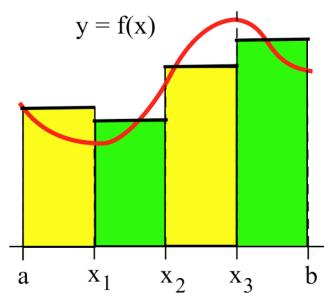
4.9 APPROXIMATING DEFINITE INTEGRALS

The Fundamental Theorem of Calculus tells how to calculate the exact value of a definite integral IF the integrand function is continuous and IF we can find an antiderivative of the integrand. In practice, however, we may need the definite integral of a function defined by a table of measurements or a graph, or of a function which does not have an elementary antiderivative. This section includes several techniques for getting approximate numerical values for definite integrals without using antiderivatives. Mathematically, exact answers are preferable and satisfying, but for most applications, a numerical answer with several digits of accuracy is just as useful.

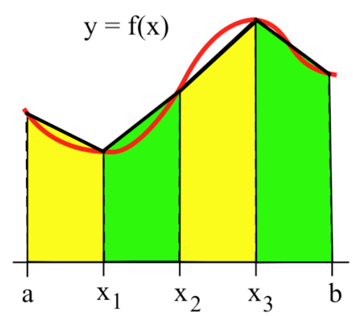
The ideas behind the approximation methods are geometrical and rather simple, but using the methods to get good approximations typically requires lots of arithmetic, something calculators are very good and quick at doing. All of these approximate methods can be easily programmed, and program listings for two of these methods are included after the Practice Answers.



Rectangles: Fitting horizontal lines
Fig. 1

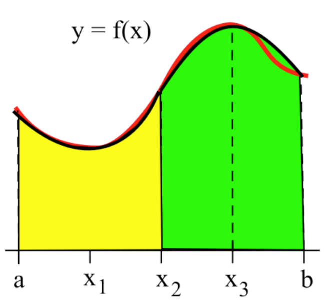
The General Approach

The methods in this section approximate the definite integral of a function f by building "easy" functions close to f and then exactly evaluating the definite integrals of the "easy" functions. If the "easy" functions are close enough to f, then the sum of the definite integrals of the "easy" functions will be close to the definite integral of f. The Left, Right and Midpoint approximations fit horizontal lines to f, the "easy" functions are constant functions, and the approximating regions are rectangles (Fig. 1). The Trapezoidal Rule fits slanted lines to f, the "easy" functions are linear, and the approximating regions are trapezoids (Fig. 2). Finally, Simpson's Rule fits parabolas to f, and the "easy" functions are quadratics (Fig. 3).



Trapezoids: Fitting slanted lines
Fig. 2

The Left and Right approximation rules are simply Riemann sums with the point c_i in each subinterval chosen to be the left or right endpoint of that subinterval. They typically require a large number of computations to get even mediocre approximations and are seldom used in practice. They and the Midpoint rule are discussed at the end of the problem set.



Parabolas: Fitting parabolas
Fig. 3

All of the methods divide the interval [a,b] into **n** **equally–long** subintervals. Each subinterval has length $\mathbf{h = \Delta x_i = \frac{b-a}{n}}$, and the points of the partition are

$x_0 = a$, $x_1 = a + h$, $x_2 = a + 2 \cdot h$, . . . , $\mathbf{x_i = a + i \cdot h}$, . . . , $x_n = a + n \cdot h = a + n(\frac{b-a}{n}) = b$.

Approximating A Definite Integral Using Trapezoids

If the graph of f is curved, then slanted lines typically come closer to the graph of f than horizontal ones do, and the slanted lines lead to trapezoidal regions (Fig. 2).

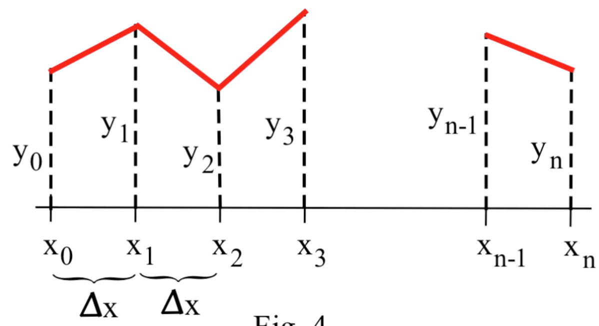


Fig. 4

The area of a trapezoid is (base)•(average height) so the area of the first trapezoid in Fig. 4 is

$$(\Delta x)\bullet\frac{y_0 + y_1}{2} = \frac{\Delta x}{2}(y_0 + y_1) .$$

Similarly, the areas of the other trapezoids are

$$\frac{\Delta x}{2}(y_1 + y_2) , \quad \frac{\Delta x}{2}(y_2 + y_3) , \quad \ldots , \quad \frac{\Delta x}{2}(y_{n-1} + y_n) .$$

The sum of the trapezoidal areas is

$$T_n = \frac{\Delta x}{2}(y_0 + y_1) + \frac{\Delta x}{2}(y_1 + y_2) + \frac{\Delta x}{2}(y_2 + y_3) + \ldots + \frac{\Delta x}{2}(y_{n-1} + y_n)$$

$$= \frac{\Delta x}{2}\{y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\} \text{ or, equivalently,}$$

$$\frac{\Delta x}{2}\left\{ f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n) \right\}.$$

Each $f(x_i)$ value, except the first $(i = 0)$ and the last $(i = n)$, is the right–endpoint height of one trapezoid and the left–endpoint height of the next trapezoid so it shows up in the calculation for two trapezoids and is multiplied by two in the formula for the trapezoidal approximation.

