Chapter 4: THE INTEGRAL

Previous chapters dealt with **Differential Calculus**. They started with the "simple" geometrical idea of the **slope of a tangent line** to a curve, developed it into a combination of theory about derivatives and their properties, techniques for calculating derivatives, and applications of derivatives. This chapter begins the development of **Integral Calculus** and starts with the "simple" geometric idea of **area**. This idea will be developed into another combination of theory, techniques, and applications.

One of the most important results in mathematics, The Fundamental Theorem of Calculus, appears in this chapter. It unifies the differential and integral calculus into a single grand structure. Historically, this unification marked the beginning of modern mathematics, and it provided important tools for the growth and development of the sciences.

The chapter begins with a look at area, some geometric properties of areas, and some applications. First we will see ways of approximating the areas of regions such as tree leaves that are bounded by curved edges and the areas of regions bounded by graphs of functions. Then we will find ways to calculate the areas of some of these regions exactly. Finally, we will explore more of the rich variety of uses of "areas".

The primary purpose of this introductory section is to help develop your intuition about areas and your ability to reason using geometric arguments about area. This type of reasoning will appear often in the rest of this book and is very helpful for applying the ideas of calculus.

AREA

The basic shape we will use is the rectangle; the area of a rectangle is (base)•(height). If the units for each side of the rectangle are "meters," then the area will have the units ("meters")•("meters") = "square meters" = m^2. The only other area formulas needed for this section are for triangles, area = $b \cdot h/2$, and for circles, area = $\pi \cdot r^2$. Three other familiar properties of area are assumed and will be used:

Addition Property: The total area of a region is the sum of the areas of the non–overlapping pieces which comprise the region. (Fig. 1)

Inclusion Property: If region B is on or inside region A, then the area of region B is less than or equal to the area of region A. (Fig. 2)

Location–Independence Property: The area of a region does not depend on its location. (Fig. 3)

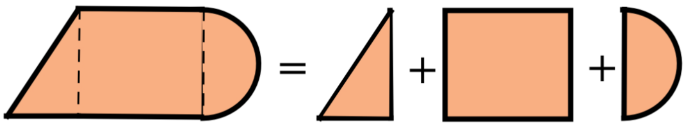


Fig. 1

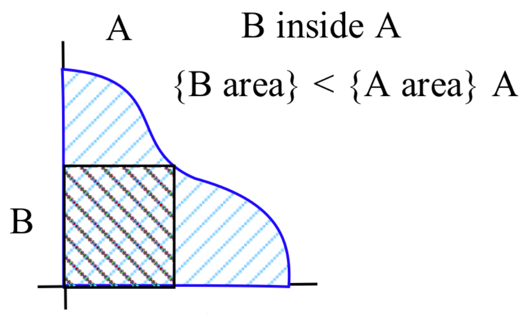


Fig. 2

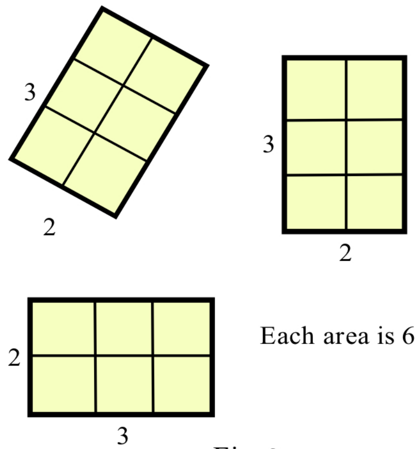


Fig. 3

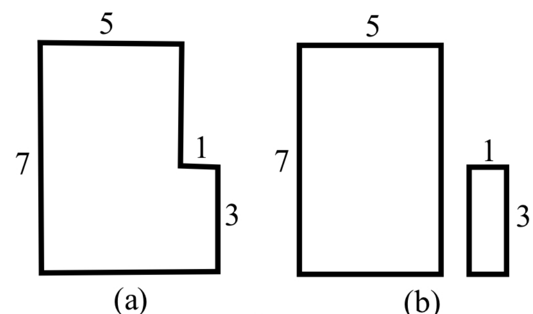


Fig. 4

Example 1: Determine the area of the region in Fig. 4a

Solution: The region can easily be broken into two rectangles, Fig. 4b, with areas 35 square inches and 3 square inches respectively, so the area of the original region is 38 square inches.

