadratic equation by clearing all fractions and collecting all of the terms on one side of the equation. What is perhaps surprising is that the graph of a general quadratic equation is always a conic section or a degenerate form of a conic section. Usually the graph of a general quadratic equation is not centered at the origin and is not symmetric about either axis, but the shape is always an ellipse, parabola, hyperbola, or degenerate form of one of these.

Even more surprising, a quick and easy calculation using just the coefficients A, B, and C of the general quadratic equation tells us the shape of its graph: ellipse, parabola, or hyperbola. The value obtained by this simple calculation is called the discriminant of the general quadratic equation.

Discriminant

The **discriminant** of the the general quadratic form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

is the value $\mathbf{B^2 - 4AC}$.

Example 4: Write each of the following in its general quadratic form and calculate its discriminant.

(a) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ (b) $3y + 7 = 2x^2 + 5x + 1$ (c) $5x^2 + 3 = 7y^2 - 2xy + 4y + 8$

Solution: (a) $9x^2 + 25y^2 - 225 = 0$ so $A = 9, C = 25, F = -225$, and $B = D = E = 0$. $B^2 - 4AC = -900$.

(b) $2x^2 + 5x - 3y - 6 = 0$ so $A = 2, D = 5, E = -3, F = -6$, and $B = C = 0$. $B^2 - 4AC = 0$.

(c) $5x^2 + 2xy - 7y^2 - 4y - 5 = 0$ so $A = 5, B = 2, C = -7, D = 0, E = -4$ and $F = -5$.
$B^2 - 4AC = 4 - 4(5)(-7) = 144$.

Practice 6: Write each of the following in its general quadratic form and calculate its discriminant.

(a) $1 = \frac{x^2}{36} - \frac{y^2}{9}$ (b) $x = 3y^2 - 5$ (c) $\frac{x^2}{16} + \frac{(y-2)^2}{25} = 1$

One very important property of the discriminant is that it is invariant under translations and rotations, its value does not change even if the graph is rigidly translated around the plane and rotated. When a graph is shifted or rotated or both, its general quadratic equation changes, but the discriminant of the new quadratic equation is the same value as the discriminant of the original quadratic equation. And we can determine the shape of the graph simply from the sign of the discriminant.

Quadratic Shape Theorem

The graph of the general quadratic equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is

an ellipse if	$B^2 - 4AC < 0$	(degenerate forms: one point or no points)
a parabola if	$B^2 - 4AC = 0$	(degenerate forms: two lines, one line, or no points)
a hyperbola if	$B^2 - 4AC > 0$	(degenerate form: pair of intersecting lines).

The proofs of this result and of the invariance of the discriminant under translations and rotations are "elementary" and just require a knowledge of algebra and trigonometry, but they are rather long and are very computational. A proof of the invariance of the discriminant under translations and rotations and of the Quadratic Shape Theorem is given in the Appendix after the problem set.

Example 5: Use the discriminant to determine the shapes of the graphs of the following equations.
(a) $x^2 + 3xy + 3y^2 = -7y - 4$ (b) $4x^2 + 4xy + y^2 = 3x - 1$ (c) $y^2 - 4x^2 = 0$.

Solution: (a) $B^2 - 4AC = 3^2 - 4(1)(3) = -3 < 0$. The graph is an ellipse.

(b) $B^2 - 4AC = 4^2 - 4(4)(1) = 0$. The graph is a parabola.

(c) $B^2 - 4AC = 0^2 - 4(-4)(1) = 16 > 0$. The graph is a hyperbola — actually a degenerate hyperbola. The graph of $0 = y^2 - 4x^2 = (y + 2x)(y - 2x)$ consists of the two lines $y = -2x$ and $y = 2x$.

Practice 7: Use the discriminant to determine the shapes of the graphs of the following equations.
(a) $x^2 + 2xy = 2y^2 + 4x + 3$ (b) $y^2 + 2x^2 = xy - 3y + 7$ (c) $2x^2 - 4xy = 3 + 5y - 2y^2$.

Sketching Standard Ellipses and Hyperbolas

The graphs of general ellipses and hyperbolas require plotting lots of points (a computer or calculator can help), but it is easy to sketch good graphs of the standard ellipses and hyperbolas. The steps for doing so are given below.

Graphing the Standard Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

1. Sketch short vertical line segments at the points $(\pm a, 0)$ on the x–axis and short horizontal line segments at the points $(0, \pm b)$ on the y–axis (Fig. 16a). Draw a rectangle whose sides are formed by extending the line segments.