Trapezoidal Approximation Rule

If f is integrable on [a,b], and [a,b] is partitioned into n subintervals of length $h = \frac{b-a}{n}$,

then the Trapezoidal approximation of $\int_a^b f(x)\,dx$ is

$$T_n = \frac{h}{2}\left\{ f(x_0) + \mathbf{2}f(x_1) + \mathbf{2}f(x_2) + \ldots + \mathbf{2}f(x_{n-1}) + f(x_n) \right\}$$

x	f(x)
1.0	4.2
1.5	3.4
2.0	2.8
2.5	3.6
3.0	3.2

Table 1

Example 1: Calculate T_4 , the Trapezoidal approximation of $\int_1^3 f(x)\,dx$, for the function values in Table 1.

Solution: The step size is $h = (b–a)/n = (3–1)/4 = 1/2$. Then

$$T_4 = \frac{h}{2} \left\{ f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4) \right\}$$
$$= \frac{.5}{2} \left\{ 4.2 + 2(3.4) + 2(2.8) + 2(3.6) + (3.2) \right\} = (.25)(27) = 6.75 .$$

Let's see how well the trapezoidal rule approximates an integral whose exact value we know, $\int_1^3 x^2\, dx = 8\frac{2}{3}$.

Example 2: Calculate T_4 , the Trapezoidal approximation of $\int_1^3 x^2\, dx$ for $n = 4$.

Solution: As in Example 1, $h = .5$ and $x_0 = 1, x_1 = 1.5, x_2 = 2, x_3 = 2.5$, and $x_4 = 3$. Then

$$T_4 = \frac{h}{2} \left\{f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right\} = \frac{.5}{2} \left\{f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + f(3)\right\}$$

$$= (.25) \left\{1 + 2(2.25) + 2(4) + 2(6.25) + 9\right\} = 8.75 .$$

Larger values for n give better approximations: $T_{20} = 8.67$ and $T_{100} = 8.6668$.

Practice 1: On a summer day, the level of the pond in Fig. 5 went down 0.1 feet because of evaporation. Use the trapezoidal rule to approximate the surface area of the pond and then calculate how much water evaporated.

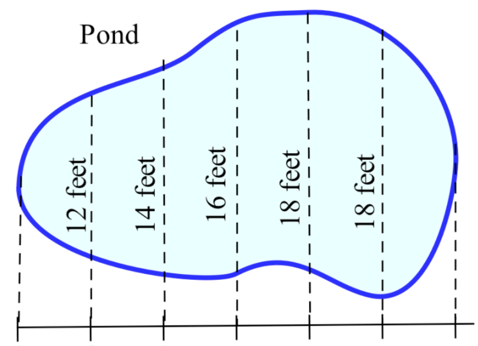
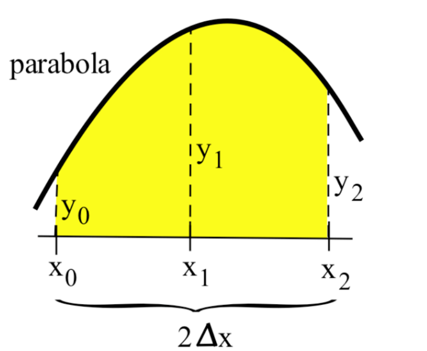


Fig. 5

Approximating A Definite Integral Using Parabolas

If the graph of f is curved, even the slanted lines may not fit the graph of f as closely as we would like, and a large number of subintervals may still be needed with the Trapezoidal rule to get a good approximation of the definite integral. Curves typically fit the graph of f better than straight lines, and the easiest nonlinear curves are parabolas.

$$\text{Area} = (2\Delta x)\left\{\frac{y_0 + 4y_1 + y_2}{6}\right\}$$



Fig/ 6

Three points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ are needed to determine the equation of a parabola, and the area under a parabolic region with evenly spaced x_i values (Fig. 6) is

$$(2\Delta x)\cdot\left\{ \frac{y_0 + 4y_1 + y_2}{6} \right\} = \frac{\Delta x}{3} \cdot\{ y_0 + 4y_1 + y_2 \}.$$

(The steps to verify this formula for parabolas are outlined in problem 32.)

