5.1 VOLUMES OF SOLIDS

The last chapter emphasized a geometric interpretation of definite integrals as "areas" in two dimensions. This section emphasizes another geometrical use of integration, calculating volumes of solid three–dimensional objects such as those shown in Fig. 1. Our basic approach is to cut the whole solid into thin "slices" whose volumes can be approximated, add the volumes of these "slices" together (a Riemann sum), and finally obtain an exact answer by taking a limit of the sums to get a definite integral.

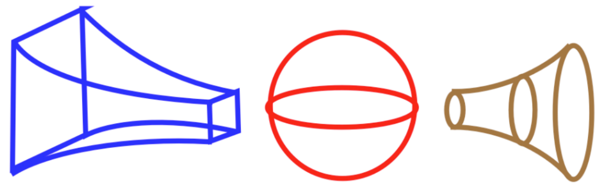


Fig. 1

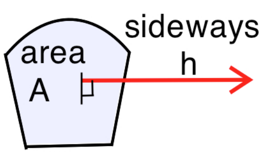


Fig. 2

The Building Blocks: Right Solids

A **right solid** is a three–dimensional shape swept out by moving a planar region A some distance h along a line perpendicular to the plane of A (Fig. 2). The region A is called a **face** of the solid, and the word "right" is used to indicate that the movement is along a line perpendicular, at a right angle, to the plane of A. Two parallel **cuts** produce one **slice** with two **faces** (Fig. 3): a slice has volume, and a face has area.

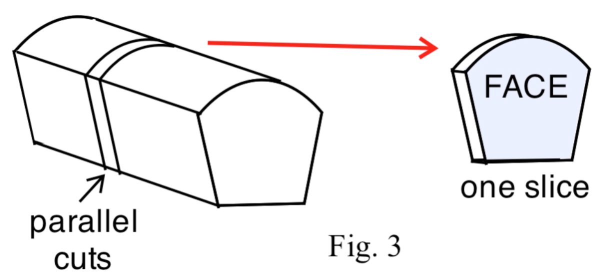


Fig. 3

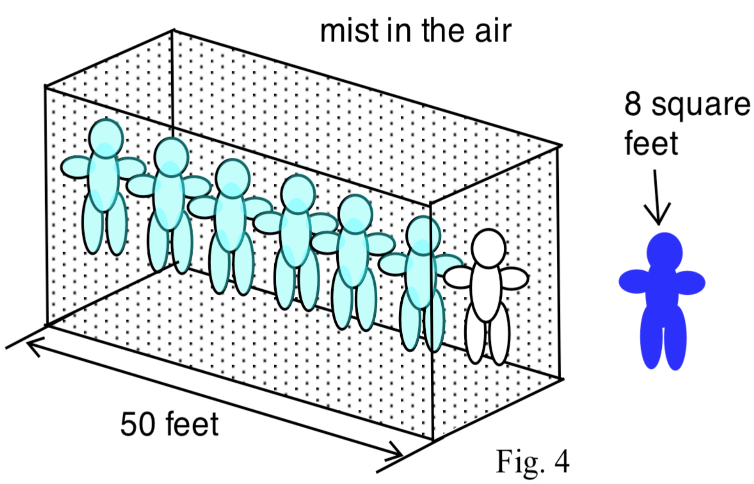


Fig. 4

Example 1: Suppose there is a fine, uniform mist in the air, and every cubic foot of mist contains 0.02 ounces of water droplets. If you run 50 feet in a straight line through this mist, how wet do you get? Assume that the front (or a cross section) of your body has an area of 8 square feet.

Solution: As you run, the front of your body sweeps out a "tunnel" through the mist (Fig. 4). The volume of the tunnel is the area of the front of your body multiplied by the length of the tunnel: volume = $(8\ \text{ft}^2)(50\ \text{ft}) = 400\ \text{ft}^3$. Since each cubic foot of mist held 0.02 ounces of water which is now on you, you swept out a total of $(400\ \text{ft}^3)\cdot(0.02\ \text{oz/ft}^3) = 8$ ounces of water. If the water was truly suspended and not falling, would it matter how fast you ran?

