4.2 THE DEFINITE INTEGRAL

Definition of The Definite Integral

Each particular Riemann sum depends on several things: the function f, the interval [a,b], the partition P of the interval, and the values chosen for c_k in each subinterval. Fortunately, for most of the functions needed for applications, as the approximating rectangles get thinner (as the mesh of P approaches 0 and the number of subintervals gets bigger) the values of the Riemann sums approach the same value independently of the particular partition P and the points c_k. For these functions, the LIMIT (as the mesh approaches 0) of the Riemann sums is the same number no matter how the c_k are chosen.

This limit of the Riemann sums is the next big topic in calculus, the definite integral. Integrals arise throughout the rest of this book and in applications in almost every field that uses mathematics.

> Definition: **The Definite Integral**
>
> If $\lim_{mesh \to 0} \left(\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \right)$ equals a finite number I
>
> then f is **integrable** on the interval [a, b] .
>
> The number I is called the **Definite Integral of f on [a,b]** and is written $\int_a^b f(x)\, dx$.

The symbol $\int_a^b f(x)\, dx$ is read as "the integral from a to b of eff of x dee x" or "the integral from a to b of f(x) with respect to x." The name of each piece of the symbol is shown in Fig. 1.

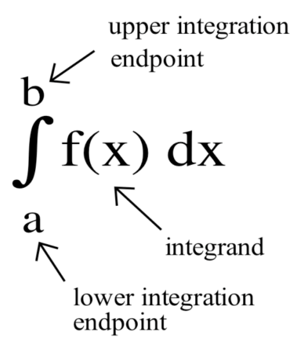

Fig. 1

Example 1: Describe the area between the graph of f(x) = 1/x, the x–axis, and the vertical lines at $x = 1$ and $x = 5$ as a limit of Riemann sums and as a definite integral.

Solution: Area = $\lim_{mesh \to 0} \left(\sum_{k=1}^{n} \frac{1}{c_k} \Delta x_k \right) = \int_1^5 \frac{1}{x}\, dx \approx 1.609$

(from Table 2 in Section 4.1).

Practice 1: Describe the area between the graph of f(x)= sin(x), the x–axis, and the vertical lines at $x = 0$ and $x = \pi$ as a limit of Riemann sums and as a definite integral.

Example 2: Using the idea of area, determine the values of

(a) $\lim\limits_{mesh \to 0} \left(\sum\limits_{k=1}^{n} (1 + c_k) \Delta x_k \right)$ on the interval [1,3] (b) $\int_0^4 (5 - x)\, dx$ (c) $\int_{-1}^{1} \sqrt{1 - x^2}\, dx$

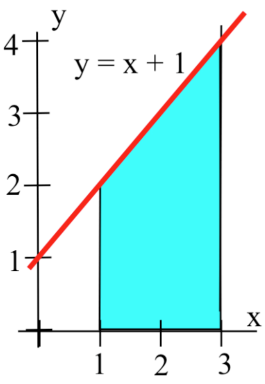

Fig. 2

Solution: (a) represents the area between the graph of $f(x) = 1+x$, the x–axis, and the vertical lines at 1 and 3 (Fig. 2), and this area equals 6 square units.

(b) represents the area between $f(x) = 5 - x$, the x–axis and the vertical lines at 0 and 4, so the integral equals 12 square units.

(c) represents the area of 1/2 of the circle $x^2 + y^2 = 1$ with radius 1 and center at (0,0), and the integral equals (circle area)/2 = $(\pi r^2)/2 = \pi/2$.

Practice 2: Using the area idea, determine the values of

(a) $\lim\limits_{mesh \to 0} \left(\sum\limits_{k=1}^{n} (2c_k) \Delta x_k \right)$ on the interval [1,3] and (b) $\int_3^8 4\, dx$.

Example 3: Represent the limit of each Riemann sum as a definite integral.