Taking the subintervals in pairs, the areas of the other parabolic regions are

$\frac{\Delta x}{3} \bullet \{ y_2 + 4y_3 + y_4 \}$, $\frac{\Delta x}{3} \bullet \{ y_4 + 4y_5 + y_6 \}, \ldots,$

$$\frac{\Delta x}{3} \bullet \{ y_{n-2} + 4y_{n-1} + y_n \}$$

so the sum of the parabolic areas (Fig. 7) is

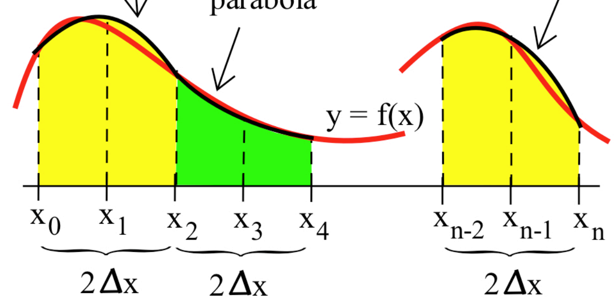


Fig. 7

$$S_n = \frac{\Delta x}{3} \bullet \{ y_0 + 4y_1 + y_2 \}$$

$$+ \frac{\Delta x}{3} \bullet \{ y_2 + 4y_3 + y_4 \}$$

$$+ \ldots + \frac{\Delta x}{3} \bullet \{ y_{n-2} + 4y_{n-1} + y_n \}$$

$$= \frac{\Delta x}{3} \bullet \{ y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n \}$$ or, equivalently,

$$\frac{\Delta x}{3} \left\{ f(x_0) + \mathbf{4}f(x_1) + \mathbf{2}f(x_2) + \mathbf{4}f(x_3) + \mathbf{2}f(x_4) + \ldots + \mathbf{2}f(x_{n-2}) + \mathbf{4}f(x_{n-1}) + f(x_n) \right\}.$$

In order to use **pairs** of subintervals, the number n of subintervals must be **even**. The coefficient pattern for a single parabola is 1–4–1, but when we put several parabolas next to each other, they share some edges and the pattern becomes 1–4–**2**–4–**2**– . . . –**2**–4–1 with the shared edges getting counted twice.

Parabolic Approximation Rule (Simpson's Rule)

If f is integrable on [a,b], and [a,b] is partitioned into an **even number** n of subintervals of length $h = \frac{b-a}{n}$, then the Parabolic approximation of $\int_a^b f(x)\,dx$ is

$$S_n = \frac{h}{3} \bullet \left\{ f(x_0) + \mathbf{4}f(x_1) + \mathbf{2}f(x_2) + \mathbf{4}f(x_3) + \mathbf{2}f(x_4) + \ldots + \mathbf{4}f(x_{n-1}) + f(x_n) \right\}$$

Example 3: Calculate S_4, Simpson's parabolic approximation of $\int_1^3 f(x)\,dx$, for the function in Table 1.

Solution: The step size is $h = (b-a)/n = (3-1)/4 = 1/2$. Then

$$S_4 = \frac{h}{3} \left\{ f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4) \right\}$$

$$= \frac{1/2}{3} \left\{ 4.2 + 4(3.4) + 2(2.8) + 4(3.6) + (3.2) \right\} = \frac{1}{6}(41) \approx 6.833 .$$

Example 4: Calculate S_4, Simpson's parabolic approximation of $\int_1^3 2^x\,dx$ for n = 4.

Solution: As in the previous Examples, h = (b–a)/n = .5 and $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, $x_3 = 2.5$, and $x_4 = 3$.

$$S_4 = \frac{h}{3}\left\{f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right\} = \frac{.5}{3}\left\{f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3)\right\}$$

$$= \left(\frac{1}{6}\right)\left\{2 + 4(2.828427) + 2(4) + 4(5.656854) + 8\right\} = \frac{1}{6}(51.941124) = 8.656854 .$$

Larger values for n give better approximations: $S_{20} = 8.656171$ and $S_{100} = 8.656170$.

Practice 2: Use Simpson's Rule to estimate the area of the pond in Fig. 5.

Which Method Is Best?

The hardest and slowest part of these approximations, whether by hand or by computer, is the evaluation of the function at the x_i values. For n subintervals, all of the methods require about the same number of function evaluations. Table 2 illustrates how closely each method approximates the definite integral of 1/x using several values of n. The values in Table 2 also show how quickly the actual error shrinks as the values of n increase: just doubling n from 4 to 8 cut the actual error of the parabolic approximation of this definite integral by a factor of 9 — a good reward for our extra work. The rest of this section discusses "error bounds" of the approximations so we can know how close our approximation is to the exact value of the integral even if we don't know the exact value.

