# 4.3 PROPERTIES OF THE DEFINITE INTEGRAL

Definite integrals are defined as limits of Riemann sums, and they can be interpreted as "areas" of geometric regions. These two views of the definite integral can help us understand and use integrals, and together they are very powerful. This section continues to emphasize this dual view of definite integrals and presents several properties of definite integrals. These properties are justified using the properties of summations and the definition of a definite integral as a Riemann sum, but they also have natural interpretations as properties of areas of regions. These properties are used in this section to help understand functions that are defined by integrals. They will be used in future sections to help calculate the values of definite integrals.

**Properties of the Definite Integral**

As you read each statement about definite integrals, examine the associated Figure and interpret the property as a statement about areas.

1. $\int_a^a f(x)\,dx = 0$ (a definition)
2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ (a definition)
3. $\int_a^b k\,dx = k\cdot(b-a)$ (Fig. 1)
4. $\int_a^b k\cdot f(x)\,dx = k\cdot\int_a^b f(x)\,dx$
5. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ (Fig. 2)

Justification of Property 3: Using area: If $k > 0$ (Fig. 1), then $\int_a^b k\,dx$ represents the area of the rectangle with base = b–a and height = k, so $\int_a^b k\,dx = (\text{height})\cdot(\text{base}) = k\cdot(b-a)$.

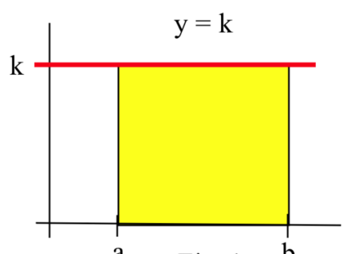

Fig. 1

Using Riemann Sums: For every partition $P = \{ x_0 = a, x_1, x_2, x_3, \ldots, x_{n-1}, x_n = b\}$ of the interval [a, b], and every choice of $c_k$, the Riemann sum is

$$\sum_{k=1}^{n} f(c_k)\cdot\Delta x_k = \sum_{k=1}^{n} k\cdot\Delta x_k = k\sum_{k=1}^{n}\Delta x_k$$

$$= k\sum_{k=1}^{n}(x_k - x_{k-1}) = k\{(x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + \ldots + (x_{n-1} - x_n)\}$$

$$= k\cdot(x_n - x_0) = k\cdot(b - a)$$

so the limit of the Riemann sums, as the mesh approaches zero, is $k\cdot(b - a)$. $\int_a^b k\,dx = k\cdot(b - a)$.

Justification of Property 4:

$$\int_a^b k\cdot f(x)\,dx = \lim_{mesh \to 0}\left(\sum_{k=1}^{n} k\cdot f(c_k)\Delta x_k\right) = k\cdot\lim_{mesh \to 0}\left(\sum_{k=1}^{n} f(c_k)\Delta x_k\right) = k\cdot\int_a^b f(x)\,dx .$$

Property 5 can be justified using Riemann sums, but Fig. 2 graphically illustrates why it is true.

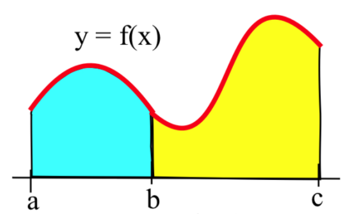

Fig. 2

**Properties of Definite Integrals of Combinations of Functions**

Properties 6 and 7 relate the values of integrals of sums and differences of functions to the sums and differences of integrals of the individual functions. These two properties will be very useful when we need the integral of a function which is the sum or difference of several terms: we can integrate each term and then add or subtract the individual results to get the integral we want. Both of these new properties have natural interpretations as statements about areas of regions.