2. Use the tangent line segments in step 1 as guide to sketching the ellipse (Fig. 16b). The graph of the ellipse is always inside the rectangle except at the 4 points that touch it.

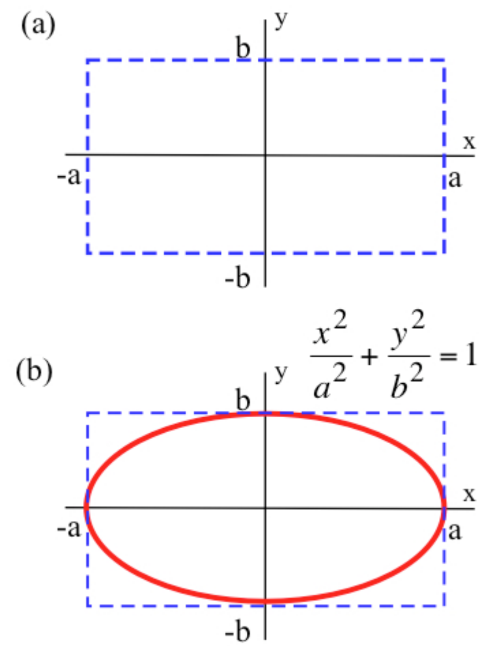


Fig. 16

Graphing the Standard Hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

1. Sketch the rectangle that intersects the x–axis at the points $(\pm a, 0)$ and the y–axis at the points $(0, \pm b)$. (Fig. 17a)
2. Draw the lines which go through the origin and the corners of the rectangle from step 1. (Fig. 17b) These lines are the asymptotes of the hyperbola.

3. For $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, plot the points $(\pm a, 0)$ on the hyperbola, and use the asymptotes from step 2 as a guide to sketching the rest of the hyperbola. (Fig. 17c)

3'. For $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, plot the points $(0, \pm b)$ on the hyperbola, and use the asymptotes from step 2 as a guide to sketching the rest of the hyperbola. (Fig. 17d)

The graph of the hyperbola is always outside the rectangle except at the 2 points which touch it.

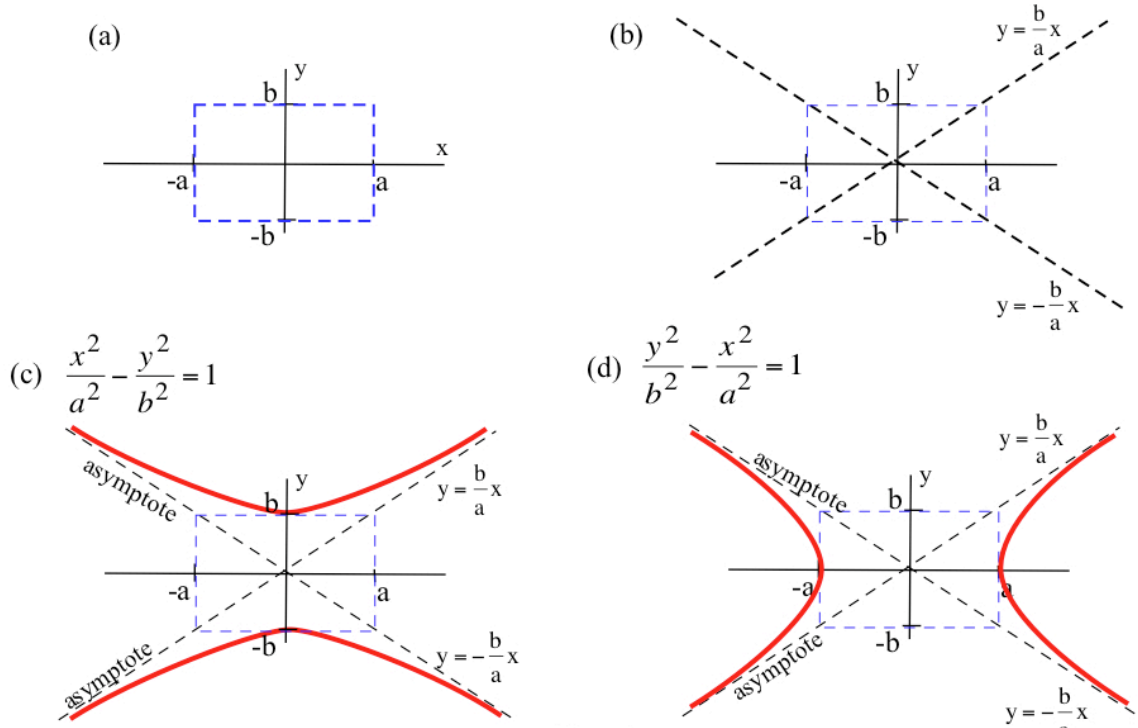


Fig. 17

Symmetry of the Conic Sections

Symmetry properties of the conic sections can simplify the task of graphing them. A parabola has one line of symmetry, so once we have graphed half of a parabola we can get the other half by folding along the line of symmetry. An ellipse and a hyperbola each have two lines of symmetry, so once we have graphed one fourth of an ellipse or hyperbola we can get the rest of the graph by folding along each line of symmetry.