If A is a rectangle (Fig. 5), then the "right solid" formed by moving A along the line is a 3–dimensional solid box B. The volume of B is

(area of A)•(distance along the line) = (base)•(height)•(width).

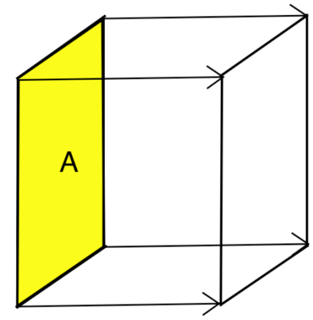


Fig. 5: Solid box

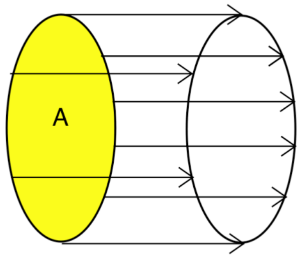


Fig. 6: Solid cylinder

If A is a circle with radius r meters (Fig. 6), then the "right solid" formed by moving A along the line h meters is a right circular cylinder with volume equal to

$$\{\text{area of A}\}\cdot\{\text{distance along the line}\}$$
$$= \{\, \pi\,(r \text{ ft})^2 \,\}\cdot\{h \text{ ft}\} = \{\, \pi\cdot r^2 \text{ ft}^2 \,\}\cdot\{\text{ h ft }\} = \pi\, r^2 h \text{ ft}^3 .$$

If we cut a right solid perpendicular to its axis (like cutting a loaf of bread), then each face (cross section) has the same two–dimensional shape and area. In general, if a 3–dimensional right solid B is formed by moving a 2–dimensional shape A along a line perpendicular to A, then the volume of B is *defined* to be

volume of B = (area of A)•(distance moved along the line perpendicular to A).

The volume of each right solid in Fig. 7 is (area of the base)•(height).

Example 2: Calculate the volumes of the right solids in Fig. 7.

Solution: (a) This cylinder is formed by moving the circular base (area = $\pi r^2 = 9\pi \text{ in}^2$) along a line perpendicular to the base for 4 inches, so the volume is $(9\pi \text{ in}^2)\cdot(4 \text{ in}) = 36\pi \text{ in}^3$.

(b) volume = (base area)•(distance along the line) = $(8 \text{ m}^2)\cdot(3 \text{ m}) = 24 \text{ m}^3$.

(c) This shape is composed to two easy right solids with volumes $V_1 = (\pi 3^2)\cdot(2) = 18\pi \text{ cm}^3$ and $V_2 = (6)(1)\cdot(2) = 12 \text{ cm}^3$, so the total volume is $(18\pi + 12) \text{ cm}^3$ or approximately 68.5 cm^3 .

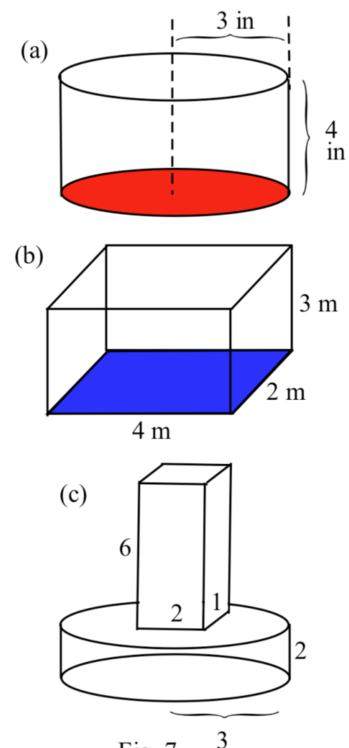


Fig. 7

Practice 1: Calculate the volumes of the right solids in Fig. 8.