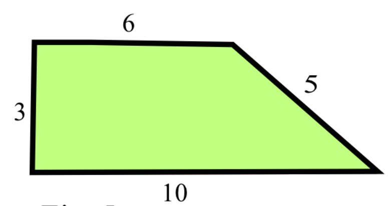


Fig. 5

Practice 1: Determine the area of the region in Fig. 5 by cutting it in two ways: (a) into a rectangle and triangle and (b) into two triangles.

We can use the three properties of area to get information about areas that are difficult to calculate exactly. Let A be the region bounded by the graph of $f(x) = 1/x$, the x–axis, and vertical lines at $x = 1$ and $x = 3$. Since the two rectangles in Fig. 6 are inside the region A and do not overlap each, the area of the rectangles, $1/2 + 1/3 = 5/6$, is less than the area of region A.

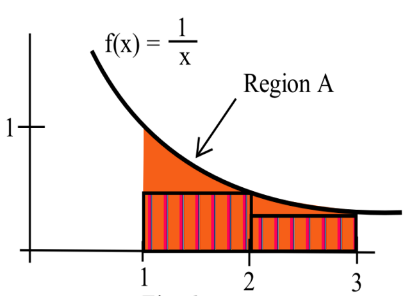


Fig. 6

Practice 2: Build two rectangles, each with base 1 unit, outside the shaded region in Fig. 6 and use their areas to make a VALID statement about the area of region A.

Practice 3: What can be said about the area of region A in Fig. 6 if we use both inside and outside rectangles with base 1/2 unit?

Example 2: In Fig. 7, there are 32 dark squares, 1 centimeter on a side, and 31 lighter squares of the same size. We can be sure that the area of the leaf is smaller than what number?

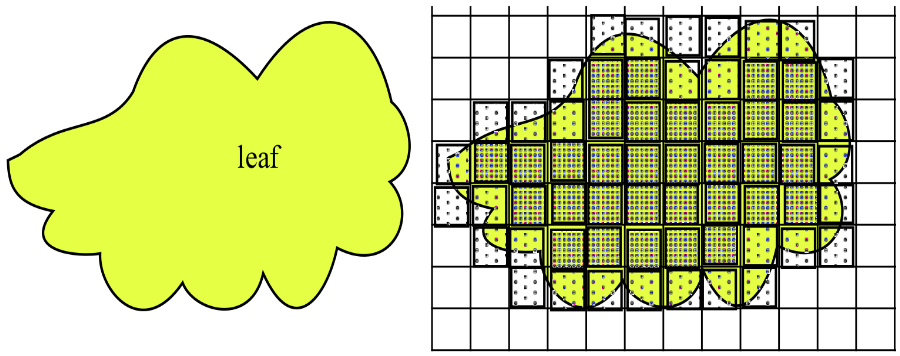


Fig. 7

Solution: The area of the leaf is smaller than $32 + 31 = 63\ \text{cm}^2$.

Practice 4: We can be sure that the area of the leaf is at least how large?

Functions can be defined in terms of areas. For the constant function $f(t) = 2$, define $A(x)$ to be the area of the rectangular region bounded by the graph of f, the t–axis, and the vertical lines at $t=1$ and $t=x$ (Fig. 8a). A(2) is the area of the shaded region in Fig. 8b, and A(2) =2. Similarly, A(3)= 4 and A(4)=6. In general, $A(x)= \text{(base)(height)} = (x-1)(2) = 2x - 2$ for any $x \geq 1$. The graph of $y=A(x)$ is shown in Fig. 8c, and $A'(x) = 2$ for every value of $x > 1$.

