4.8 USING TABLES TO FIND ANTIDERIVATIVES

The inside covers of this book show the patterns for many antiderivatives, and there are reference books which contain many more than the ones here. The table of integrals is to help you now while you are learning calculus and to serve as a reference later when you are using calculus. Think of the tables as a dictionary — something to use when you need to spell a difficult word or need the meaning of a new word. It would be difficult to write a letter if you had to look up the spelling of every word, and it will be difficult to learn and do calculus if you have to look up every antiderivative. Tables of antiderivatives are limited and often take longer to use than finding an antiderivative from scratch, but they can also be very valuable and useful. This section shows how to transform some integrals into forms in the tables and how to use the "recursive" formulas in the tables.

The first examples and practice problems illustrate some of the techniques used to change an integral into a standard form. **These techniques are useful whether that standard form resides in a table or in your head.**

Example 1: Use the table to find $\int \frac{1}{9+x^2}\,dx$.

Solution: Table entry number 35, $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

is the integral pattern of our problem for $a = 3$, so by replacing the a with 3 we have

$$\int \frac{1}{9+x^2}\,dx = \int \frac{1}{3^2+x^2}\,dx = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C.$$

Practice 1: Use the table to find $\int \frac{1}{25-x^2}\,dx$. Notice that a small change in the form of the integrand (from + in the example to – in the practice problem) can lead to a very different looking result. {Table entry 37: $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$ }

Sometimes the choice for a constant in the antiderivative pattern appears unusual.

Example 2: Use the table to find $\int \frac{1}{5+x^2}\,dx$.

Solution: The table entry 35, $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$, for the first example still works if a^2 is 5. Then $a = \sqrt{5}$,

$$\int \frac{1}{5+x^2}\,dx = \int \frac{1}{(\sqrt{5})^2+x^2}\,dx = \frac{1}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) + C.$$

Practice 2: Use the table to find $\int \frac{1}{7-x^2}\,dx$.

Sometimes algebraic manipulations are needed to change the integrand we have into one that exactly matches the pattern in the table.

Example 3: Use the table to find $\int \frac{1}{9+4x^2}\,dx$.

Solution: The pattern of this problem is basically an arctangent, but we need to **exactly** match the pattern in the table. For this problem we can use algebra or change of variables.

Algebra: We need x^2 rather than $4x^2$, and we can get it by factoring a 4 from each piece of the denominator: $\frac{1}{9+4x^2} = \frac{1}{4}\frac{1}{(9/4)+x^2}$. Then

$$\int \frac{1}{9+4x^2}\,dx = \frac{1}{4}\int \frac{1}{(9/4)+x^2}\,dx = \frac{1}{4}\int \frac{1}{(3/2)^2+x^2}\,dx \qquad \text{now put } a = \frac{3}{2}$$

$$= \frac{1}{4}\frac{1}{(3/2)}\arctan\left(\frac{x}{(3/2)}\right) + C = \frac{1}{6}\arctan\left(\frac{2x}{3}\right) + C.$$

Change of variable: Put $u = 2x$. Then $du = 2\,dx$, $dx = \frac{1}{2}\,du$ and $u^2 = 4x^2$ so

$$\int \frac{1}{9+4x^2}\,dx = \frac{1}{2}\int \frac{1}{9+u^2}\,du = \frac{1}{2}\cdot\frac{1}{3}\arctan\left(\frac{u}{3}\right) + C = \frac{1}{6}\arctan\left(\frac{2x}{3}\right) + C.$$

Practice 3: Use the table to find $\int \frac{1}{25-9x^2}\,dx$.

Sometimes we have to change the variable.

Example 4: Use the table to find $\int \frac{e^x}{9+e^{2x}}\,dx$.

Solution: Putting $u = e^x$, then $du = e^x\,dx$ so our integrand is transformed:

$$\int \frac{e^x}{9+e^{2x}}\,dx = \int \frac{1}{9+(e^x)^2}(e^x\,dx)$$

$$= \int \frac{1}{3^2+u^2}\,du = \frac{1}{3}\arctan\left(\frac{u}{3}\right) + C = \frac{1}{3}\arctan\left(\frac{e^x}{3}\right) + C.$$

Practice 4: Use the table to find $\int \frac{\cos(x)}{25 - \sin^2(x)}\, dx$.

How should you recognize whether algebra or a change of variable is needed? Experience and practice, practice, practice.