6. $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$ (Fig. 3)

7. $\int_a^b f(x) - g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Justification of Property 6: $\int_a^b f(x) + g(x)\,dx$

$$= \lim_{mesh \to 0}\left(\sum_{k=1}^{n}(f(c_k) + g(c_k))\Delta x_k\right)$$

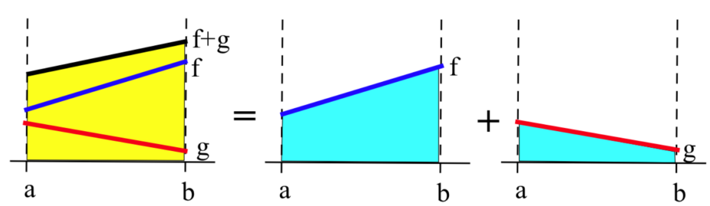

Fig. 3

=

$$\lim_{mesh \to 0} \left( \sum_{k=1}^{n} f(c_k)\Delta x_k + \sum_{k=1}^{n} g(c_k)\Delta x_k \right)$$

$$= \lim_{mesh \to 0} \left( \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \right) + \lim_{mesh \to 0} \left( \sum_{k=1}^{n} g(c_k) \cdot \Delta x_k \right) = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \,.$$

**Practice 1**: $\int_1^4 f(x)\,dx = 7$, and $\int_1^4 g(x)\,dx = 3$. Evaluate $\int_1^4 f(x)-g(x)\,dx$ .

Property 8 says that if one function is larger than another function on an interval, then the definite integral of a larger function on that interval is bigger than the definite integral of the smaller function. This property then leads to Property 9 which provides a quick method for determining bounds on how large and small a particular integral can be.

8. If $f(x) \le g(x)$ for all $x$ in $[a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$ . (Fig. 4)

9. $(b-a)\cdot(\text{min of } f \text{ on } [a,b]) \le \int_a^b f(x)\,dx \le (b-a)\cdot(\text{max of } f \text{ on } [a,b])$ . (Fig. 5)

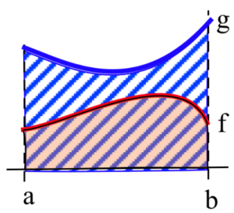

Fig. 4

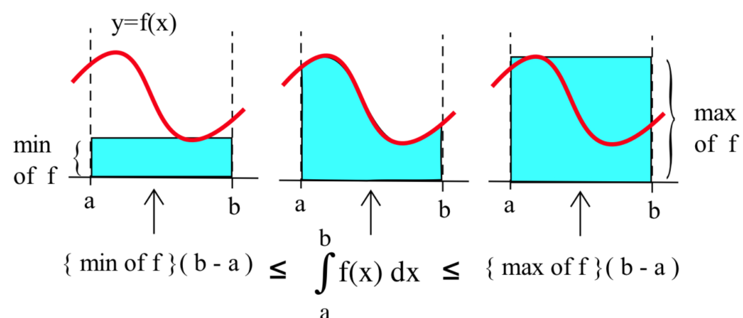

Fig. 5

Justification of Property 8:

Fig. 4 illustrates that if f and g are both positive and $f(x) \le g(x)$ for all $x$ in $[a,b]$, then the area of region F is smaller than the area of region G and $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$ .

Similar sketches for the situations when f or g are sometimes or always negative illustrate that Property 9 is always true, but we can avoid all of the different cases by using Riemann sums.

Using Riemann Sums: If the same partition and sampling points $c_k$ are used to get Riemann sums for f and g, then $f(c_k) \le g(c_k)$ for each k and

$$\sum_{k=1}^{n} f(c_k)\Delta x_k \le \sum_{k=1}^{n} g(c_k)\Delta x_k \text{ so } \lim_{mesh \to 0}\left(\sum_{k=1}^{n} f(c_k)\Delta x_k\right) \le \lim_{mesh \to 0}\left(\sum_{k=1}^{n} g(c_k)\Delta x_k\right).$$

Justification of Property 9: Property 9 follows easily from Property 8.

Let $g(x) = M = (\text{max of } f \text{ on } [a,b])$. Then $f(x) \le M = g(x)$ for all $x$ in $[a,b]$ so

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx = \int_a^b M\,dx = (b-a)\cdot M .$$

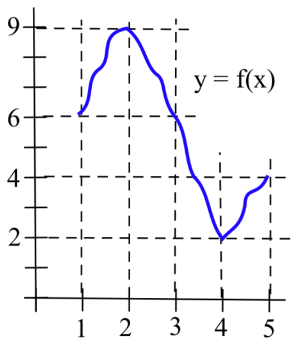

Fig. 6

**Example 1**: Determine lower and upper bounds for the value of $\int_1^5 f(x)\,dx$ in Fig. 6.

Solution: If $1 \le x \le 5$, then $2 \le f(x) \le 9$ so a lower bound is $(b-a)\cdot(\text{min of } f \text{ on } [a,b]) = (4)(2) = 8$.