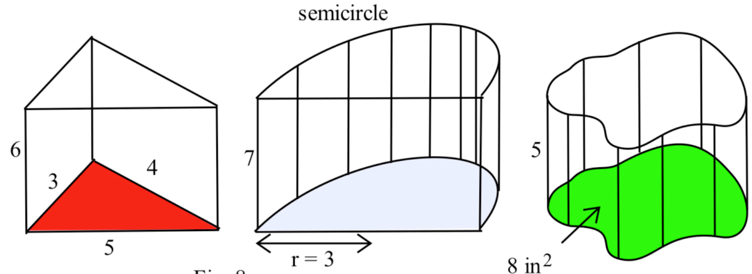


Fig. 8

Volumes of General Solids

A general solid can be cut into slices which are almost right solids. An individual slice may not be exactly a right solid since its cross sections may have different areas. However, if the cuts are close together, then the cross sectional areas will not change much within a single slice. Each slice will be almost a right solid, and its volume will be almost the volume of a right solid.

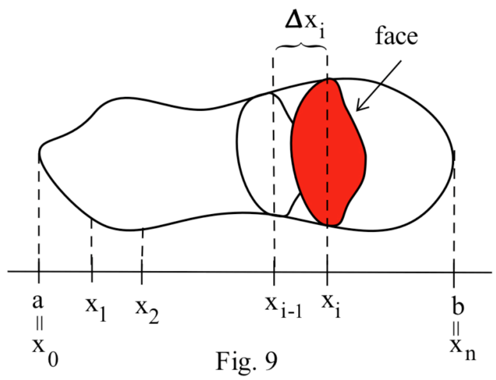


Fig. 9

Suppose an x–axis is positioned below the solid shape (Fig. 9), and let A(x) be the **area of the face** formed when the solid is cut at x perpendicular to the x–axis. If P = { x_0=a, x_1, x_2, . . . , x_n = b} is a partition of [a,b], and the solid is cut at each x_i, then each slice of the solid is almost a right solid, and the volume of each slice is approximately

(area of a face of the slice)•(thickness of the slice) ≈ $A(x_i)\ \Delta x_i$.

The total volume V of the solid is approximately the sum of the volumes of the slices:

$$V = \sum \{\text{volume of each slice}\} \approx \sum A(x_i)\Delta x_i \quad \text{which is a Riemann sum.}$$

The limit, as the mesh of the partition approaches 0 (taking thinner and thinner slices), of the Riemann sum is the definite integral of A(x):

$$V \approx \sum A(x_i)\Delta x_i \longrightarrow \int_a^b A(x)\,dx \quad .$$

Volume By Slices Formula

If S is a solid and A(x) is the area of the face formed by a cut at x and perpendicular to the x–axis ,

then the volume V of the part of S above the interval [a,b] is $V = \int_a^b A(x)\,dx$.

If S is a solid (Fig. 10), and A(y) is the area of a face formed by a cut at y perpendicular to the ***y*–axis**, then the volume of a slice with thickness Δy_i is approximately $A(y_i) \cdot \Delta y_i$. The volume of the part of S between cuts at c and d on the y–axis is

$$V = \int_c^d A(y)\ dy\ .$$

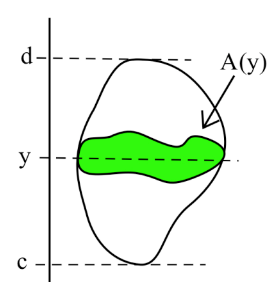


Fig. 10

Example 3: For the solid in Fig. 11, the face formed by a cut at x is a rectangle with a base of 2 inches and a height of cos(x) inches. (a) Write a formula for the approximate volume of the slice between x_{i-1} and x_i . (b) Write and evaluate an integral for the volume of the solid for x between 0 and π/2.

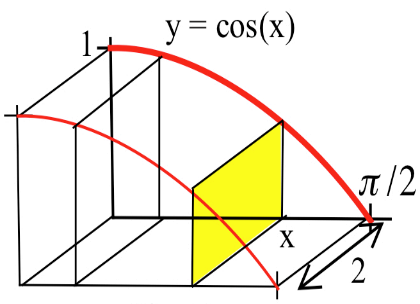


Fig. 11

Solution:

(a) The volume of the slice ≈ (area of the face)•(thickness)

= (base)•(height)•(thickness)

= (2 in)•($\cos(x_i)$ in)•(Δx_i in)

= $2\cos(x_i)\ \Delta x_i$ in^3 .