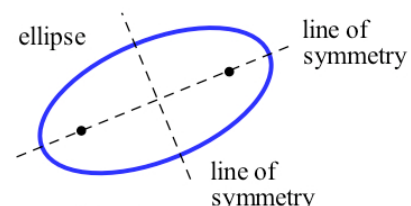


Fig. 18

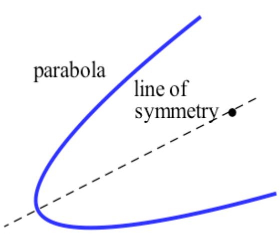


Fig. 19

- The parabola is symmetric about the line through the focus and the vertex (Fig. 18).
- The ellipse is symmetric about the line through the two foci. It is also symmetric about the perpendicular bisector of the line segment through the two foci (Fig. 19).
- The hyperbola is symmetric about the line through the two foci and about the perpendicular bisector of the line segment through the two foci (Fig. 20).

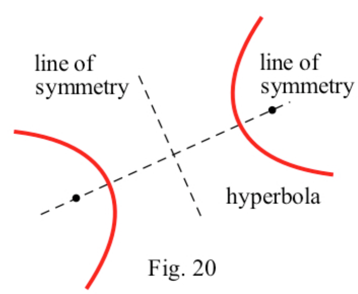


Fig. 20

The Conic Sections as "Shadows of Spheres"

There are a lot of different shapes at the beach on a sunny day, even conic sections. Suppose we have a sphere resting on a flat surface and a point radiating light.

- If the point of light is higher than the top of the sphere, then the shadow of the sphere is an ellipse (Fig. 21).
- If the point of light is exactly the same height as the top of the sphere, then the shadow of the sphere is a parabola (Fig. 22).
- If the point of light is lower than the top of the sphere, then the shadow of the sphere is one branch of a hyperbola (Fig. 23).

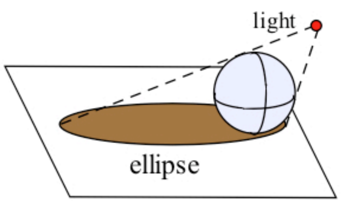


Fig. 21

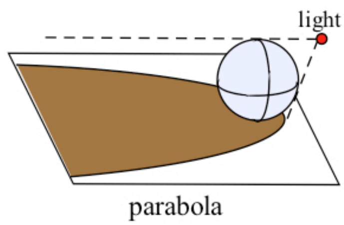


Fig. 22

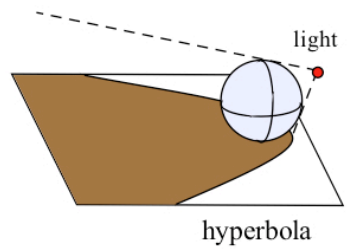


Fig. 23

PROBLEMS

1. What is the shape of the graph of the set of points whose distances from (6,0) and (–6,0) always add up to 20? Find an equation for the graph.

2. What is the shape of the graph of the set of points whose distances from (2,0) and (–2,0) always add up to 20? Find an equation for the graph.

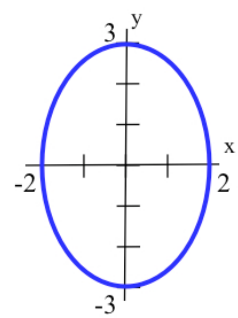


Fig. 24

3. What is the shape of the graph of the set of points whose distance from the point (0,5) is equal to the distance from the point to the line $y = -5$? Find an equation for the graph.

4. What is the shape of the graph of the set of points whose distance from the point (2,0) is equal to the distance from the point to the line $x = -4$? Find an equation for the graph.