Table 2: Approximating $\int_1^5 \frac{1}{x}\,dx = \ln 5 = 1.609437912$

Using n=4 (h= (5–1)/4 = 1)

method	approximation	M	error bound	actual error
T_4	**1.6833333**	**2**	**.6666666**	**.07389542**
S_4	**1.6222222**	**24**	**.5333333**	**.01278431**
L_4	2.083333333	1	2	.47389542
R_4	1.283333333	1	2	.32610458
M_4	1.574603175	2	.33333333	.03483474

Using n=8 (h= (5–1)/8 = 1/2)

method	approximation	M	error bound	actual error
T_8	**1.628968254**	**2**	**.1666666**	**.01953034**
S_8	**1.610846561**	**24**	**.0333333**	**.00140865**
L_8	1.828968254	1	1	.21953034
R_8	1.428968254	1	1	.18046966
M_8	1.599844394	2	.08333333	.00959352

Using n=20 (h= (5–1)/20 = 1/5)

method	approximation	M	error bound	actual error
T_{20}	**1.612624844**	**2**	**.0266667**	**.00318693**
S_{20}	**1.609486789**	**24**	**.0008533**	**.00004888**
L_{20}	1.692624844	1	.4	.08318693
R_{20}	1.532624844	1	.4	.07681307
M_{20}	1.607849324	2	.01333333	.00158859

How Good Are the Approximations?

The approximation rules are valuable by themselves, but they are particularly useful because there are "error bound" formulas that guarantee how close the approximations are to the exact values of the integrals. It is useful to know that an integral is "about 3.7," but we can have more confidence if we know that the integral is "within .0001 of 3.7 ." Then we can decide if our approximation is good enough for the job at hand or if we need to improve it. The formulas for the error bounds can also be solved to determine how many subintervals are needed to guarantee that our approximation is within some specified distance of the exact answer. There is no reason to use 1000 subintervals if 18 will give the needed accuracy. Unfortunately, the formulas for the error bounds require information about the derivatives of the integrands, so we can not use these formulas to determine error bounds for the approximations of integrals of functions defined by tables of values.

Error Bound for Trapezoidal Approximation

If the second derivative of f is continuous on [a,b] and $M_2 \geq \{$ maximum of $| f''(x) |$ on [a,b] $\}$,

then the "error" of the T_n approximation is $\left| \int_a^b f(x)\,dx - T_n \right| \leq \frac{(b-a)^3}{12\,n^2} \cdot M_2$ = "error bound."

The "error bound" formula $\mathbf{\frac{(b-a)^3}{12\,n^2} \cdot M_2}$ for the Trapezoidal approximation is a "guarantee:" the actual error is guaranteed to be no larger than the error bound. In fact, the actual error is usually much smaller than the error bound. The word "error" does not indicate a mistake, it means the deviation or distance from the exact answer.

Example 5: We can be certain that the T_{10} approximation of $\int_0^2 \sin(x^2)\,dx$ is **within** what distance of the exact value of the integral?

Solution: $b - a = 2$, $n = 10$, $f(x) = \sin(x^2)$, and $f''(x) = -4x^2 \cdot \sin(x^2) + 2 \cdot \cos(x^2)$ is continuous on $[0, 2]$. The graph of $f''(x)$ is given in Fig. 8. Even though we may not know the exact maximum value M_2 of $| f''(x) |$ on $[0, 2]$, it is clear from the graph that $M_2 \leq 11$. Then

$$\text{"actual error"} \leq \text{"error bound"} = \frac{(b-a)^3}{12\,n^2} \cdot M_2$$

$$= \frac{(2)^3}{12\,(10)^2} \cdot (11) = \frac{88}{1200} < 0.074$$

so we can be certain that our T_{10} approximation of the definite integral is within .074 of the exact value:

$$T_{10} - 0.074 \leq \int_0^2 \sin(x^2)\,dx \leq T_{10} + 0.074 .$$

$T_{10} = 0.7959247$, so we can be certain that the value of the integral is between 0.722 and 0.870 .

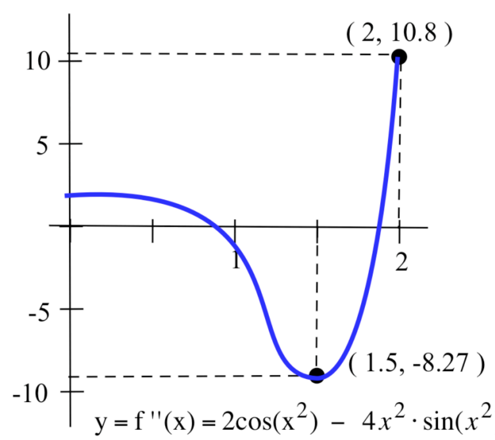


Fig. 8

Practice 3: Find an error bound for the T_{12} approximation of $\int_2^5 \frac{1}{x}\,dx$.

Example 6: How large must n be to be certain that T_n is within 0.001 of $\int_0^2 \sin(x^2)\,dx$?