(a) $\lim\limits_{mesh \to 0} \left(\sum\limits_{k=1}^{n} (3 + c_k) \Delta x_k \right)$ on [1,4] (b) $\lim\limits_{mesh \to 0} \left(\sum\limits_{k=1}^{n} \sqrt{c_k} \cdot \Delta x_k \right)$ on [0,9]

Solution: (a) $\int_1^4 (3 + x)\, dx$ (b) $\int_0^9 \sqrt{x}\, dx$

Example 4: Represent each shaded area in Fig. 3 as a definite integral. (Do not evaluate the definite integral, just translate the picture into symbols.)

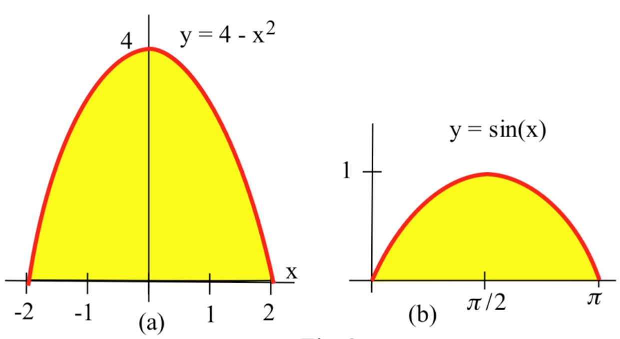

Fig. 3

Solution: (a) $\int_{-2}^{2} (4 - x^2)\, dx$ (b) $\int_{\pi/2}^{\pi} \sin(x)\, dx$

The value of a definite integral $\int_a^b f(x)\, dx$ depends only on the function f being integrated and on the interval [a, b]. The variable $\boldsymbol{x}$ in $\int_a^b f(\boldsymbol{x})\, d\boldsymbol{x}$ is a "dummy variable" and replacing it with another variable does not change the value of the integral. The following integrals each represent the integral of the function f on the interval [a,b], and they are all equal:

$$\int_a^b f(\boldsymbol{x})\, d\boldsymbol{x} = \int_a^b f(\boldsymbol{t})\, d\boldsymbol{t} = \int_a^b f(\boldsymbol{w})\, d\boldsymbol{w} = \int_a^b f(\boldsymbol{z})\, d\boldsymbol{z} .$$

Definite Integrals of Negative Functions

A definite integral is a limit of Riemann sums, and Riemann sums can be made from any integrand function f, positive or negative, continuous or discontinuous. The definite integral still has a geometric meaning even if the function is sometimes (or always) negative, and definite integrals of negative functions also have interpretations in applications.

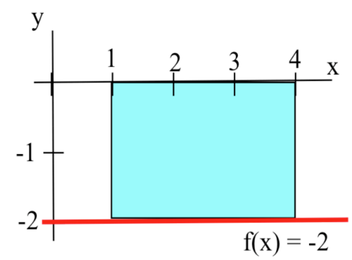

Fig. 4

Example 5: Find the definite integral of $f(x) = -2$ on the interval $[1,4]$.

Solution: $\sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} (-2)\cdot\Delta x_k = -2\cdot\sum_{k=1}^{n} \Delta x_k = -2\cdot(3) = -6$

for every partition P and every choice of values for c_k so

$$\int_1^4 -2\,dx = \lim_{mesh\to 0}\left(\sum_{k=1}^{n} f(c_k)\cdot\Delta x_k\right) = \lim_{mesh\to 0}(-6) = -6 .$$

The **area** of the region in Fig. 4 is 6 units, but because the region is **below** the x–axis, the value of the **integral** is -6 .

If	the graph of $f(x)$ is **below** the x–axis for $a \le x \le b$ (f is **negative**),
then	$\int_a^b f(x)\,dx = -$ { area below the x–axis for $a \le x \le b$ } , a negative number.

If $f(t)$ is the rate of population change (people/year) for a town, then negative values of f would indicate that the population of the town was getting smaller, and the definite integral (now a negative number) would be the **change** in the population, a decrease, during the time interval.