5. Give the standard equation for the ellipse in Fig. 24.

6. Give the standard equation for the ellipse in Fig. 25.

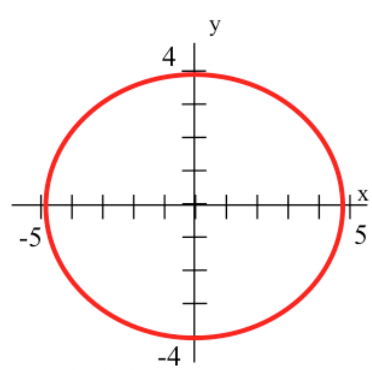


Fig. 25

7. What lines are linear asymptotes for the hyperbola $4x^2 - 9y^2 = 36$, and where are the foci?

8. What lines are linear asymptotes for the hyperbola $25x^2 - 4y^2 = 100$, and where are the foci?

9. What lines are linear asymptotes for the hyperbola $5y^2 - 3x^2 = 15$, and where are the foci?

10. What lines are linear asymptotes for the hyperbola $5y^2 - 3x^2 = 120$, and where are the foci?

In problems 11–16, rewrite each equation in the form of the general quadratic equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ and then calculate the value of the discriminant. What is the shape of each graph?

11. (a) $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$ (b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

12. (a) $\dfrac{x^2}{4} - \dfrac{y^2}{25} = 1$ (b) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

13. $x + 2y = 1 + \dfrac{3}{x - y}$

14. $y = \dfrac{5 + 2y - x^2}{4x + 5y}$

15. $x = \dfrac{7x - 3 - 2y^2}{2x + 4y}$

16. $x = \dfrac{2y^2 + 7x - 3}{2x + 5y}$

Problems 17–20 illustrate that a small change in the value of just **one** coefficient in the quadratic equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ can have a dramatic effect on the shape of the graph. Determine the shape of the graph for each formula.

17. (a) $2x^2 + \mathbf{3}xy + 2y^2$ + (*terms for x, y, and a constant*) = 0.
    (b) $2x^2 + \mathbf{4}xy + 2y^2$ + (*terms for x, y, and a constant*) = 0.
    (c) $2x^2 + \mathbf{5}xy + 2y^2$ + (*terms for x, y, and a constant*) = 0.
    (d) What are the shapes if the coefficients of the xy term are 3.99, 4, and 4.01?

18. (a) $\mathbf{1}x^2 + 4xy + 2y^2$ + (*terms for x, y, and a constant*) = 0.
    (b) $\mathbf{2}x^2 + 4xy + 2y^2$ + (*terms for x, y, and a constant*) = 0.
    (c) $\mathbf{3}x^2 + 4xy + 2y^2$ + (*terms for x, y, and a constant*) = 0.
    (d) What are the shapes if the coefficients of the x^2 term are 1.99, 2, and 2.01?

19. (a) $x^2 + 4xy + \mathbf{3}y^2$ + (*terms for x, y, and a constant*) = 0.
    (b) $x^2 + 4xy + \mathbf{4}y^2$ + (*terms for x, y, and a constant*) = 0.
    (c) $x^2 + 4xy + \mathbf{5}y^2$ + (*terms for x, y, and a constant*) = 0.
    (d) What are the shapes if the coefficients of the y^2 term are 3.99, 4, and 4.01?

20. Just changing a single sign can also dramatically change the shape of the graph.

(a) $x^2 + 2xy + y^2 + (\textit{terms for } x, y, \textit{and a constant}) = 0$.

(b) $x^2 + 2xy - y^2 + (\textit{terms for } x, y, \textit{and a constant}) = 0$.

21. Find the volume obtained when the region enclosed by the ellipse $\frac{x^2}{2^2} + \frac{y^2}{5^2} = 1$ is rotated

(a) about the x–axis, and (b) about the y–axis.

22. Find the volume obtained when the region enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is rotated

(a) about the x–axis, and (b) about the y–axis.

23. Find the volume obtained when the region enclosed by the hyperbola $\frac{x^2}{2^2} - \frac{y^2}{5^2} = 1$ and the vertical line $x = 10$ is rotated (a) about the x–axis, and (b) about the y–axis.

24. Find the volume obtained when the region enclosed by the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the vertical line $x = L$ (Fig. 26) is rotated (a) about the x–axis, and (b) about the y–axis. (Assume $a < L$.)

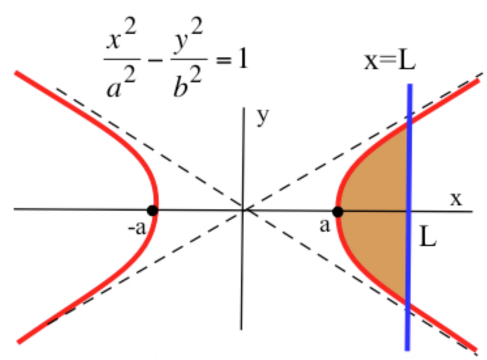


Fig. 26

25. Find the ratio of the area of the shaded parabolic region in Fig. 27 to the area of the rectangular region.

26. Find the ratio of the volumes obtained when the parabolic and rectangular regions in Fig. 27 are rotated about the y–axis.