Solution: The "allowable error" of 0.001 is given, and we are asked to find n. From Example 5 we know that $M_2 \le 11$, so we want the error bound to be less than the allowable error of 0.001. Then $.001 \ge$ "error bound" $= \frac{(2)^3}{12\,n^2}\cdot(11) = \frac{88}{12}\,\frac{1}{n^2} = \frac{22}{3n^2}$. Solving for n, we have $n^2 \ge \frac{22}{0.003} > 7334$ so $n \ge \sqrt{7334} \approx 85.6$. Since n must be an integer, we can be certain that $\mathbf{T_{86}}$ is within 0.001 of $\int_0^2 \sin(x^2)\,dx$. $T_{86} \approx 0.80465$, so we can be certain that the exact value of the integral is between 0.80**3**65 and 0.80**5**65 . As is usually the case, T_{86} is even closer than 0.001 to the exact value, $|\,T_{86} - \text{exact value}\,| \approx 0.00012$.

Practice 4: How large must n be to be certain that T_n is within 0.001 of $\int_2^5 \frac{1}{x}\,dx$?

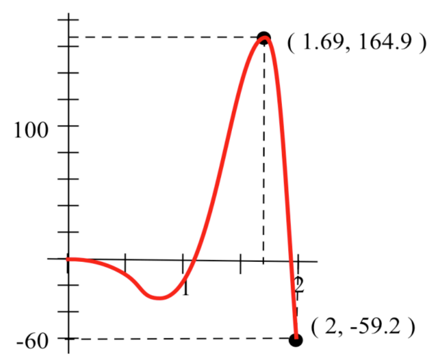
Error Bound for Simpson's Parabolic Approximation

If the fourth derivative of f is continuous on [a,b] , and $M_4 \ge$ {maximum of $|\,f^{(4)}(x)\,|$ on [a,b]},

then the "error" of the S_n approximation is $\left|\int_a^b f(x)\,dx - S_n\right| \le \frac{(b-a)^5}{180\,n^4}\cdot M_4$ = "error bound."

Example 7: Find an error bound for the S_{10} approximation of $\int_0^2 \sin(x^2)\,dx$.

Solution: $b - a = 2$, $n = 10$, $f(x) = \sin(x^2)$, and $f^{(4)}(x) = (16x^4 - 12)\sin(x^2) - 48x^2\cos(x^2)$ is continuous on [0, 2] . From Fig. 9, the graph of $f^{(4)}(x)$ on [0, 2] , we know that $M_4 \le 165$. Then



$y = f^{(4)}(x) = (16x^4 - 12)\sin(x^2) - 48x^2\cos(x^2)$

Fig. 9

$$\text{"error"} \le \frac{(b-a)^5}{180\,n^4}\cdot M_4 = \frac{(2)^5}{180\,(10)^4}\,(165) = \frac{5280}{1800000} < 0.003$$

so we can be certain that our S_{10} approximation of the definite integral is within 0.003 of the exact value:

$$S_{10} - 0.003 \le \int_0^2 \sin(x^2)\,dx \le S_{10} + 0.003 .$$

$S_{10} = 0.80537615$, so we are certain that the exact value of the integral is between 0.80**2**37615 and 0.80**8**37615 . Notice that we got a much narrower guarantee using S_{10} compared to using T_{10} to approximate the integral .

Example 8: How large must n be to be certain that S_n is within 0.001 of $\int_0^2 \sin(x^2)\,dx$?

Solution: We are given an "allowable error" of 0.001 and are asked to find n. From Fig. 9 we know that $M_4 \le 165$, so we want the error bound to be less than the allowable error of 0.001. Then

$0.001 \ge \text{"error bound"} = \frac{(2)^5}{180\, n^4}(165) = \frac{5280}{180\, n^4}$. Solving for n, we have

$n^4 \ge \frac{5280}{(.001)180} \ge 29{,}333.34$ so $n \ge \sqrt[4]{29333.34} \approx 13.08$. Since n must be an **even** integer, we can take n = 14 and be certain that S_{14} is **within** 0.001 of $\int_0^2 \sin(x^2)\,dx$. In fact, $S_{14} = 0.8049239$ is even closer than 0.001 to the exact value, $|\, S_{14} - \text{exact value} \,| \approx 0.00015$.

A variety of other methods for approximating definite integrals can be found in most books on Numerical Analysis. Definite integrals occur often in applied problems, and these approximation methods can get us the numerical answers we need even if we can't find an antiderivative of the integrand. If you have a programmable calculator, program Simpson's rule. It will be useful in Chapter 5.

PROBLEMS

For problems 1 and 2, use the values given in Table 3 to approximate the value of $\int_2^6 f(x)\,dx$.

1. Calculate T_4 and S_4 .

2. Calculate T_8 and S_8 .

For problems 3 and 4, use the values given in Table 4 to approximate the value of $\int_{-3}^{1} g(x)\,dx$.

3. Calculate T_8 and S_8 .

4. Calculate T_4 and S_4 .

x	f(x)
2.0	2.1
2.5	2.7
3.0	3.8
3.5	2.3
4.0	0.3
4.5	–1.8
5.0	–0.9
5.5	0.5
6.0	2.2

Table 3

x	g(x)
–3.0	4.2
–2.5	1.8
–2.0	0.7
–1.5	1.5
–1.0	3.4
–0.5	4.3
0	3.5
0.5	–0.3
1.0	–4.6

Table 4

For problems 5 – 10, calculate (a) T_4, (b) S_4, and (c) the exact value of the integral.