(b) Volume = $\int_a^b A(x)\,dx = \int_0^{\pi/2} 2\cos(x)\,dx = 2\sin(x)\Big|_0^{\pi/2}$

$= 2\sin(\pi/2) - 2\sin(0) = 2$ in^3 .

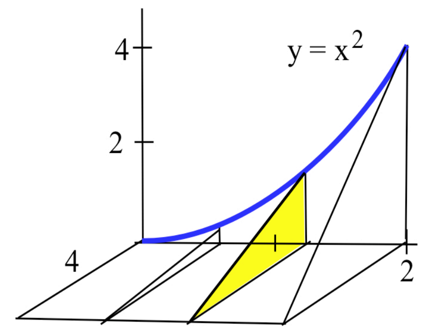


Fig. 12

Practice 2: For the solid in Fig. 12, the face formed by a cut at x is a triangle with a base of 4 inches and a height of x^2 inches.
(a) Write a formula for the approximate volume of the slice between x_{i-1} and x_i . (b) Write and evaluate an integral for the volume of the solid for x between 1 and 2.

Example 4: For the solid in Fig. 13, each face formed by a cut at x is a circle with <u>diameter</u> $\sqrt{x}$.

(a) Write a formula for the approximate volume of the slice between x_{i-1} and x_i .

(b) Write and evaluate an integral for the volume of the solid for x between 1 and 4.

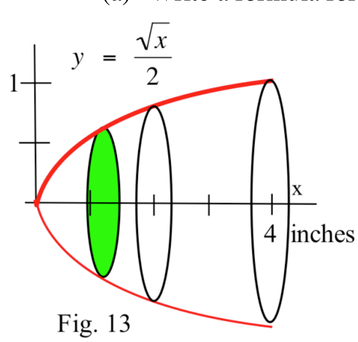


Fig. 13

Solution: (a) Each face is a circle with diameter $\sqrt{x_i}$, and the area of the circle is

$A(x_i) = \pi\cdot(\text{radius})^2 = \pi(1/2\ \text{diameter})^2 = \pi(\ 1/2\sqrt{x_i}\ )^2 = \pi x_i/4$.

The volume of the slice ≈ (area of the face)•(thickness) = $(\pi x_i/4)\cdot(\Delta x_i)$

(b) Volume = $\int_a^b A(x)\,dx = \int_1^4 \frac{\pi x}{4}\,dx = \frac{\pi}{4}\cdot\frac{x^2}{2}\Big|_1^4 = \frac{\pi}{4}\frac{16}{2} - \frac{\pi}{4}\frac{1}{2} = \frac{15\pi}{8} \approx$ 5.89 in^3.

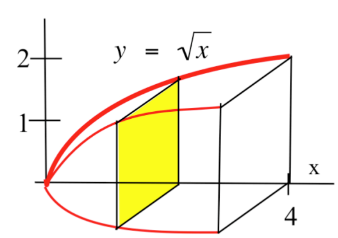


Fig. 14

Practice 3: For the solid in Fig. 14, each face formed by a cut at x is a square with height $\sqrt{x}$.

(a) Write a formula for the approximate volume of the slice between x_{i-1} and x_i .

(b) Write and evaluate an integral for the volume of the solid for x between 1 and 4.

Example 5: Find the volume of the square–based pyramid in Fig. 15.

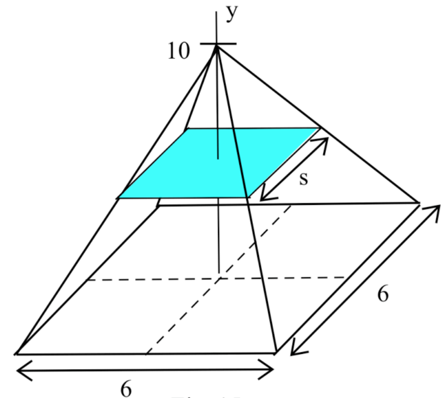


Fig. 15

Solution: Each cut perpendicular to the y–axis yields a square face, but in order to find the area of each square we need a formula for the length of one side s of the square as a function of y, the location of the cut. Using similar triangles (Fig. 16), we know that

$$\frac{s}{10-y} = \frac{6}{10} \quad \text{so} \quad s = \frac{6}{10}(10-y) .$$

The rest of the solution is straightforward.