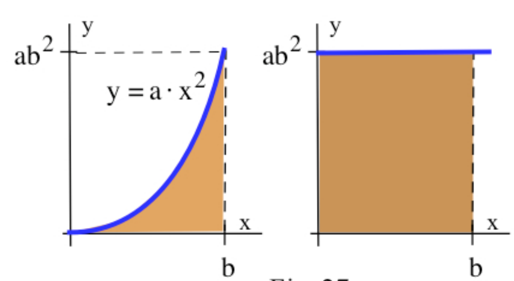


Fig. 27

String Constructions of Ellipses, Parabolas, and Hyperbolas (Optional)

All of the conic sections can be drawn with the help of some pins and string, and the directions and figures show how it can be done. For each conic section, you are asked to determine and describe why each construction produces the desired shape.

Ellipse: Pin the two ends of the string to a board so the string is not taut. Put the point of a pencil in the bend in the string (Fig. 28), and, keeping the string taut, draw a curve.

27. How is the distance between the vertices of the ellipse related to the length of the string?

28. Explain why this method produces an ellipse, a set of points whose distances from the two fixed points (foci) always sum to a constant. What is the constant?

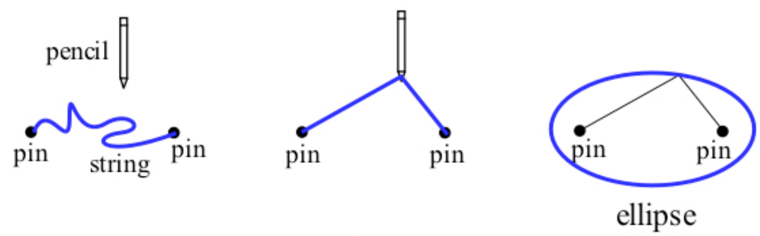


Fig. 28

29. What happens to the shape of the ellipse as the two foci are moved closer together (and the piece of string stays the same length)? Draw several ellipses using the same piece of string and different fixed points, and describe the results.

Parabola: Pin one end of the string to a board and the other end to the corner of a T–square bar that is the same length as the string. Put the point of a pencil in the bend in the string (Fig. 29) and keep the string taut. As the T–square is slid sideways, the pencil draws a curve.

30. Explain why this method produces a parabola, a set of points whose distance from a fixed point (one end of the string) is equal to the distance from a fixed line (the edge of the table).

31. What happens if the length of the string is slightly shorter than the length of the T–square bar? Draw several curves with several slightly shorter pieces of string and describe the results. What shapes are the curves?

32. Find a way to use pins, string and a pencil to sketch the graph of a hyperbola.

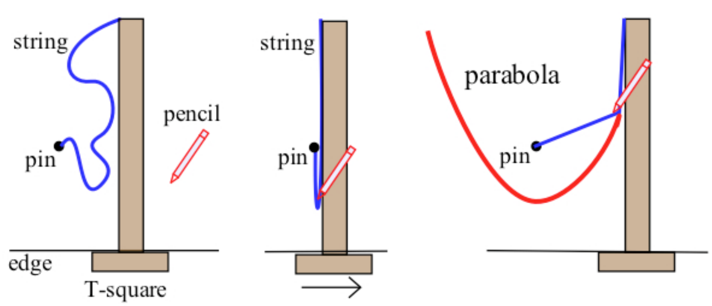


Fig. 29

Section 9.5 **PRACTICE Answers**

Practice 1: $F_1 = (3,0)$, $F_2 = (-3,0)$, and $P = (x,y)$. We want $\text{dist}(F_1, P) + \text{dist}(F_2, P) = 10$ so

$$\text{dist}((x,y),(3,0)) + \text{dist}((x,y),(-3,0)) = 10 \text{ and}$$
$$\sqrt{(x-3)^2 + y^2} + \sqrt{(x+3)^2 + y^2} = 10.$$

Moving the second radical to the right side and squaring, we get

$$(x-3)^2 + y^2 = 100 - 20\sqrt{(x+3)^2 + y^2} + (x+3)^2 + y^2 \text{ and}$$
$$x^2 - 6x + 9 + y^2 = 100 - 20\sqrt{(x+3)^2 + y^2} + x^2 + 6x + 9 + y^2 \text{ so}$$
$$-12x - 100 = -20\sqrt{(x+3)^2 + y^2}.$$

Dividing each side by –2 and then squaring, we have

$$36x^2 + 600x + 2500 = 100(x^2 + 6x + 9 + y^2) \text{ so}$$
$$1600 = 64x^2 + 100y^2 \text{ and}$$
$$1 = \frac{64x^2}{1600} + \frac{100y^2}{1600} = \frac{x^2}{25} + \frac{y^2}{16}.$$

Practice 2: $a = 13$ and $b = 5$ so the vertices of the ellipse are $(13, 0)$ and $(-13, 0)$. The value of c is $\sqrt{169 - 25} = 12$ so the foci are $(12, 0)$ and $(-12,0)$. The length of the semimajor axis is 13.