5. $\displaystyle\int_1^3 x \, dx$

6. $\displaystyle\int_0^2 (1 - x) \, dx$

7. $\displaystyle\int_{-1}^1 x^2 \, dx$

8. $\displaystyle\int_2^6 \frac{1}{x} \, dx$

9. $\displaystyle\int_0^\pi \sin(x) \, dx$

10. $\displaystyle\int_0^1 \sqrt{x} \, dx$

For problems 11 – 16, calculate (a) T_6 and (b) S_6.

11. $\displaystyle\int_0^2 \frac{1}{1 + x^3} \, dx$

12. $\displaystyle\int_1^2 2^x \, dx$

13. $\displaystyle\int_{-1}^1 \sqrt{4 - x^2} \, dx$

14. $\displaystyle\int_0^1 e^{-x^2} \, dx$

15. $\displaystyle\int_1^4 \frac{\sin(x)}{x} \, dx$

16. $\displaystyle\int_0^1 \sqrt{1 + \sin(x)} \, dx$

For problems 17 – 23, calculate (a) the error bound for T_4, (b) the error bound for S_4, (c) the value of n so the error bound for T_n is less than 0.001, and (d) the value of n so the error bound for S_n is less than 0.001.

17. $\displaystyle\int_1^3 x \, dx$

18. $\displaystyle\int_0^2 (1 - x) \, dx$

19. $\displaystyle\int_{-1}^1 x^3 \, dx$

20. $\displaystyle\int_2^6 \frac{1}{x} \, dx$

21. $\displaystyle\int_0^\pi \sin(x) \, dx$

22. $\displaystyle\int_1^4 \sqrt{x} \, dx$

23. A friend has asked you to help calculate the area of a piece of land located between a river and a road (Fig. 10). Estimate the area.

24. Estimate the area of the island in Fig. 11.

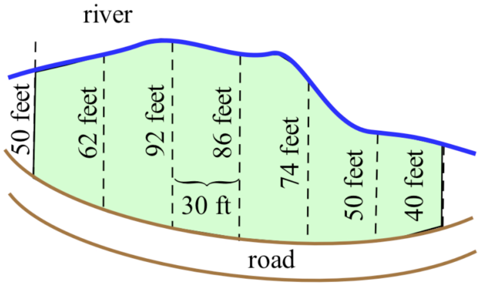


Fig. 10

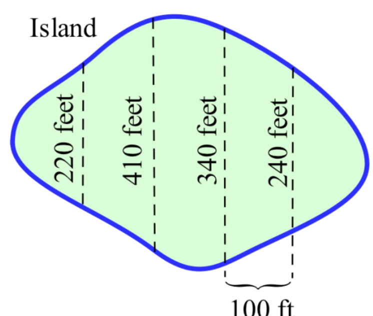


Fig. 11

25. The average depth of the reservoir in Fig. 12 is 22 feet. Estimate the amount of water in the reservoir.

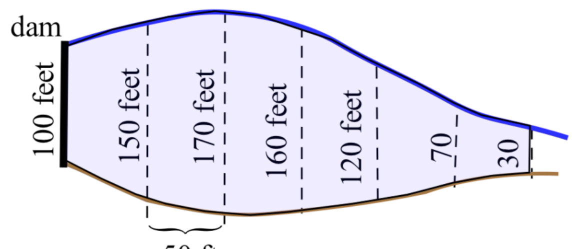


Fig. 12

26. Table 5 shows the speedometer readings for a car at one minute intervals. Estimate how far the car traveled (a) during the first 5 minutes of the trip and (b) during the first 10 minutes of the trip.

Table 5: Time (minutes) and Velocity (feet/minute) for a car.

Time	0	1	2	3	4	5	6	7	8	9	10
Velocity	0	2000	3000	5000	5000	6000	5200	4400	3000	2000	1200

27. Table 6 shows the speed of a jogger at one minute intervals. Estimate how far the jogger ran during the workout.

Table 6: Time (minutes) and Velocity (feet/minute) for a jogger.

Time	0	1	2	3	4	5	6	7	8	9	10
Velocity	0	420	540	300	500	580	520	440	360	260	180

28. Use the error formula for Simpson's rule to show that the parabolic approximation is the exact value of the integral if the integrand is a polynomial of degree 3 or less, $ax^3 + bx^2 + cx + d$.

29. A trapezoidal region (Fig. 13) with base b and heights h_1 and h_2 (assume $h_1 \le h_2$) can be cut into a rectangle with base b and height h_1 and a triangle with base b and height $h_1 - h_2$. Show that the sum of the area of the rectangle and the area of the triangle is $b \cdot \{ \frac{h_1 + h_2}{2} \}$.