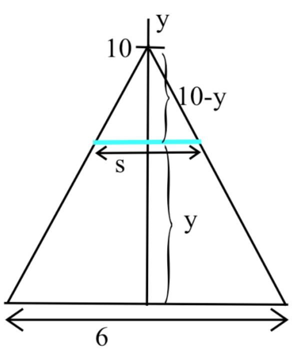


Fig. 16

$$A(y) = (\text{side})^2 = \{ \tfrac{3}{5}(10-y) \}^2 = \tfrac{9}{25}(100 - 20y + y^2) \quad \text{and}$$

$$V = \int_0^{10} A(y)\,dy = \int_0^{10} \tfrac{9}{25}(100 - 20y + y^2)\,dy = \tfrac{9}{25}(100y - 10y^2 + \tfrac{y^3}{3}) \Big|_0^{10}$$

$$= \tfrac{9}{25}(1000 - 1000 + \tfrac{1000}{3}) - (0) = \tfrac{9}{25}\,\tfrac{1000}{3} = 120 \text{ ft}^3 .$$

Example 6: A solid is built between the graphs of $f(x) = x+1$ and $g(x) = x^2$ by building squares with heights (sides) equal to the vertical distance between the graphs of f and g (Fig. 17). Find the volume of this solid for $0 \le x \le 2$.

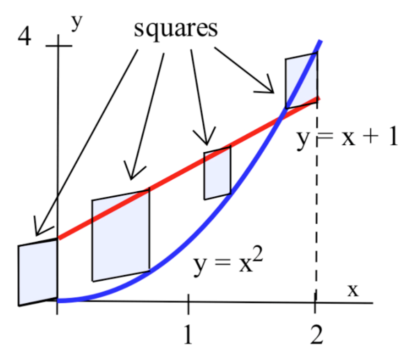


Fig. 17

Solution: The area of a square face is $A(x) = (\text{side})^2$, and the length of a side is either $f(x)–g(x)$ or $g(x)–f(x)$, depending on which function is higher at x. Fortunately, the side is squared in the area formula so it does not matter which is taller, and $A(x) = \{ f(x) - g(x) \}^2$. Then

$$V = \int_a^b A(x)\,dx = \int_0^2 \{ f(x) - g(x) \}^2\,dx = \int_0^2 \{ (x+1) - x^2 \}^2\,dx = \int_0^2 \{ (x+1) - x^2 \}^2\,dx$$

$$= \int_0^2 (1 + 2x - x^2 - 2x^3 + x^4)\,dx = x + x^2 - \frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5} \Big|_0^2 = \frac{26}{15} = 1\ \frac{11}{15} .$$

We saw earlier that areas can have nongeometric interpretations such as distance and total accumulation. Similarly, volumes can have nongeometric interpretations. If x represents an age in years, and $f(x)$ is the number of females in a population with age exactly equal to x, then the "area," $\int_a^b f(x)\,dx$, is the total number of females with ages between a and b (Fig. 18). If the birth rate for females of age x is $r(x)$, with units "births per female per year," (Fig. 19) then the "volume" of the solid in Fig. 20 is $C = \int_a^b r(x)\cdot f(x)\,dx$. C is the number of births during a year to females between the ages a and b , and the units of C will be "births."