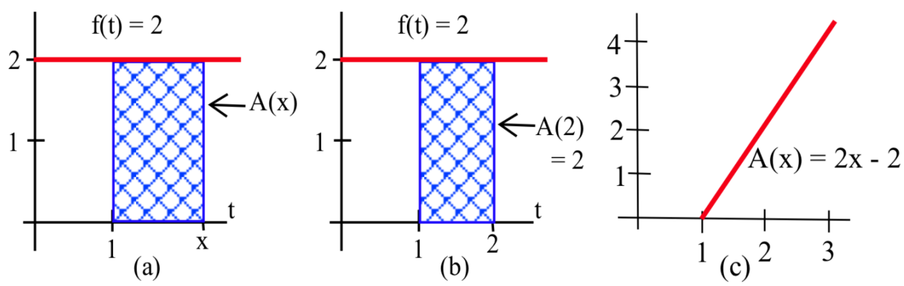
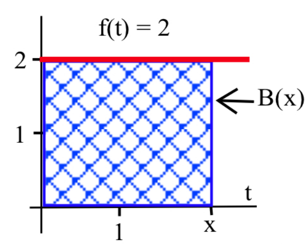


Fig. 8

Practice 5: For $f(t)=2$, define $B(x)$ to be the area of the region bounded by the graph of f, the t–axis, and vertical lines at $t = 0$ and $t = x$. Fill in the table in Fig. 9 with the values of B. How are the graphs of $y = A(x)$ and $y = B(x)$ related?



x	B(x)
0	
0.5	
1	
2	

Fig. 9

Sometimes it is useful to move regions around. The area of a parallelogram is obvious if we move the triangular region from one side of the parallelogram to fill the region on the other side and ending up with a rectangle (Fig. 10). At first glance, it is difficult to estimate the total area of the shaded regions in Fig. 11a . However, if we slide all of them into a single column (Fig. 11b), then it is easy to determine that the shaded area is less than the area of the enclosing rectangle = (base)(height) = (1)(2) = 2.

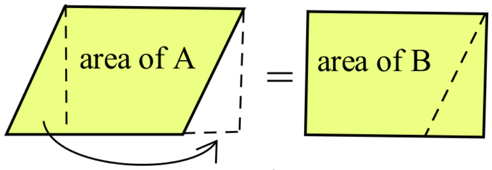


Fig. 10

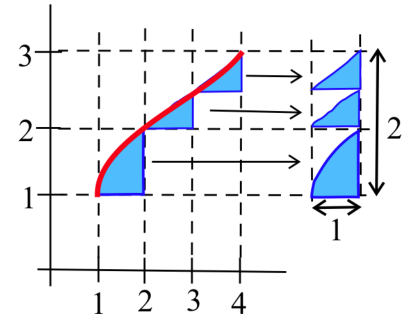


Fig. 11

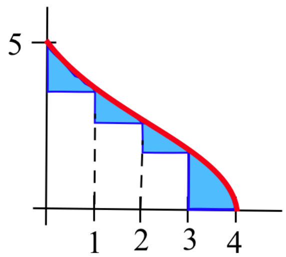


Fig. 12

Practice 6: The total area of the shaded regions in Fig. 12 is less than what number?

SOME APPLICATIONS OF "AREA"

One reason "areas" are so useful is that they can represent quantities other than simple geometric shapes. For example, if the units of the base of a rectangle are "hours" and the units of the height are "miles/hour", then the units of the "area" of the rectangle are ("hours")•("miles/hour") = "miles," a measure of distance (Fig. 13a). Similarly, if the base units are "centimeters" and the height units are "grams" (Fig. 13b), then the "area" units are "gram•centimeters," a measure of work.

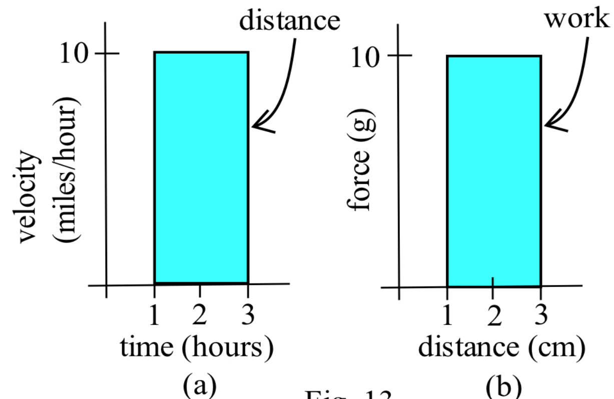


Fig. 13

Distance as an "Area"

In Fig. 14, f(t) is the velocity of a car in "miles per hour", and t is the time in "hours." Then the shaded "area" will be (base)•(height) = (3 hours)•(20 miles/hour) = 60 miles, the distance traveled by the car in the 3 hours from 1 o'clock until 4 o'clock.