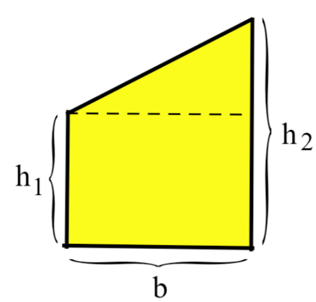


Fig. 13

30. Let f(m) be the minimum value of f on the interval $[x_0, x_1]$. Let f(M) be the maximum value of f on $[x_0, x_1]$. And let $h = x_1 - x_0$. Show that the trapezoidal value, $h \cdot \{ \frac{f(x_0) + f(x_1)}{2} \}$, is between $h \cdot f(m)$ and $h \cdot f(M)$. From this result, it can be shown that the trapezoidal approximation is between the lower and upper Riemann sums for f. Since the limit (as h approaches 0) of these Riemann sums is the definite integral of f , we can conclude that the limit of the trapezoidal sums is the value of the definite integral.

31. Let f(m) be the minimum value of f on the interval $[x_0, x_2]$. Let f(M) be the maximum of f on $[x_0, x_2]$. And let $h = x_1 - x_0 = x_2 - x_1$. Show that the parabolic value,

$2h \cdot \left\{ \frac{f(x_0) + 4f(x_1) + f(x_2)}{6} \right\}$ is between $2h \cdot f(m)$ and $2h \cdot f(M)$. From this result, it can be shown that the parabolic approximation is between the lower and upper Riemann sums for f. Since the limit (as h approaches 0) of these Riemann sums is the definite integral of f, we can conclude that the limit of the parabolic sums is the value of the definite integral.

32. This problem leads you through the steps to show that the area under a parabolic region with evenly spaced x_i values ($x_0 = m-h$, $x_1 = m$, $x_2 = m+h$) as in Fig. 14 is

$$\frac{h}{3} \cdot \{ f(x_0) + 4f(x_1) + f(x_2) \} = \frac{h}{3} \cdot \{ y_0 + 4y_1 + y_2 \}$$

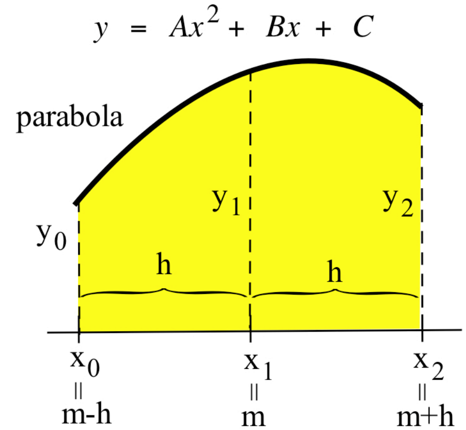


Fig. 14

(a) For $f(x) = Ax^2 + Bx + C$, a parabola, verify that

$$\int_{m-h}^{m+h} f(x)\,dx = \frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx \Big|_{x=m-h}^{x=m+h}$$

$$= \mathbf{2Am^2h + \frac{2}{3}Ah^3 + 2Bmh + 2Ch}.$$

(b) Expand $y_0 = f(m-h) = A(m-h)^2 + B(m-h) + C$,

$y_1 = f(m) = Am^2 + Bm + C$, and

$y_2 = f(m+h) = A(m+h)^2 + B(m+h) + C$. Then verify that

$$\frac{h}{3} \cdot \{ y_0 + 4y_1 + y_2 \} = 2h \left\{ \frac{f(m-h) + 4f(m) + f(m+h)}{6} \right\}$$

$$= \mathbf{2Am^2h + \frac{2}{3}Ah^3 + 2Bmh + 2Ch}.$$

(c) Compare the results of parts (a) and (b) to conclude that for any parabola $f(x) = Ax^2 + Bx + C$,

$$\int_{m-h}^{m+h} f(x)\,dx = 2h \left\{ \frac{f(m-h) + 4f(m) + f(m+h)}{6} \right\} = \frac{h}{3} \cdot \{ y_0 + 4y_1 + y_2 \}.$$

Rectangular Approximations: Left Endpoint, Right Endpoint, and Midpoint Rules

The rectangular approximation methods fit horizontal lines to the integrand. The approximating regions are rectangles, and the sum of the areas of the rectangular regions is a Riemann sum. The Left and Right Endpoint Rules are easy to understand and use, but they typically require a very large number of subintervals to provide good approximations of a definite integral. The Midpoint Rule uses the value of the function at the midpoint of each subinterval. If these midpoint values of f are available, for example when f is given by a formula, then the Midpoint Rule is often more efficient than the Trapezoidal rule. The rectangular approximation rules and their error bounds are given below.

Left endpoint: $L_n = h \cdot \{ f(x_0) + f(x_1) + f(x_2) + \ldots + f(x_{n-1}) \}$

Right endpoint: $R_n = h \cdot \{ f(x_1) + f(x_2) + f(x_3) + \ldots + f(x_n) \}$

Midpoint Rule: $M_n = h \cdot \{ f(A) + f(A + h) + f(A + 2h) + \ldots + f(A + (n-1)h) \}$ where $A = x_0 + \frac{h}{2}$.