An upper bound is

$(b-a)\cdot(\text{max of } f \text{ on } [a,b]) = (4)(9) = 36$: $8 \le \int_1^5 f(x)\,dx \le 36$. This range, from 8 to 36 is rather wide. Property 9 is not useful for finding the exact value of the integral, but it is very easy to use and it can help us avoid an unreasonable value for an integral.

**Practice 2**: Determine a lower bound and an upper bound for the value of $\int_3^5 f(x)\,dx$ in Fig. 6.

**Functions Defined by Integrals**

If one of the endpoints $a$ or $b$ of the interval $[a, b]$ changes, then the value of the integral $\int_a^b f(t)\,dt$ typically changes. A definite integral of the form $\int_a^x f(t)\,dt$ defines a function of $x$, and functions defined by definite integrals in this way have interesting and useful properties. The next examples illustrate one of them: the derivative of a function defined by an integral is closely related to the integrand, the function "inside" the integral.

**Example 2**: For the function $f(t) = 2$, define $A(x)$ to be the area of the region bounded by $f$, the $t$–axis, and vertical lines at $t = 1$ and $t = x$ (Fig. 7).

(a) Evaluate $A(1), A(2), A(3), A(4)$.

(b) Find an algebraic formula for $A(x)$ for $x \ge 1$.

(c) Calculate $\frac{d}{dx} A(x)$.

(d) Describe $A(x)$ as a definite integral.

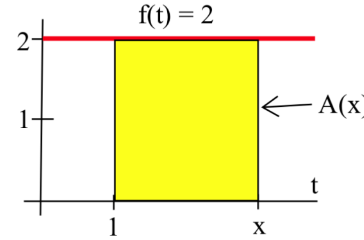

Fig. 7

Solution : (a) $A(1) = 0, A(2) = 2, A(3) = 4, A(4) = 6$ .

(b) A(x) = area of a rectangle = (base)•(height) = $(x-1)\cdot(2) = 2x-2$.

(c) $\frac{d}{dx} A(x) = \frac{d}{dx}(2x-2) = 2$ . (d) $A(x) = \int_1^x 2\,dt$ .

**Practice 3**: Answer the questions in the previous Example for $f(x) = 3$ .

**Example 3**: For the function $f(t) = 1+t$, define $B(x)$ to be the area of the region bounded by the graph of f, the $t$–axis, and vertical lines at $t = 0$ and $t = x$ (Fig. 8).

(a) Evaluate B(0), B(1), B(2), B(3).

(b) Find an algebraic formula for $B(x)$ for $x \ge 0$.

(c) Calculate $\frac{d}{dx} B(x)$.

(d) Describe $B(x)$ as a definite integral.

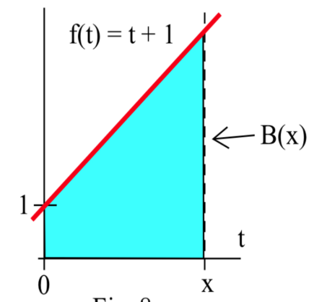


Fig. 8

Solution: (a) B(0) = 0, B(1) = 1.5, B(2) = 4, B(3) = 7.5

(b) B(x) = area of trapezoid = (base)•(average height) = $(x)\cdot\left(\frac{1+(1+x)}{2}\right) = x + \frac{x^2}{2}$ .

(c) $\frac{d}{dx} B(x) = \frac{d}{dx}\left(x + \frac{x^2}{2}\right) = 1 + x$ . (d) $B(x) = \int_0^x 1+t\,dt$

**Practice 4**: Answer the questions in the previous Example for $f(t) = 2t$ .

A curious "coincidence" appeared in each of these Examples and Practice problems: the derivative of the function defined by the integral was the same as the integrand, the function "inside" the integral. Stated another way, the function defined by the integral was an "antiderivative" of the function "inside" the integral. In section 4.4 we will see that this "coincidence" is a property of functions defined by the integral. And it is such an important property that it is called The Fundamental Theorem of Calculus, part I. Before we go on to the Fundamental Theorem of Calculus, however, there is an "existence" question to consider: Which functions can be integrated?