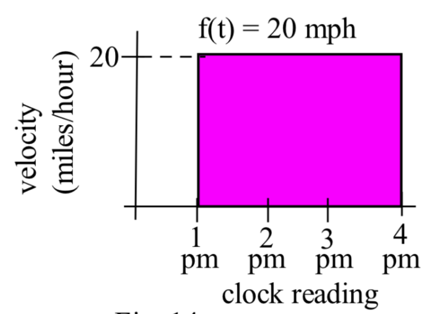


Fig. 14

Distance as an "Area"

If f(t) is the (positive) forward velocity of an object at time t, then the "area" between the graph of f and the t–axis and the vertical lines at times t = a and t = b will be the distance that the object has moved forward between times a and b.

This "area as distance" can make some difficult distance problems much easier.

Example 3: A car starts from rest (velocity = 0) and steadily speeds up so that 20 seconds later it's speed is 88 feet per second (60 miles per hour). How far did the car travel during those 20 seconds?

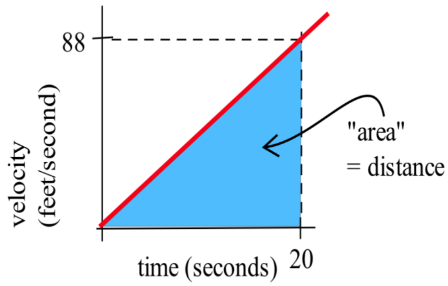


Fig. 15

Solution: We can answer the question using the techniques of chapter 3 (try it). But if "steadily" means that the velocity increases linearly, then it is easier to use Fig. 15 and the idea of "area as distance."

The "area" of the triangular region represents the distance traveled, so

$$\text{distance} = \frac{1}{2}(\text{base}) \cdot (\text{height}) = \frac{1}{2}(20 \text{ seconds}) \cdot (88 \text{ feet/second})$$

$$= 880 \text{ feet.}$$

Practice 7: A train traveling at 45 miles per hour (66 feet/second) takes 60 seconds to come to a complete stop. If the train slowed down at a steady rate (the velocity decreased linearly), how many feet did the train travel while coming to a stop?

Practice 8: You and a friend start off at noon and walk in the same direction along the same path at the rates shown in Fig. 16.

(a) Who is walking faster at 2 pm? Who is ahead at 2 pm?

(b) Who is walking faster at 3 pm? Who is ahead at 3 pm?

(c) When will you and your friend be together? (Answer in words.)

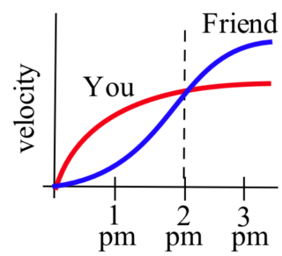


Fig. 16

Total Accumulation as "Area"

In the previous examples, the function represented a **rate** of travel (miles per hour), and the area represented the **total** distance traveled. For functions representing other **rates** such as the production of a factory (bicycles per day), or the flow of water in a river (gallons per minute) or traffic over a bridge (cars per minute), or the spread of a disease (newly sick people per week), the area will still represent the **total** amount of something.