Example 6: In 1980 there were 12,000 ducks nesting around a lake, and the **rate** of population change is shown in Fig. 5. Write a definite integral to represent the total change in the duck population from 1980 to 1990, and estimate the population in 1990.

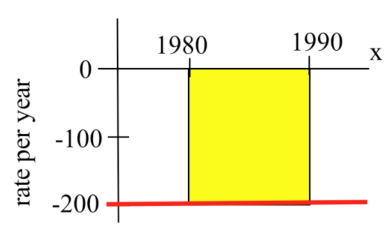

Fig. 5

Solution: Total **change** in population

$$= \int_{1980}^{1990} f(t)\,dt = -\{\text{area between f and axis}\}$$

$$\approx -\{200 \text{ ducks/year}\}\cdot\{10 \text{ years}\} = -2000 \text{ ducks.}$$

Then {1990 duck population} = {1980 population} + {change from 1980 to 1990}

= {12,000} + { –2000} = 10,000 ducks.

If $f(t)$ is the velocity of a car in the positive direction along a straight line at time t (miles/hour), then negative values of f indicate that the car is traveling in the negative direction. The definite integral of f (the integral is a negative number) is the change in position of the car during the time interval, how far the car traveled in the negative direction.

Practice 3: A bug starts at the location $x = 12$ on the x–axis at 1 pm walks along the axis with the velocity shown in Fig. 6. How far does the bug travel between 1 pm and 3 pm, and where is the bug at 3 pm?

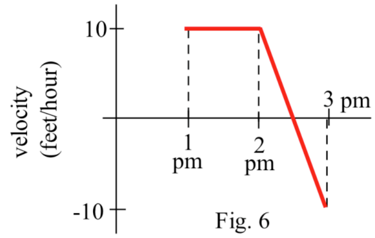

Fig. 6

Frequently our integrand functions will be positive some of the time and negative some of the time. If f represents the rate of population increase, then the integral of the positive parts of f will be the increase in population and the integral of the negative parts of f will be the decrease in population. Altogether, the integral of f over the whole time interval will be the **total (net) change** in the population.

$$\int_a^b f(x)\,dx = \{\text{ area above axis }\} - \{\text{ area below axis }\}$$

Example 7: Use Fig. 7 to calculate $\int_0^2 f(x)\,dx$, $\int_2^4 f(x)\,dx$, $\int_4^5 f(x)\,dx$, $\int_0^5 f(x)\,dx$.

Solution: $\int_0^2 f(x)\,dx = 2$, $\int_2^4 f(x)\,dx = \mathbf{-5}$, $\int_4^5 f(x)\,dx = 2$, and

$$\int_0^5 f(x)\,dx = \{\text{area above}\} - \{\text{area below}\} = \{2+2\} - \{5\} = -1.$$

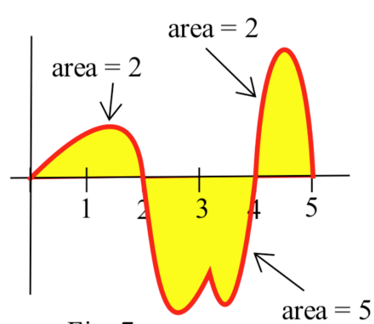

Fig. 7

Practice 4: Use geometric reasoning to evaluate $\int_0^{2\pi} \sin(x)\,dx$.

If f is a velocity, then the integrals on the intervals where f is positive measure the distances moved forward; the integrals on the intervals where f is negative measure the distances moved backward; and the integral over the whole time interval is the **total (net) change** in position, the distance moved forward minus the distance moved backward.

Practice 5: A car is driven with the velocity west shown in Fig. 8.

(a) Between noon and 6 pm how far does the car travel?

(b) At 6 pm, where is the car relative to its starting point (its position at noon)?