Practice 3: $\text{dist}(P, \text{focus}) = \text{dist}(P, \text{directrix})$ so $\text{dist}((x,y), (0,2)) = \text{dist}((x,y), \text{line } y=-2)$:

$$\sqrt{(x-0)^2 + (y-2)^2} = y + 2.$$

Squaring, we get $x^2 + y^2 - 4y + 4 = y^2 + 4y + 4$ so $x^2 = 8y$ or $y = \frac{1}{8}x^2$.

Practice 4: This is similar to Practice 3: $\text{dist}(P, \text{focus}) = \text{dist}(P, \text{directrix})$ so $\text{dist}((x,y), (\frac{1}{4a}, 0)) = \text{dist}((x,y), \text{line } x = -\frac{1}{4a})$. Then

$$\sqrt{\left(x - \frac{1}{4a}\right)^2 + (y - 0)^2} = x + \frac{1}{4a}.$$ Squaring each side we get

$$x^2 - 2x\frac{1}{4a} + \frac{1}{16a^2} + y^2 = x^2 + 2x\frac{1}{4a} + \frac{1}{16a^2} \text{ so } y^2 = \frac{1}{a}x \text{ and } x = ay^2.$$

Practice 5: The graphs are shown in Fig. 30. Both hyperbolas have the same linear asymptotes: $y = \frac{4}{5}x$ and $y = -\frac{4}{5}x$.

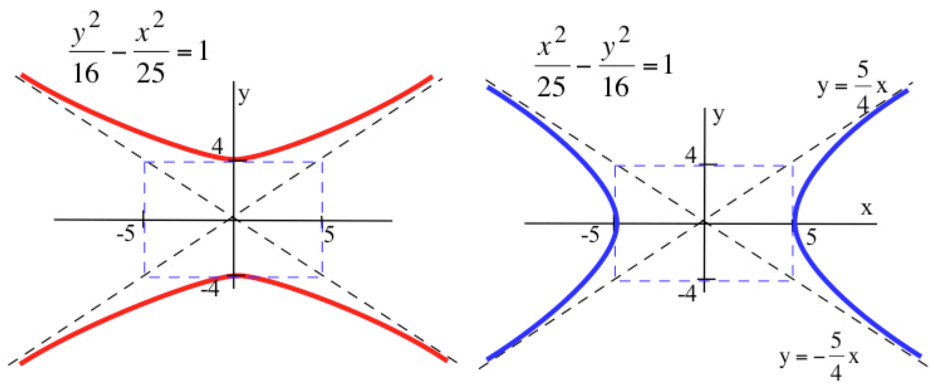


Fig. 30

Practice 6: (a) $324 = 9x^2 - 36y^2$ so $9x^2 - 36y^2 - 324 = 0$.

$A = 9$, $B = 0$, and $C = -36$ so $D = 0 - 4(9)(-36) = \mathbf{576}$.

(b) $0x^2 + 0xy + 3y^2 - x - 5 = 0$.

$A = 0$, $B = 0$, and $C = 3$ so $D = 0 - 4(0)(3) = \mathbf{0}$.

(c) $25x^2 + 16(y-2)^2 = 400$ so $25x^2 + 16y^2 - 64y + 48 - 400 = 0$.

$A = 25$, $B = 0$, and $C = 16$ so $D = 0 - 4(25)(16) = \mathbf{-1600}$.

Practice 7: (a) $x^2 + 2xy - 2y^2 - 4x - 3 = 0$.

$A = 1$, $B = 2$, $C = -2$ so $D = 4 - 4(1)(-2) = 12 > 0$: hyperbola.

(b) $2x^2 - 1xy + 1y^2 + 3y - 7 = 0$.

$A = 2$, $B = -1$, $C = 1$ so $D = 1 - 4(2)(1) = -7 < 0$: ellipse.

(c) $2x^2 - 4xy + 2y^2 - 5y - 3 = 0$.

$A = 2$, $B = -4$, and $C = 2$ so $D = 16 - 4(2)(2) = 0$: parabola.