> **"Area" as a Total Accumulation**
>
> If f(t) represents a positive **rate** (in units per time interval) at time t, then the "area" between the graph of f and the t–axis and the vertical lines at times t=a and t=b will be the total units which accumulate between times a and b. (Fig. 17)

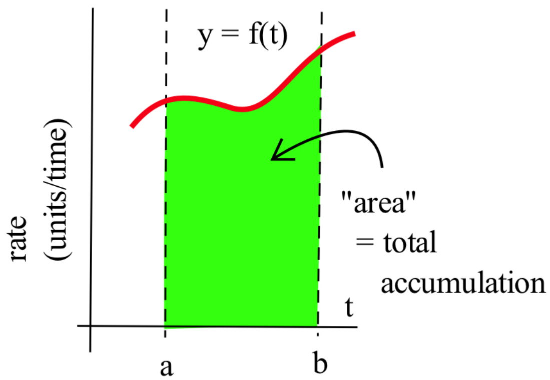


Fig. 17

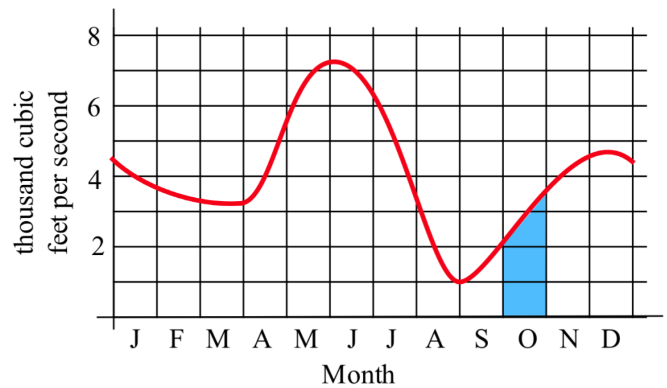


Fig. 18

For example, Fig. 18 shows the flow rate (cubic feet per second) of water in the Skykomish river at the town of Goldbar in Washington state. The area of the shaded region represents the total volume (cubic feet) of water flowing past the town during the month of October:

Total water = "area"

= area of rectangle + area of the triangle

$\approx$ (2000 cubic feet/sec)(30 days) + $\frac{1}{2}$(1500 cf/s)(30 days) = (2750 cubic feet/sec)(30 days)

= (2750 cubic feet/sec)(2,592,000 sec) $\approx 7.128 \times 10^9$ cubic feet

(For comparison, the flow over Niagara Falls is about 2.12×10^5 cf/s.)

Practice 9: Fig. 19 shows the number of telephone calls made per hour on a Tuesday. Approximately how many calls were made between 9 pm and 11 pm?

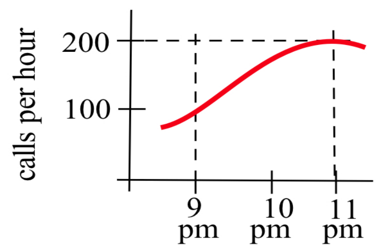


Fig. 19

PROBLEMS

1. Calculate the areas of the shaded regions in Fig. 20.

2. Calculate the area of the trapezoidal region in Fig. 21 by breaking it into a triangle and a rectangle.

3. Break Fig. 22 into a triangle and rectangle and verify that the total area of the trapezoid is $b \cdot (\frac{h+H}{2})$.

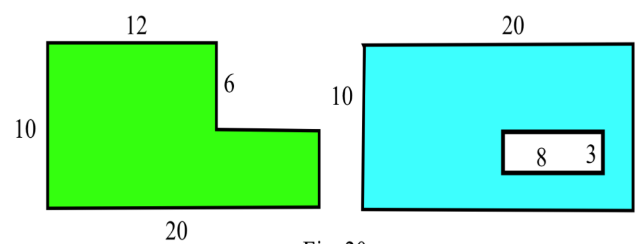


Fig. 20

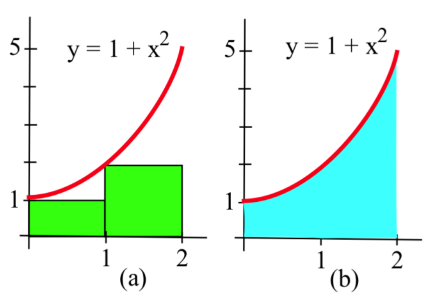


Fig. 23

4. (a) Calculate the sum of the rectangular areas in Fig. 23a.

   (b) What can we say about the area of the shaded region in Fig. 23b?