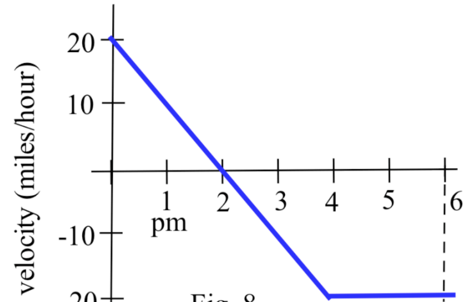

Fig. 8

Units For the Definite Integral

We have already seen that the "area" under a graph can represent quantities whose units are not the usual geometric units of square meters or square feet. For example, if x is a measure of time in "seconds" and $f(x)$ is a velocity with units "feet/second", then Δx has the units "seconds" and $f(x)\cdot\Delta x$ has the units ("feet/second")("seconds") = "feet," a measure of distance. Since each Riemann sum $\sum f(x)\cdot\Delta x$ is a sum of "feet" and the definite integral is the limit of the Riemann sums, the definite integral, has the same units, "feet".

If the units of $f(x)$ are "square feet" and the units of x are "feet", then $\int_a^b f(x)\,dx$ is a number with the units ("square feet")·("feet") = "cubic feet," a measure of volume. If $f(x)$ is a force in grams, and x is a distance in centimeters, then $\int_a^b f(x)\,dx$ is a number with the units "gram·centimeters," a measure of work.

In general, the units for the definite integral $\int_a^b f(x)\,dx$ are (units for $f(x)$)·(units for x). A quick check of the units can help avoid errors in setting up an applied problem.

PROBLEMS

In problems 1 – 4 , rewrite the limit of each Riemann sum as a definite integral.

1. $\lim_{mesh \to 0}\left(\sum_{k=1}^{n}(2+3c_k)\Delta x_k\right)$ on the interval [0, 4]

2. $\lim_{mesh \to 0}\left(\sum_{k=1}^{n}\cos(5c_k)\Delta x_k\right)$ on [0,11]

3. $\lim_{mesh \to 0}\left(\sum_{k=1}^{n}(c_k)^3\Delta x_k\right)$ on [2, 5]

4. $\lim_{mesh \to 0}\left(\sum_{k=1}^{n}\sqrt{c_k}\,\Delta x_k\right)$ on [1, 4]

In problems 5 – 10, represent the area of each bounded region as a definite integral. (Do **not** evaluate the integral, just translate the area into an integral.)

5. The region bounded by $y = x^3$, the x–axis, the line $x = 1$, and $x = 5$.

6. The region bounded by $y = \sqrt{x}$, the x–axis, and the line $x = 9$.

7. The region bounded by $y = x \cdot \sin(x)$, the x–axis, the line $x = 1/2$, and $x = 2$.

8. The shaded region in Fig. 9.

9. The shaded region in Fig. 10.

10. The shaded region in Fig. 10 for $2 \le x \le 3$.

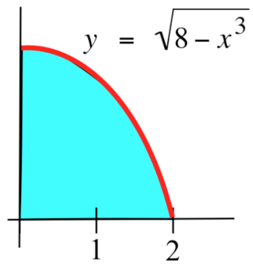

Fig. 9

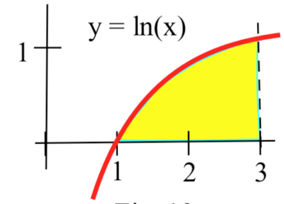

Fig. 10

In problems 11 – 15 , represent the area of each bounded region as a definite integral, and use geometry to determine the value of the definite integral.