Appendix for 9.5: Conic Sections

Deriving the Standard Forms from Distance Definitions of the Conic Sections

Ellipse

Ellipse An ellipse is the set of all points P so the **sum** of the distances of P from two fixed points (called foci) **is a constant**.

If F_1 and F_2 are the foci (Fig. 40), then for every point P on the ellipse, the distance from P to F_1 PLUS the distance from P to F_2 is a constant: $PF_1 + PF_2 = \text{constant}$. If the center of the ellipse is at the origin and the foci lie on the x–axis at $F_1 = (c, 0)$ and $F_2 = (-c, 0)$, we can translate the words into a formula:

$PF_1 + PF_2 = \text{constant}$ becomes $\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$.

(Calling the constant 2a simply makes some of the later algebra easier.)

By moving the second radical to the right side of the equation, squaring each side, and simplifying, we get

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$

so $x^2 - 2xc + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2xc + c^2 + y^2$

and $xc + a^2 = a\sqrt{(x+c)^2 + y^2}$.

Squaring each side again and simplifying, we get

$(xc + a^2)^2 = a^2 \{ (x+c)^2 + y^2 \}$ so $x^2c^2 + 2xca^2 + a^4 = a^2x^2 + 2xca^2 + a^2c^2 + a^2y^2$

and $a^2(a^2 - c^2) = x^2(a^2 - c^2) + y^2a^2$.

Finally, dividing each side by $a^2(a^2 - c^2)$, we get $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - c^2} = 1$.

By setting $b^2 = a^2 - c^2$, we have $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, the standard form of the ellipse.

Hyperbola

Hyperbola: A hyperbola is the set of all points P so the **difference** of the distances of P from the two fixed points (foci) **is a constant**.

If F_1 and F_2 are the foci (Fig. 9), then for every point P on the hyperbola, the distance from P to F_1 MINUS the distance from P to F_2 is a constant: $PF_1 - PF_2$ = constant (Fig. 42). If the center of the hyperbola is at the origin and the foci lie on the x–axis at $F_1 = (c, 0)$ and $F_2 = (-c, 0)$, we can translate the words into a formula:

$PF_1 - PF_2$ = constant becomes $\sqrt{(x - c)^2 + y^2} - \sqrt{(x + c)^2 + y^2} = 2a.$

(Calling the constant 2a simply makes some of the later algebra easier.)

The algebra which follows is very similar to that used for the ellipse.

Moving the second radical to the right side of the equation, squaring each side, and simplifying, we get

$$\sqrt{(x - c)^2 + y^2} = 2a + \sqrt{(x + c)^2 + y^2}$$

$$(x - c)^2 + y^2 = 4a^2 + 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2 + y^2$$

so $x^2 - 2xc + c^2 + y^2 = 4a^2 + 4a\sqrt{(x + c)^2 + y^2} + x^2 + 2xc + c^2 + y^2$

and $xc + a^2 = -a\sqrt{(x + c)^2 + y^2}$.

Squaring each side again and simplifying, we get

$(xc + a^2)^2 = a^2 \{ (x + c)^2 + y^2 \}$ so $x^2c^2 + 2xca^2 + a^2 = a^2x + 2xca^2 + a^2c^2 + a^2y^2$

and $x^2(c^2 - a^2) - y^2a^2 = a^2(c^2 - a^2)$.

Finally, dividing each side by $a^2(c^2 - a^2)$, we get $\dfrac{x^2}{a^2} + \dfrac{y^2}{c^2 - a^2} = 1$.

By setting $b^2 = c^2 - a^2$, we have $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, the standard form of the hyperbola.

Invariance Properties of the Discriminant

The **discriminant** of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is $d = \mathbf{B^2 - 4AC}$.

The Discriminant $B^2 - 4AC$ is invariant under translations (shifts):

If a point (x, y) is shifted h units up and k units to the right, then the coordinates of the new point are $(x', y') = (x+h, y+k)$. To show that the discriminant of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is invariant under translations, we need to show that the discriminant of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ and the discriminant of $A(x')^2 + B(x')(y') + C(y')^2 + Dx' + Ey' + F = 0$ are equal for $x' = x + h$ and $y' = y + k$.