### Which Functions Are Integrable?

This important question was finally answered in the 1850s by Georg Riemann, a name that should be familiar by now. Riemann proved that a function must be badly discontinuous to not be integrable.

**Every continuous function is integrable.**

If f is **continuous** on the interval [a,b],

then $\lim_{mesh \to 0}\left(\sum_{k=1}^{n} f(c_k)\cdot\Delta x_k\right)$ is always the same finite number , $\int_a^b f(x)\,dx$ ,

so f is **integrable** on [a,b] .

In fact, a function can even have any finite number of breaks and still be integrable.

**Every bounded, piecewise continuous function is integrable.**

If f is defined and bounded ($-M \le f(x) \le M$ for some M) for all x in [a,b] and continuous except at a finite number of points in [a,b] ,

then $\lim_{mesh \to 0}\left(\sum_{k=1}^{n} f(c_k)\Delta x_k\right)$ is always the same finite number, $\int_a^b f(x)\,dx$ ,

so f is integrable on [a,b] .

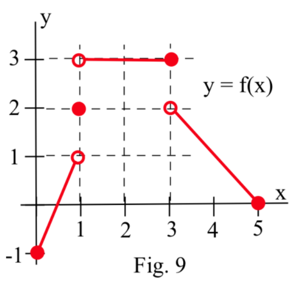


Fig. 9

The function f in Fig. 9 is always between –3 and 3 (in fact, always between –1 and 3) so it is bounded , and it is continuous except at 2 and 3. As long as the values of f(2) and f(3) are finite numbers, their actual values will not effect the value of the definite integral, and

$$\int_0^5 f(x)\,dx = 0 + 6 + 2 = 8.$$

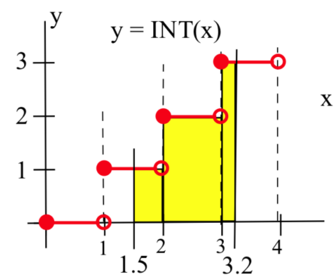


Fig. 10

**Practice 5**: Evaluate $\int_{1.5}^{3.2} \text{INT}(x)\,dx$ . (Fig. 10)

Fig. 11 summarizes the relationships among differentiable, continuous, and integrable functions:

- Every differentiable function is continuous, but there are continuous functions which are not differentiable. (example: $|x|$ is continuous but not differentiable at $x = 0$.)
- Every continuous function is integrable, but there are integrable functions which are not continuous. (example: the function in Fig. 9 is integrable on [0, 5] but is not continuous at 2 and 3.)
- Finally, as shown in the optional part of this section, there are functions which are not integrable.

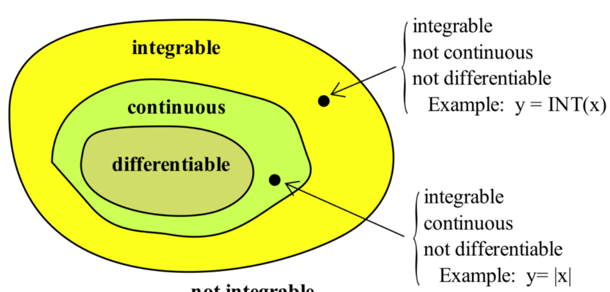


Fig. 11

**A Nonintegrable Function**

If f is continuous or piecewise continuous on [a,b], then f is integrable on [a,b] . Fortunately, the functions we will use in the rest of this book are all integrable as are the functions you are likely to need for applications. However, there are functions for which the limit of the Riemann sums does not exist, and those functions are not integrable.

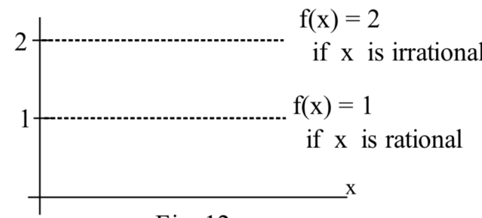

Fig. 12

**A nonintegrable function:**

The function $f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ 2 & \text{if } x \text{ is an irrational number} \end{cases}$ (Fig. 12)

is not integrable on [0,3].