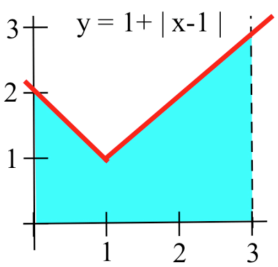

Fig. 11

11. The region bounded by $y = 2x$, the x–axis, the line $x = 1$, and $x = 3$.

12. The region bounded by $y = 4 - 2x$, the x–axis, and the y–axis.

13. The region bounded by $y = | x |$, the x–axis, and the line $x = -1$.

14. The shaded region in Fig. 11.

15. The shaded region in Fig. 12.

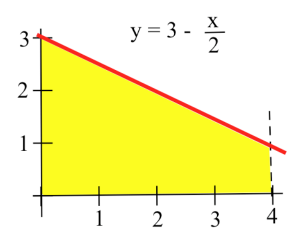

Fig. 12

16. Fig. 13 shows the graph of f and the areas of several regions. Evaluate:

(a) $\int_0^3 f(x)\,dx$ (b) $\int_3^5 f(x)\,dx$ (c) $\int_5^7 f(x)\,dx$

(d) $\int_0^5 | f(x) |\,dx$ (e) $\int_3^7 f(x)\,dx$

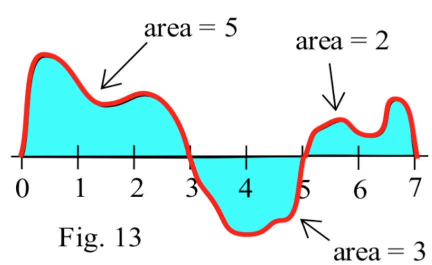

Fig. 13

17. Fig. 14 shows the graph of g and the areas of several regions.

Evaluate : (a) $\int_1^3 g(x)\,dx$ (b) $\int_3^4 g(x)\,dx$

(c) $\int_4^8 g(x)\,dx$ (d) $\int_1^8 g(x)\,dx$ (e) $\int_3^8 | g(x) |\,dx$

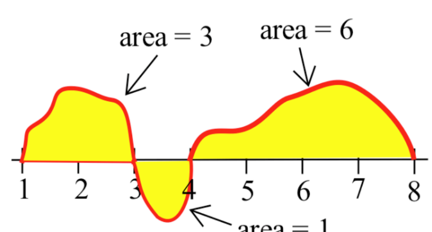

Fig. 14

18. Fig. 15 shows the graph of h and the areas of several regions. Evaluate:

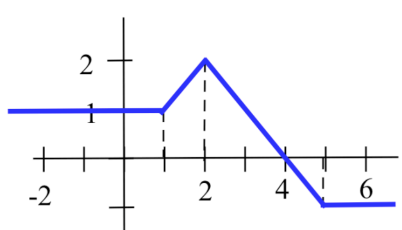

Fig. 15

(a) $\int_{-2}^{1} h(x)\,dx$ (b) $\int_4^6 h(x)\,dx$

(c) $\int_{-2}^{6} h(x)\,dx$ (d) $\int_{-2}^{4} h(x)\,dx$ (e) $\int_{-2}^{4} | h(x) |\,dx$

In problems 19 – 20, your velocity (in feet per minute) along a straight path is shown. (a) Sketch the graph of your location.

(b) How many feet did you walk in 8 minutes?

(c) Where, relative to your starting location, are you after 8 minutes?

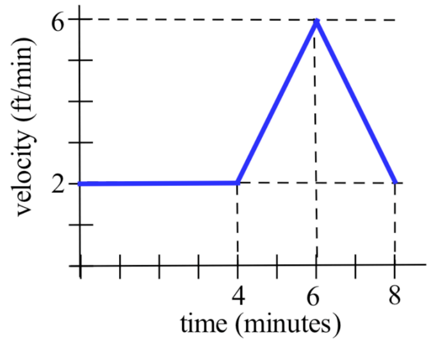

Fig. 16

19. Your velocity is shown in Fig. 16.

20. Your velocity is shown in Fig. 17.

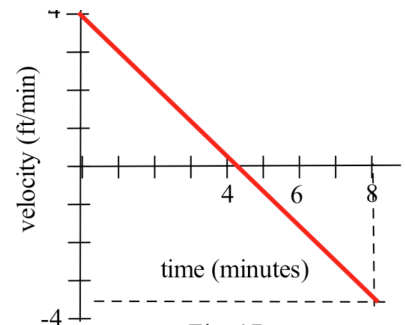

Fig. 17

In problems 21 – 27, the units are given for x and a $f(x)$. Give the units of $\int_a^b f(x)\,dx$.