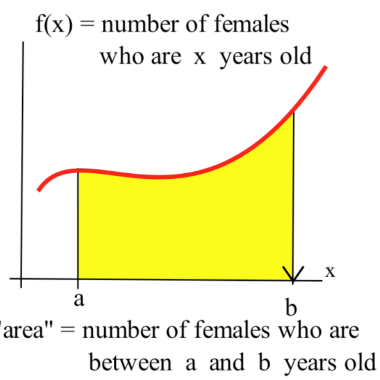


Fig. 18

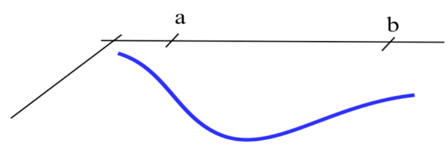


Fig. 19

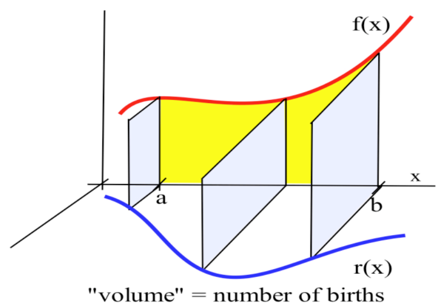


Fig. 20

Volumes of Revolved Regions

When a region is revolved around a line (Fig. 21) a right solid is formed. The face of each slice of the revolved region is a circle so the formula for the area of the face is easy: $A(x)$ = area of a circle = $\pi(\text{radius})^2$ where the radius is often a function of the location x. Finding a formula for the changing radius requires care.

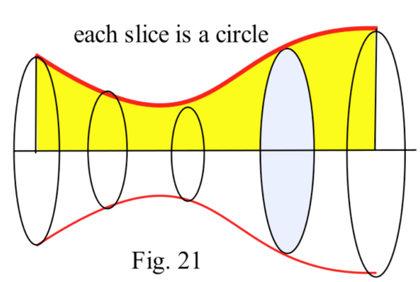


Fig. 21

Example 7: For $0 \le x \le 2$, the area between the graph of $f(x) = x^2$ and the horizontal line $y = 1$ is revolved about the horizontal line $y=1$ to form a solid (Fig. 22). Calculate the volume of the solid.

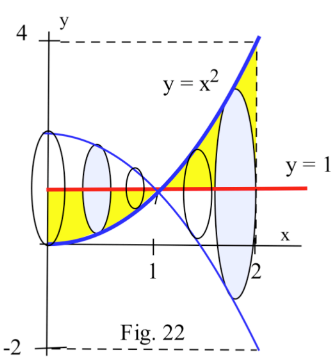


Fig. 22

Solution: The radius function is shown in the figure for several values of x. If $0 \le x \le 1$, then $r(x) = 1 - x^2$, and if $1 \le x \le 2$ then $r(x) = x^2 - 1$. Fortunately, however, $A(x) = \pi\,\{r(x)\}^2$ always uses the square of $r(x)$ and the squares of $1-x^2$ and x^2-1 are equal.

$A(x) = \pi \{ r(x) \}^2 = \pi \{ x^2 - 1 \}^2 = \pi \{ x^4 - 2x^2 + 1 \}$, and

$$V = \int_0^2 \pi\{ x^4 - 2x^2 + 1 \} \, dx = \pi\left(\frac{x^5}{5} - \frac{2}{3} x^3 + x \right) \Big|_0^2 = \frac{46}{15} \pi \approx 9.63 .$$

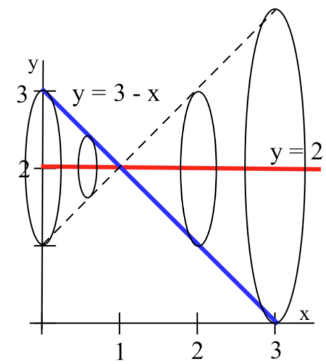


Fig. 23

Practice 4: A solid of revolution is formed when the region between $f(x) = 3 - x$ and the horizontal line $y = 2$ is revolved about the line $y=2$ for $0 \le x \le 3$ (Fig. 23). Find the volume of the solid.

Volumes of Revolved Regions ("Disks")

If the region formed between f, the horizontal line $y = L$, and the interval $[a, b]$ is revolved about the horizontal line $y = L$, (Fig. 24)

then the volume is $V = \int_a^b A(x) \, dx = \int_a^b \pi \cdot (\text{radius})^2 \, dx = \int_a^b \pi\{ f(x) - L \}^2 \, dx$.