The discriminant of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is equal to $B^2 - 4AC$.
Replacing x' with $x+h$ and y' with $y+k$,

$$\begin{aligned}
&A(x')^2 + B(x')(y') + C(y')^2 + Dx' + Ey' + F \\
&= A(x+h)^2 + B(x+h)(y+k) + C(y+k)^2 + D(x+h) + E(y+k) + F \\
&= A(x^2 + 2xh + h^2) + B(xy + xk + yh + hk) + C(y^2 + 2yk + k^2) + D(x+h) + E(y+k) + F \\
&= Ax^2 + Bxy + Cy^2 + (2Ah + Bk + D)x + (Bh + 2Ck + E)y + (Ah^2 + Bhk + Ck^2 + Dh + Ek + F).
\end{aligned}$$

The discriminant of this final formula is $B^2 - 4AC$, the same as the discriminant of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F$. In fact, a translation does not change the values of the coefficients of x^2, xy, and y^2 (the values of A, B, and C) so the discriminant is unchanged.

The Discriminant $B^2 - 4AC$ is invariant under rotation by an angle θ:

If a point (x, y) is rotated about the origin by an angle of θ, then the coordinates of the new point are $(x', y') = (x\cdot\cos(\theta) - y\cdot\sin(\theta), x\cdot\sin(\theta) + y\cdot\cos(\theta))$. To show that the discriminant of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is invariant under rotations, we need to show that the discriminant of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ and the discriminant of $A(x')^2 + B(x')(y') + C(y')^2 + Dx' + Ey' + F = 0$ are equal when $x' = x\cdot\cos(\theta) - y\cdot\sin(\theta)$ and $y' = x\cdot\sin(\theta) + y\cdot\cos(\theta)$.

The discriminant of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is equal to $B^2 - 4AC$.
Replacing x' with $x\cdot\cos(\theta) - y\cdot\sin(\theta) = x\cdot c - y\cdot s$ and y' with $x\cdot\sin(\theta) + y\cdot\cos(\theta) = x\cdot s + y\cdot c$

$$\begin{aligned}
&A(x')^2 + B(x')(y') + C(y')^2 + Dx' + Ey' + F = 0 \\
&= A(xc - ys)^2 + B(xc - ys)(xs + yc) + C(xs + yc)^2 + \ldots \text{(terms without } x^2\text{, } xy\text{, and } y^2\text{)} \\
&= A(x^2c^2 - 2xysc + y^2s^2) + B(x^2sc - xys^2 + xyc^2 - y^2sc) + C(x^2s^2 + 2xysc + y^2c^2) + \ldots \\
&= (Ac^2 + Bsc + Cs^2)x^2 + (-2Asc - Bs^2 + Bc^2 + 2Csc)xy + (As^2 + Bsc + Cc^2)y^2 + \ldots
\end{aligned}$$

Then $A' = Ac^2 + Bsc + Cs^2$, $B' = -2Asc - Bs^2 + Bc^2 + 2Csc$, and $C' = As^2 + Bsc + Cc^2$, so the new discriminant is

$(B')^2 - 4(A')(C')$
$= (-2Asc - Bs^2 + Bc^2 + 2Csc)^2 - 4(Ac^2 + Bsc + Cs^2)(As^2 + Bsc + Cc^2)$
$= \{ s^4(B^2) + s^3c(4AB - 4BC) + s^2c^2(4A^2 - 8AC - 2B^2 + 4C^2) + sc^3(-4AB + 4BC) + c^4(B^2) \}$
$\quad - 4\{ s^4(AC) + s^3c(AB - BC) + s^2c^2(A^2 - B^2 + C^2) + sc^3(-AB + BC) + c^4(AC) \}$
$= s^4(B^2 - 4AC) + s^2c^2(2B^2 - 8AC) + c^4(B^2 - 4AC)$
$= (B^2 - 4AC)(s^4 + 2s^2c^2 + c^4) = (B^2 - 4AC)(s^2 + c^2)(s^2 + c^2) = (B^2 - 4AC)$, the original discriminant.

The invariance of the discriminant under translation and rotation shows that any conic section can be translated so its "center" is at the origin and rotated so its axis is the x–axis without changing the value of the discriminant: the value of the discriminant depends strictly on the shape of the curve, not on its location or orientation. When the axis of the conic section is the x–axis, the standard quadratic equation
$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
does not have an xy term (B=0) so we only need to investigate the reduced form
$Ax^2 + Cy^2 + Dx + Ey + F = 0$.

1) $A = C = 0$ (discriminant $d=0$). A straight line. (special case: no graph)

2) $A = C \neq 0$ ($d<0$). A circle. (special cases: a point or no graph)

3) $A = 0, C \neq 0$ or $A \neq 0, C = 0$ ($d=0$): A parabola. (special cases: 2 lines, 1 line, or no graph)

4) A and C both positive or both negative ($d<0$): An Ellipse. (special cases: a point or no graph)

5) A and C have opposite signs ($d>0$): A Hyperbola. (special case: a pair of intersecting lines)