21. x is time in "seconds", and $f(x)$ is velocity in "meters per second."

22. x is time in "hours", and $f(x)$ is a flow rate in "gallons per hour."

23. x is a position in "feet", and $f(x)$ is an area in "square feet."

24. x is a time in "days", and $f(x)$ is a temperature in "degrees Celsius."

25. x is a height in "meters", and $f(x)$ is a force in "grams."

26. x is a position in "inches", and $f(x)$ is a density in "pounds per inch."

27. x is a time in "seconds", and $f(x)$ is an acceleration in "feet per second per second (ft/s^2)."

Example 8: For $f(x) = x^2$, divide the interval $[0,2]$ into n equally–wide subintervals and evaluate the lower sum and the limit of the lower sum as $n \to \infty$.

Solution: $\Delta x = \frac{b-a}{n} = \frac{2}{n}$, $x_i = \frac{2}{n}\, i$ (for $i = 0, 1, 2, \ldots, n$) and $f(x_i) = (x_i)^2 = \frac{4}{n^2}\, i^2$. Since f is increasing on $[0, 2]$, the minimum value of f on each subinterval occurs at the left edge of that subinterval, so $m_i = x_{i-1}$. Then

$$LS = \sum_{i=1}^{n} f(m_i)\Delta x = \sum_{i=1}^{n} (m_i)^2 \frac{2}{n} = \sum_{i=1}^{n} (x_{i-1})^2 \frac{2}{n}) = \sum_{i=1}^{n} (\frac{2}{n}(i-1))^2 \frac{2}{n}$$

$$= \sum_{i=1}^{n} \frac{4}{n^2}(i^2 - 2i + 1)\frac{2}{n} = \frac{8}{n^3} \sum_{i=1}^{n} (i^2 - 2i + 1))$$

$$= \frac{8}{n^3}\left\{ \sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{k=1}^{n} 1 \right\} = \frac{8}{n^3}\left\{ \frac{n(n+1)(2n+1)}{6} - 2\,\frac{n(n+1)}{2} + n \right\}$$

$$= \frac{8}{n^3}\left\{ \frac{2n^3 - 3n^2 + n}{6} \right\} = \frac{8}{6}\left\{ \frac{2n^3 - 3n^2 + n}{n^3} \right\} = \frac{8}{6}\left\{ 2 - \frac{3}{n} + \frac{1}{n^2} \right\}$$

$$\longrightarrow \frac{8}{6}\{2\} = \frac{8}{3}. \qquad \text{We can be certain that } \int_0^2 x^2\,dx \ \geq\ 8/3.$$

Practice 6: For $f(x) = x^2$, divide the interval $[0,2]$ into n equally–wide subintervals. Verify that $\Delta x = 2/n$, M_i = right edge $= x_i = \frac{2}{n}\,i$, $f(M_i) = \frac{4}{n^2}\,i^2$, and that the upper sum is $\sum_{i=1}^{n} \frac{4}{n^2}\,i^2\,\frac{2}{n}$. Show that the limit of the upper sum, as $n \to \infty$, is $8/3$.

From the previous Example and Practice problem, we know that

$$\frac{8}{3} \leq \int_0^2 x^2\,dx \leq \frac{8}{3} \quad \text{so we can conclude that } \int_0^2 x^2\,dx = \frac{8}{3}.$$

A much easier method for evaluating this integral will be presented in Section 4.4 .

28. For $f(x) = 3 + x$, partition the interval $[0,2]$ into n equally wide subintervals of length $\Delta x = 2/n$. Write the lower sum for this function and partition, and calculate the limit of the lower sum as $n \to \infty$.
 (b) Write the upper sum for this function and partition and find the limit of the upper sum as $n \to \infty$.