This is called the "disk" method because the shape of each thin slice is a circular disk. If the region between f and the x–axis (L=0) is revolved about the x–axis, then the previous formula reduces to

$$V = \int_a^b \pi \{ f(x) \}^2 \, dx .$$

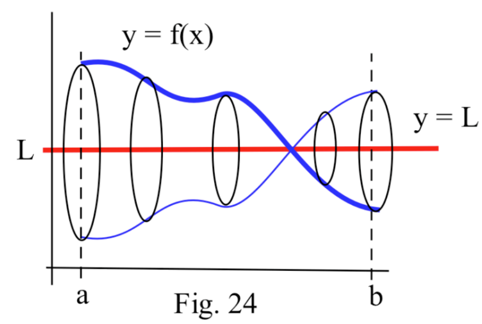


Fig. 24

Example 8: Find the volume generated when the region between one arch of the sine curve $(0 \le x \le \pi)$ and the x–axis is revolved about (a) the x–axis and (b) the line y=1/2.

Solution: (a) $V = \int_a^b \pi \cdot (\text{radius})^2 \, dx = \int_0^\pi \pi\{ \sin(x) \}^2 \, dx = \pi \int_0^\pi \sin^2(x) \, dx = \frac{\pi}{2} \int_0^\pi 1 - \cos(2x) \, dx$

$$= \frac{\pi}{2} \left\{ x - \frac{\sin(2x)}{2} \right\} \Big|_0^\pi = \frac{\pi}{2} \{ \pi - 0 \} = \frac{\pi^2}{2} \approx 4.93 .$$

(b) $V = \int_a^b \pi \cdot (\text{radius})^2 \, dx = \int_0^\pi \pi\{ \sin(x) - \frac{1}{2} \}^2 \, dx = \pi \int_0^\pi \{ \sin^2(x) - \sin(x) + \frac{1}{4} \} \, dx$

$$= \pi \left\{ \frac{\pi}{2} - 2 + \frac{\pi}{4} \right\} \approx 1.12 .$$

Practice 5: Find the volumes swept out when

(a) the region between $f(x) = x^2$ and the x–axis, for $0 \le x \le 2$, is revolved about the x–axis, and

(b) the region between $f(x) = x^2$ and the line y=2, for $0 \le x \le 2$, is revolved about the line y=2.

Example 9: Given that $\int_1^5 f(x)\,dx = 4$ and $\int_1^5 \{ f(x) \}^2\,dx = 7$. Represent the volumes of the solids (a), (b) and (c) in Fig. 25 as definite integrals and evaluate the integrals.

Solution: (a) $V = \int_1^5 \pi\cdot(\text{ radius })^2\,dx = \int_1^5 \pi\cdot\{ f(x) \}^2\,dx = \pi \int_1^5 f^2(x)\,dx = 7\pi$.

(b) $V = \int_1^5 \pi\cdot(\text{ radius })^2\,dx = \int_1^5 \pi\cdot\{ f(x) - (-1) \}^2\,dx = \pi \int_1^5 \{ f^2(x) + 2f(x) + 1 \}\,dx$

$$= \pi \left\{ \int_1^5 f^2(x)\,dx + 2 \int_1^5 f(x)\,dx + \int_1^5 1\,dx \right\} = \pi \{ 7 + 2\cdot 4 + 4 \} = 19\pi .$$

(c) $V = \int_1^5 \pi\cdot(\text{ radius })^2\,dx = \int_1^5 \pi\cdot\{ f(x) / \mathbf{2} \}^2\,dx = \frac{\pi}{\mathbf{4}} \int_1^5 f^2(x)\,dx = \frac{7\pi}{4}$.

Practice 6: Set up and evaluate the integral for the volume of (d) in Fig. 25.