Using "Recursive" Formulas

A recursive formula in the table is one that gives one antiderivative in terms of another antiderivative. Usually the new antiderivative is simpler than the original one. For example, table entry #19 gives the pattern for the antiderivative of $\sin^n(ax)$ in terms of the antiderivative of $\sin^{n-2}(ax)$. If we start with the integral of $\sin^5(x)$, then we can use the table to get an answer in terms of the integral of $\sin^3(x)$. Using the recursion formula again, we can get the integral of $\sin^3(x)$ in terms of the integral of $\sin(x)$, which is easy to integrate.

Table entry 19: $\int \sin^n(ax)\, dx = \frac{-\sin^{n-1}(ax)\cdot\cos(ax)}{na} + \frac{n-1}{n} \int \sin^{n-2}(ax)\, dx$

Example 5: Use the given recursive formula to evaluate $\int \sin^3(5x)\, dx$.

Solution: In this example n = 3 and a = 5. Then

$$\int \sin^3(5x)\, dx = \frac{-\sin^2(5x)\cdot\cos(5x)}{3\cdot 5} + \frac{2}{3} \int \sin(5x)\, dx$$

$$= \frac{-\sin^2(5x)\cdot\cos(5x)}{15} - \frac{2}{3}\frac{1}{5} \cos(5x) + C.$$

Practice 5: Use the recursion formula in the table to evaluate $\int \cos^3(7x)\, dx$.

PROBLEMS

Use the integral table for the following problems.

1. $\int \frac{1}{4 + x^2}\, dx$
2. $\int \frac{5}{4 + x^2}\, dx$
3. $\int 2x + \frac{2}{25 + x^2}\, dx$
4. $\int \frac{1}{4 - x^2}\, dx$
5. $\int \frac{2}{9 - x^2}\, dx$
6. $\int \cos(x) + \frac{3}{25 - x^2}\, dx$
7. $\int \frac{1}{3 + x^2}\, dx$
8. $\int \frac{5}{7 + x^2}\, dx$
9. $\int e^x + \frac{7}{2 + x^2}\, dx$

10. $\int \frac{1}{\sqrt{4-x^2}}\ dx$

11. $\int \frac{3}{\sqrt{5-x^2}}\ dx$

12. $\int \frac{3}{\sqrt{4-x^2}}\ dx$

13. $\int \frac{1}{4+25x^2}\ dx$

14. $\int \frac{2}{\sqrt{9-16x^2}}\ dx$

15. $\int \frac{5}{\sqrt{1-4x^2}}\ dx$

16. $\int \sec(x+5)\ dx$

17. $\int \frac{2}{\sqrt{1+9x^2}}\ dx$

18. $\int x \cdot \sec(2x^2+7)\ dx$

19. $\int \ln(x+1)\ dx$

20. $\int \ln(3x-1)\ dx$

21. $\int 3x \cdot \ln(5x^2+7)\ dx$

22. $\int e^x \ln(e^x - 3)\ dx$

23. $\int \cos(x) \ln(\sin(x))\ dx$

24. $\int \frac{2}{\sqrt{x^2-9}}\ dx$

25. $\int \sqrt{4+x^2}\ dx$

26. $\int \sqrt{9+x^2}\ dx$

27. . $\int \sqrt{16+x^2}\ dx$

28. $\int_0^1 \frac{1}{4+x^2}\ dx$

29. $\int_1^3 2x + \frac{2}{25+x^2}\ dx$

30. $\int_0^2 \frac{2}{9-x^2}\ dx$

31. $\int_{-1}^1 \frac{1}{3+x^2}\ dx$

32. $\int_0^1 e^x + \frac{7}{2+x^2}\ dx$

33. $\int_1^2 \frac{3}{\sqrt{5-x^2}}\ dx$

34. $\int_0^1 \frac{1}{4+25x^2}\ dx$

35. $\int_0^{0.1} \frac{5}{\sqrt{1-4x^2}}\ dx$

36. $\int_0^1 \frac{1}{\sqrt{9-4x^2}}\ dx$

37. $\int_0^6 \ln(x+1)\ dx$

38. $\int_0^3 3x \cdot \ln(5x^2+7)\ dx$

39. $\int_0^{\pi/2} \cos(x) \ln(2+\sin(x))\ dx$

40. $\int_0^2 \sqrt{4+x^2}\ dx$

41. $\int_{-3}^3 \sqrt{9+x^2}\ dx$

42. $\int_0^1 \sqrt{16+x^2}\ dx$

In problems 43 – 48 , use the recursion formulas in the table.

43. $\int \sin^3(x)\ dx$

44. $\int \cos^3(x)\ dx$

45. $\int \cos^5(x)\ dx$

46. $\int \tan^3(x)\ dx$

47. $\int x^2 \cos(x)\ dx$

48. $\int x^2 \sin(x)\ dx$

49. Before doing any calculations, predict which do you expect to be larger; the average value of $\sin(x)$ or of $\sin^2(x)$ on the interval $[0, \pi]$? Then calculate each average to see if your prediction was correct.