29. For $f(x) = x^3$, partition the interval $[0,2]$ into n equally wide subintervals of length $\Delta x = 2/n$. Write the lower sum for this function and partition, and calculate the limit of the lower sum as $n \to \infty$.
 (b) Write the upper sum for this function and partition and find the limit of the upper sum as $n \to \infty$.

30. For $f(x) = \sqrt{x}$, partition the interval $[0, 9]$ by taking $x_i = \frac{9}{n^2}\,i^2$ for $i = 0, 1, 2, \ldots, n$. Then $\Delta x_i = x_i - x_{i-1} = \frac{9}{n^2}\,i^2 - \frac{9}{n^2}(i-1)^2 = \frac{9}{n^2}(2i+1)$ and $f(x_i) = \sqrt{x_i} = \frac{3}{n}\,i$. Write the upper sum for this function and partition (take $c_i = x_i$ for $i = 1, 2, \ldots, n$) , use the summation formulas in the previous section for sums of powers to simplify the summation, and take the limit of the result as $n \to \infty$.
 (b) Follow the same steps for the lower sum (take $c_i = x_i$ for $i = 0, 2, \ldots, n-1$).

Section 4.2 **PRACTICE Answers**

Practice 1: Area = $\lim_{mesh \to 0} \left(\sum_{k=1}^{n} \sin(c_k) \Delta x_k \right) = \int_0^{\pi} \sin(x)\, dx$.

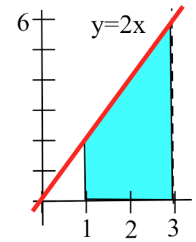

Fig. 18

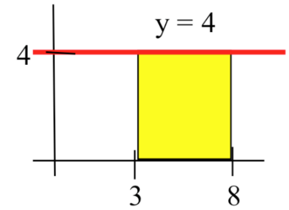

Fig. 19

Practice 2: $\lim_{mesh \to 0} \left(\sum_{k=1}^{n} (2c_k) \Delta x_k \right) =$ Shaded area in Fig. 18 = 8

$\int_3^8 4\, dx$ = shaded area in Fig. 19 = 20 .

Practice 3: (a) Total distance = 12.5 feet forward and 2.5 feet backward = 15 feet total travel.

(b) The bug ends up 10 feet forward of it's starting position at x = 12 so the bug's final location is at x = 22.

Practice 4: Between x = 0 and x = 2π, the graph of y = sin(x) (Fig. 20) has the same area above the x–axis as below the x–axis so the definite integral is 0: $\int_0^{2\pi} \sin(x)\, dx = 0$.

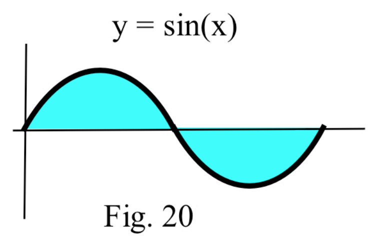

Fig. 20

Practice 5: (a) 20 miles west (from noon to 2 pm) plus 60 miles east (from 2 to 6 pm) is a total travel distance of 80 miles. (At 4 pm the driver is back at the starting position after driving 40 miles = 20 miles west and then 20 miles east.)

(b) The car is 40 miles **east** of the starting location. (East is the "negative" of west.)

Practice 6: $\Delta x = \frac{2-0}{n} = \frac{2}{n}$. $M_i = \frac{2}{n}\, i$ so $f(M_i) = \{ \frac{2}{n}\, i \}^2 = \frac{4}{n^2}\, i^2$. Then

$$US = \sum_{i=1}^{n} f(M_i)\Delta x = \sum_{i=1}^{n} \frac{4}{n^2}\, i^2\, \frac{2}{n} = \frac{8}{n^3} \left\{ \sum_{i=1}^{n} i^2 \right\}$$

$$= \frac{8}{n^3} \left\{ \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{2}{12} n \right\} = \frac{8}{3} + \frac{4}{n} + \frac{16}{12} \frac{1}{n^2} \longrightarrow \frac{8}{3} \text{ as } n \text{ approaches infinity.}$$