(The points A, A + h, A + 2h, . . . are the **midpoints** of the subintervals.)

The "error bound" for L_n and R_n is $\frac{(b-a)^2}{2n} \cdot M_1$ where $M_1 \geq$ {maximum of $| f'(x) |$ on [a,b]}.

The "error bound" for M_n is $\frac{(b-a)^3}{\mathbf{24}\, n^2} \cdot M_2$ where $M_2 \geq$ {maximum of $| f''(x) |$ on [a,b]}. This is half the error bound of T_n , the trapezoidal approximation.

For problems 33 – 38, calculate (a) L_4 , (b) R_4 , (c) M_4 , and (d) the exact value of the integral.

33. $\int_1^3 x \, dx$

34. $\int_0^2 (1 - x) \, dx$

35. $\int_{-1}^1 x^2 \, dx$

36. $\int_2^6 \frac{1}{x} \, dx$

37. $\int_0^\pi \sin(x) \, dx$

38. $\int_0^1 \sqrt{x} \, dx$

39. Show that the trapezoidal approximation is the average of the left and right endpoint approximations: $T_n = (L_n + R_n)/2$.

40. Which endpoint rule will give a better approximation of $\int_a^b f(x) \, dx$ if f is concave up on [a, b]?

Calculator Problems

The following definite integrals arise in applications, but they do not have easy antiderivatives. Use Simpson's Rule with n = 10 and n = 40 to approximate their values. (Is S_{40} very different from S_{10} ?)

41. $\int_{-1}^2 \sqrt{1 + 4 \cdot x^2} \, dx$. This is the length of the curve $y = x^2$ from (–1, 1) to (2, 4).

42. $\int_0^\pi \sqrt{1 + \cos^2(x)} \, dx$. This is the length of one arch of the curve y = sin(x).

43. $\displaystyle\int_0^{2\pi} \sqrt{16\cdot\sin^2(x) + 9\cdot\cos^2(x)}\ dx$.

This is the length of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

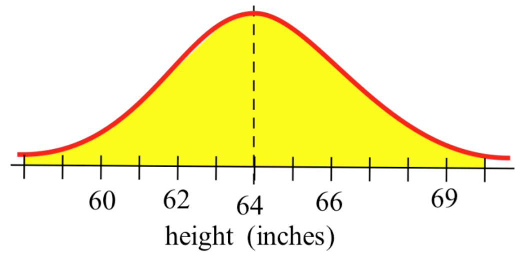


Fig. 15

44. $\displaystyle\frac{100}{\sqrt{2\pi}} \int_{60}^{69} \mathrm{EXP}\left(-\{(x-64)/2.5\}^2/2 \right) dx$. $\mathrm{EXP}(x) = e^x$.

This is the percentage of adult females who are between 60 and 69 inches tall (Fig. 15). Approximate the value of this integral.

45. Approximate the percentage of adult females who are between 61 and 64 inches tall.

Section 4.9 Practice Answers

Practice 1: Using the Trapezoidal rule to approximate the surface area of the pond in Fig. 5,

$$T \approx \frac{5\text{ feet}}{2} \cdot \{ 0 + 2\cdot12 + 2\cdot14 + 2\cdot16 + 2\cdot18 + 2\cdot18 + 0 \text{ feet} \} = 390 \text{ ft}^2.$$

Then volume = (surface area)(depth) $\approx (390\text{ ft}^2)(0.1\text{ ft}) = 39\text{ ft}^3$.

Practice 2: Using Simpson's Rule to estimate the area of the pond in Fig. 5,

$$S \approx \frac{5\text{ feet}}{3} \{ 0 + 4\cdot12 + 2\cdot14 + 4\cdot16 + 2\cdot18 + 4\cdot18 + 0 \text{ feet}\} = 413.3 \text{ ft}^2.$$

Practice 3: $f(x) = \frac{1}{x}$, $b - a = 3$, $n = 12$, $f''(x) = \frac{2}{x^3}$. On the interval [2,5], $| f''(x) | = | \frac{2}{x^3} | \le \frac{2}{2^3} = \frac{1}{4}$

so we can take $M_2 = \frac{1}{4}$. Then $| \text{error} | \le \frac{(b-a)^3}{12n^2} \cdot M_2 = \frac{3^3}{12(12)^2} \frac{1}{4} = \frac{27}{6912} \approx$ **0.004** .

Practice 4: "error bound" $= \frac{(b-a)^3}{12n^2} \cdot M_2 = \frac{(3)^3}{12n^2} \cdot \frac{1}{4} = \frac{27}{48n^2}$. Setting this "error bound" equal to 0.001

and solving for n, we get $n = \sqrt{\frac{27}{48\cdot(0.001)}} = \sqrt{562.5} \approx 23.7$. Put $n = 24$.

We can be certain that T_{24} is within 0.001 of the exact value of the integral. (We can not guarentee that T_{23} is within 0.001 of the exact value of the integral, but it probably is.)