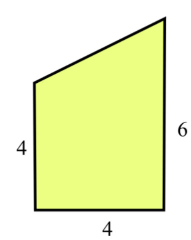


Fig. 21

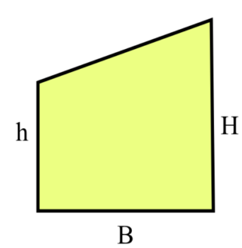


Fig. 22

5. (a) Calculate the sum of the rectangular areas in Fig. 24a.
   (b) From part (a), what can we say about the area of the shaded region in Fig. 24b?

6. (a) Calculate the sum of the areas of the shaded regions in Fig. 24c.
   (b) From part (a), what can we say about the area of the shaded region in Fig. 24b?

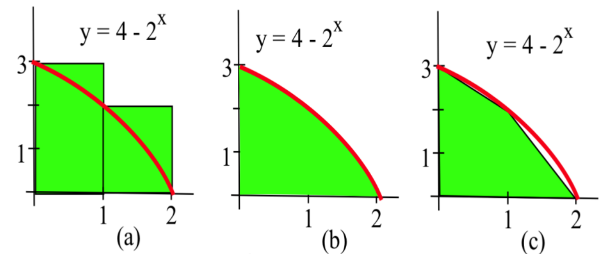


Fig. 24

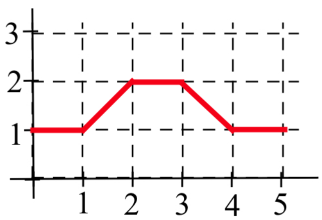


Fig. 25

7. Let A(x) represent the area bounded by the graph and the horizontal axis and vertical lines at t=0 and t=x for the graph in Fig. 25. Evaluate A(x) for x = 1, 2, 3, 4, and 5.

8. Let B(x) represent the area bounded by the graph and the horizontal axis and vertical lines at t=0 and t=x for the graph in Fig. 26. Evaluate B(x) for x = 1, 2, 3, 4, and 5.

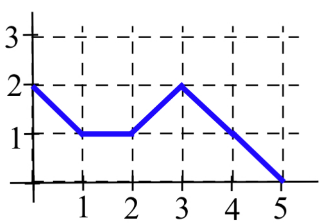


Fig. 26

9. Let C(x) represent the area bounded by the graph and the horizontal axis and vertical lines at t=0 and t=x for the graph in Fig. 27. Evaluate C(x) for x = 1, 2, and 3 and find a formula for C(x).

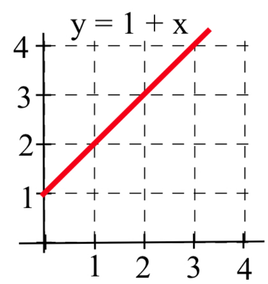


Fig. 27

10. Let A(x) represent the area bounded by the graph and the horizontal axis and vertical lines at t=0 and t=x for the graph in Fig. 28. Evaluate A(x) for x = 1, 2, and 3 and find a formula for A(x).

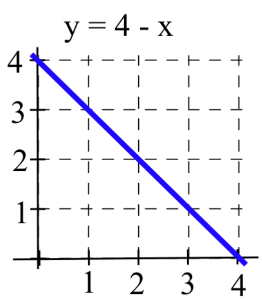


Fig. 28

11. A car had the velocity given in Fig. 29. How far did the car travel from t=0 to t=30 seconds?

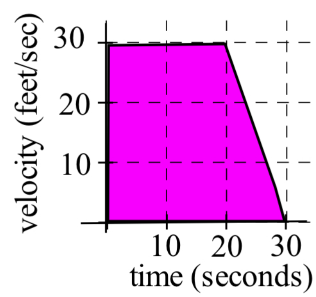


Fig. 29

12. A car had the velocity given in Fig. 30. How far did the car travel from t=0 to t=30 seconds?

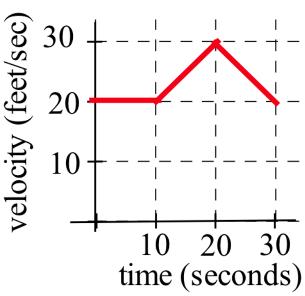


Fig. 30

13. The velocities of two cars are shown in Fig. 31.
    (a) From the time the brakes were applied, how many seconds did it take each car to stop? (b) From the time the brakes were applied, which car traveled farther until it came to a complete stop?