50. Find the area between $f(x) = \ln(x)$ and the x–axis for $1 \le x \le C$ when $C = e, 10, 100, 200$.

51. Find the average value of $f(x) = \ln(x)$ on the interval $1 \le x \le C$ when $C = e, 10, 100, 200$.

52. Before doing any calculations, predict which of the following integrals you expect to be the largest?
(a) $\int_0^1 e^x \, dx$ (b) $\int_0^1 x e^x \, dx$ (c) $\int_0^1 x^2 e^x \, dx$. Then calculate the value of each integral.

53. Before doing any calculations, predict which of the following integrals you expect to be the largest?
(a) $\int_1^2 e^x \, dx$ (b) $\int_1^2 x e^x \, dx$ (c) $\int_1^2 x^2 e^x \, dx$. Then calculate the value of each integral.

54. Before doing any calculations, predict which of the following integrals you expect to be the largest?
(a) $\int_0^\pi \sin(x) \, dx$ (b) $\int_0^\pi x \sin(x) \, dx$ (c) $\int_0^\pi x^2 \sin(x) \, dx$. Then calculate the value of each integral.

55. Evaluate $\int_0^C \frac{2}{1 + x^2} \, dx$ for $C = 1, 10, 20,$ and 30. Before doing the calculation, estimate the value of the integral when $C = 40$.

Section 4.8 **Practice Answers**

Practice 1: The integral $\int \frac{1}{25 - x^2} \, dx$ matches table entry 37: $\int \frac{1}{a^2 - x^2} \, dx = \frac{1}{2a} \ln\left| \frac{a+x}{a-x} \right| + C$

when we put $a = 5$, so $\int \frac{1}{25 - x^2} \, dx = \frac{1}{2 \cdot 5} \ln\left| \frac{5 + x}{5 - x} \right| + C$.

Practice 2: The integral $\int \frac{1}{7 - x^2} \, dx$ matches table entry 37: $\int \frac{1}{a^2 - x^2} \, dx = \frac{1}{2a} \ln\left| \frac{a+x}{a-x} \right| + C$

when $a = \sqrt{7}$, so $\int \frac{1}{7 - x^2} \, dx = \frac{1}{2\sqrt{7}} \ln\left| \frac{\sqrt{7} + x}{\sqrt{7} - x} \right| + C$.

Practice 3: $\int \frac{1}{25 - 9x^2} \, dx = \frac{1}{9} \int \frac{1}{(25/9) - x^2} \, dx = \frac{1}{9} \int \frac{1}{(5/3)^2 - x^2} \, dx$

so use #37 with $a = 5/3$. Then

$$\int \frac{1}{25 - 9x^2} \, dx = \frac{1}{9} \int \frac{1}{(5/3)^2 - x^2} \, dx = \frac{1}{9} \frac{1}{2(5/3)} \ln\left| \frac{5/3 + x}{5/3 - x} \right| + C$$

$$= \frac{1}{30} \ln \left| \frac{5/3 + x}{5/3 - x} \right| + C \text{ or } \frac{1}{30} \ln \left| \frac{5 + 3x}{5 - 3x} \right| + C$$

Practice 4: $\int \frac{\cos(x)}{25 - \sin^2(x)} \, dx$. Put $u = \sin(x)$. Then $du = \cos(x)\, dx$ so

$$\int \frac{\cos(x)}{25 - \sin^2(x)} \, dx = \int \frac{1}{25 - u^2} \, du = \frac{1}{2 \cdot 5} \ln \left| \frac{5 + u}{5 - u} \right| + C = \frac{1}{10} \ln \left| \frac{5 + \sin(x)}{5 - \sin(x)} \right| + C$$

Practice 5: We can evaluate $\int \cos^3(7x)\, dx$ by using the recursion formula from the Table with $n = 3$:

20. $\int \cos^n(ax)\, dx = \frac{\cos^{n-1}(ax)\sin(ax)}{na} + \frac{n-1}{n} \int \cos^{n-2}(ax)\, dx$. Then

$$\int \cos^3(7x)\, dx = \frac{\cos^2(7x) \cdot \sin(7x)}{7 \cdot 3} + \frac{2}{3} \int \cos(7x)\, dx$$

$$= \frac{\cos^2(7x) \cdot \sin(7x)}{21} + \frac{2}{3} \cdot \frac{1}{7} \cdot \sin(7x) + C$$