Proof: For any partition P, suppose that you, a very rational person, always select values of $c_k$ which are rational numbers. (Every subinterval contains rational numbers and irrational numbers, so you can always pick $c_k$ to be a rational number.)

Then $f(c_k) = 1$, and your Riemann sum, YS, is always

$$YS_P = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{\mathbf{n}} \mathbf{1}\Delta x_k = 3.$$

Suppose your friend, however, always selects values of $c_k$ which are irrational numbers. Then $f(c_k) = 2$, and your friend's Riemann sum, FS, is always

$$FS_P = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{\mathbf{n}} \mathbf{2}\Delta x_k = \mathbf{2}\sum_{k=1}^{n} \Delta x_k = 6.$$

Then $\lim_{mesh \to 0} YS_P = 3$ and $\lim_{mesh \to 0} FS_P = 6$ so $\lim_{mesh \to 0} \left( \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \right)$

does not exist, and this f is not integrable on [0,3] or on any other interval either.

**PROBLEMS:**

Problems 1 – 20 refer to the graph of f in Fig. 13. Use the graph to determine the values of the definite integrals. (The bold numbers represent the **area** of each region.)

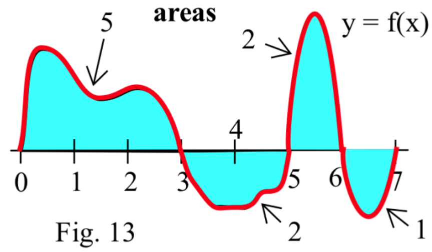

Fig. 13

1. $\int_0^3 f(x)\,dx$ 2. $\int_3^5 f(x)\,dx$ 3. $\int_2^2 f(x)\,dx$ 4. $\int_6^7 f(w)\,dw$ 5. $\int_0^5 f(x)\,dx$

6. $\int_0^7 f(x)\,dx$ 7. $\int_3^6 f(t)\,dt$ 8. $\int_5^7 f(x)\,dx$ 9. $\int_3^0 f(x)\,dx$ 10. $\int_5^3 f(x)\,dx$

11. $\int_6^0 f(x)\,dx$ 12. $\int_0^3 2f(x)\,dx$ 13. $\int_4^4 f^2(s)\,ds$ 14. $\int_0^3 1+f(x)\,dx$ 15. $\int_0^3 x+f(x)\,dx$

16. $\int_3^5 3+f(x)\,dx$ 17. $\int_0^5 2+f(x)\,dx$ 18. $\int_3^5 |\,f(x)\,|\,dx$ 19. $\int_0^5 |\,f(x)\,|\,dx$ 20. $\int_7^3 1+|\,f(x)\,|\,dx$

Problems 21 – 30 refer to the graph of g in Fig. 14. Use the graph to evaluate the integrals.

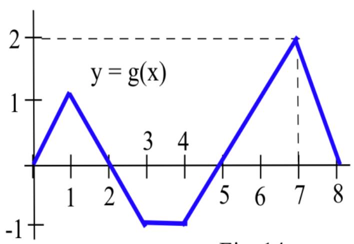

Fig. 14

21. $\int_0^2 g(x)\,dx$ 22. $\int_1^3 g(t)\,dt$ 23. $\int_0^5 g(x)\,dx$ 24. $\int_4^2 g(x)\,dx$

25. $\int_0^8 g(s)\,ds$ 26. $\int_1^4 |\,g(x)\,|\,dx$ 27. $\int_0^3 2g(t)\,dt$ 28. $\int_5^8 1+g(x)\,dx$

29. $\int_6^3 g(u)\,du$ 30. $\int_0^8 t+g(t)\,dt$

For problems 31 – 34 , use the constant functions $f(x) = 4$ and $g(x) = 3$ on the interval [0,2]. Calculate each integral and verify that the value obtained in part (a) is **not** equal to the value in part (b).