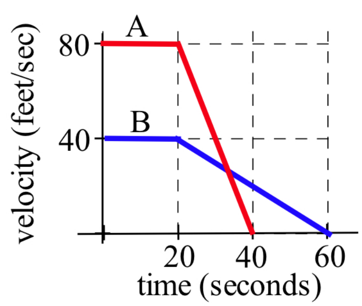


Fig. 31

14. Police chase: A speeder traveling 45 miles per hour (in a 25 mph zone) passes a stopped police car which immediately takes off after the speeder. If the police car speeds up steadily to 60 miles/hour in 10 seconds and then travels at a steady 60 miles/hour, **how long** and **how far** before the police car catches the speeder who continued traveling at 45 miles/hour? (Fig. 32)

15. What are the units for the "area" of a rectangle with the given base and height units?

Base units	Height units	"Area" units
miles per second	seconds	
hours	dollars per hour	
square feet	feet	
kilowatts	hours	
houses	people per house	
meals	meals	

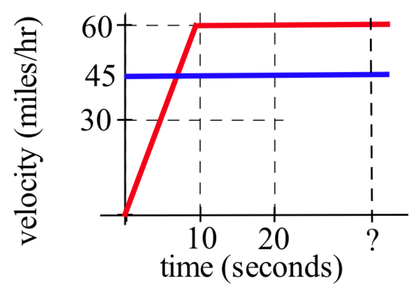


Fig. 32

Section 4.0 **PRACTICE Answers**

Practice 1: area $= 3(6) + \frac{1}{2}(4)(3) = 24$ and area $= \frac{1}{2}(3)(10) + \frac{1}{2}(6)(3) = 24$.

Practice 2: outside rectangular area $= (1)(1) + (1)(\frac{1}{2}) = 1.5$

Practice 3: Using rectangles with base = 1/2. Inside: area $= \frac{1}{2}\left(\frac{2}{3} + \frac{1}{2} + \frac{2}{5} + \frac{1}{3}\right) = \frac{57}{60} \approx 0.95$

Outside: area $= \frac{1}{2}\left(1 + \frac{2}{3} + \frac{1}{2} + \frac{2}{5}\right) = \frac{72}{60} \approx 1.2$

The area of the region is between 0.95 and 1.2 .

Practice 4: The area of the leaf is larger than the area of the dark rectangles, 32 cm^2.

Pr 5:

x	B(x)
0	0
1/2	1
1	2
2	4

$y = B(x) = 2x$ is a line with slope 2 so it is parallel to $y = A(x) = 2x - 2$.

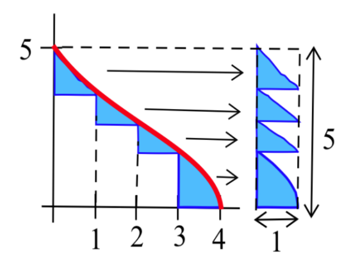


Fig. 33

Practice 6: See Fig. 33. Area < area of the rectangle enclosing the shifted regions = 5

Practice 7: See Fig. 34. Distance = area of shaded region $= \frac{1}{2}$(base)(height)

$= \frac{1}{2}$(60 seconds)(66 feet/second) = 1980 feet.

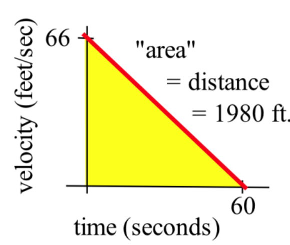


Fig. 34

Practice 8: (a) At 2 pm both are walking at the same velocity. You are ahead.

(b) At 3 pm your friend is walking faster than you, but you are still ahead. (The "area" under your velocity curve is larger than the "area" under your friend's.)

(c) You and your friend will be together on the trail when the "areas" (distances) under the two velocity graphs are equal.

Practice 9: Total calls = "area" under rate curve from 9 am to 11 am ≈ 300 calls.