31.(a) $\int_0^2 f(x)dx \cdot \int_0^2 g(x)dx$ (b) $\int_0^2 f(x)\cdot g(x)\,dx$ 32.(a) $\int_0^2 f(x)dx \Big/ \int_0^2 g(x)dx$ (b) $\int_0^2 f(x)/g(x)\,dx$

33.(a) $\int_0^2 f^2(x)dx$ (b) $\left(\int_0^2 f(x)dx\right)^2$ 34.(a) $\int_0^2 \sqrt{f(x)}\,dx$ (b) $\sqrt{\int_0^2 f(x)\,dx}$

For problems 35 – 42 , sketch the graph of the integrand function and use it to help evaluate the integral.

35 $\int_0^4 |x|\,dx$ 36. $\int_0^4 1 + |t|\,dt$ 37. $\int_{-1}^2 |x|\,dx$ 38. $\int_0^2 |x| - 1\,dx$

39. $\int_1^3 \text{INT}(u)\,du$ 40. $\int_1^{3.5} \text{INT}(x)\,dx$ 41. $\int_1^3 2 + \text{INT}(t)\,dt$ 42. $\int_3^1 \text{INT}(x)\,dx$

For problems 43 – 46, (a) Sketch the graph of $y = A(x) = \int_0^x f(t)\,dt$ and (b) sketch the graph of $y = A'(x)$.

43. $f(x) = x$ . 44. $f(x) = x - 2$ . 45. f in Fig. 15. 46. f in Fig. 16.

For problems 47 – 50, state whether the function is
(a) continuous on [1,4], (b) differentiable on [1,4] , and (c) integrable on [1,4].

47. f in Fig. 15, 48. f in Fig. 16. 49. f in Fig. 17. 50. f in Fig. 18.

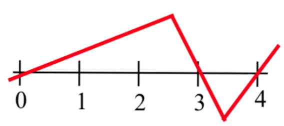

Fig. 15

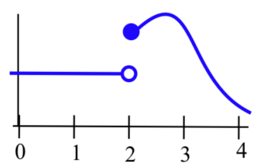

Fig. 16

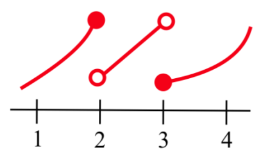

Fig. 17

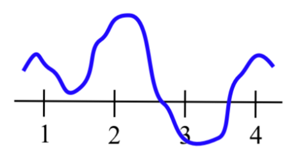

Fig. 18

51. Write the total distance traveled by the car in Fig. 19 between 1 pm and 4 pm as a definite integral and estimate the value of the integral.

52. Write the total distance traveled by the car in Fig. 19 between 3 pm and 6 pm as a definite integral and estimate the value of the integral.

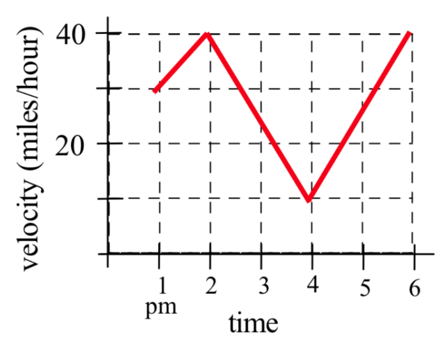

Fig. 19

**Section 4.3** **PRACTICE Answers**

**Practice 1**: $\int_1^4 f(x) - g(x)\,dx = 7 - 3 = 4$ .

**Practice 2**: $(2)(\text{min. of f on }[3,5]) = 4 \le \int_3^5 f(x)\,dx \le 2(\text{max. of f on }[3,5]) = 12$ .

**Practice 3**: (a) $A(1) = 0, A(2) = 3, A(3) = 6, A(4) = 9$ (b) $A(x) = (x - 1)(3) = 3x - 3$

(c) $\frac{d}{dx} A(x) = 3$ (d) $A(x) = \int_1^x 3\,dx$

**Practice 4**: (a) $B(0) = 0, B(1) = 1, B(2) = 4, B(3) = 9$ (b) $B(x) = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(x)(2x) = x^2$

. (c) $\frac{d}{dx} B(x) = \frac{d}{dx} x^2 = 2x$ (d) $B(x) = \int_0^x 2t\,dt$

**Practice 5**: The integral = the shaded area in Fig. 10 = (0.5)(1) + (1)(2) + (0.2)(3) = 3.1 .