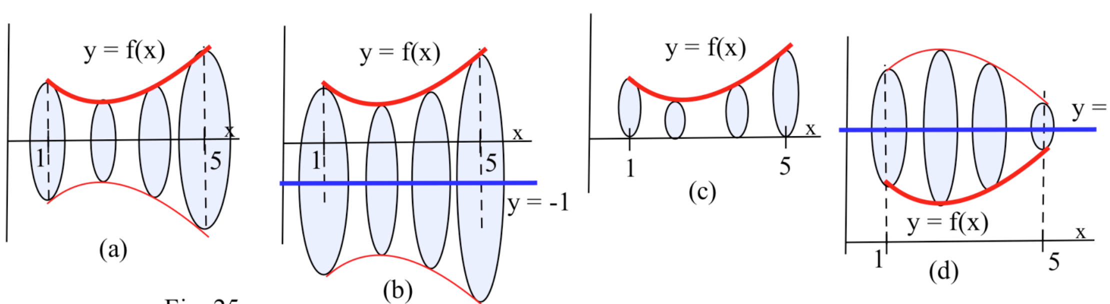


Fig. 25

Solids With Holes

The previous ideas and techniques can also be used to find the volumes of solids with holes in them. If A(x) is the area of the face formed by a cut at x, then it is still true that the volume is $V = \int_a^b A(x)\,dx$. However, if the solid has holes, then some of the faces will also have holes and a formula for $A(x)$ may be more complicated.

Sometimes it is easier to work with two integrals and then subtract: (i) calculate the volume S of the solid without the hole, (ii) calculate the volume H of the hole, and (iii) subtract H from S.

Example 10: Calculate the volume of the solid in Fig. 26.

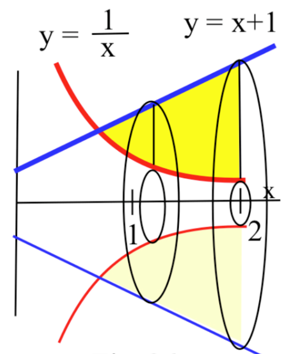


Fig. 26

Solution: The face for a slice at x , has area

$$\begin{aligned} A(x) &= \{\text{area of large circle}\} - \{\text{area of small circle}\} \\ &= \pi\{\text{large radius}\}^2 - \pi\{\text{small radius}\}^2 \\ &= \pi\{\, x+1 \,\}^2 - \pi\{1/x\}^2 = \pi(x^2 + 2x + 1 - 1/x^2). \end{aligned}$$

Then

$$\text{Volume} = \int_a^b A(x)\,dx = \int_1^2 \pi(x^2 + 2x + 1 - 1/x^2)\,dx$$

$$= \pi\{ \tfrac{1}{3} x^3 + x^2 + x + 1/x \} \Big|_1^2 \approx 18.33 .$$

Alternately, the volume of the solid with the large circular faces is $\int_1^2 \pi(x^2 + 2x + 1)\,dx = \frac{19\pi}{3} \approx 19.90$, and the volume of the hole is $\int_1^2 \pi(1/x^2)\,dx = \frac{\pi}{2} \approx 1.57$ so the volume we want is $19.90 - 1.57 = 18.33$.

Practice 7: Calculate the volume of the solid in Fig. 27 .

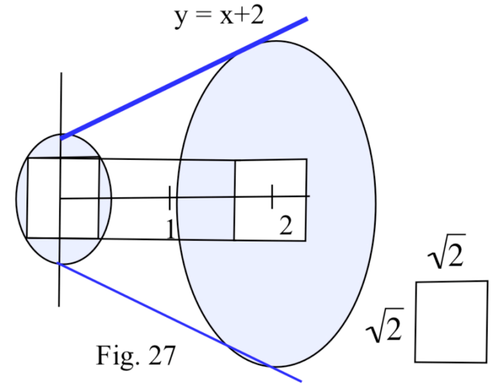


Fig. 27

WRAP UP

At first, all of these volumes may seem overwhelming — there are so many possible solids and formulas and different cases. If you concentrate on the differences, it is very complicated. Instead, focus on the pattern of **cutting, finding areas of faces, volumes of slices, and adding**. With that pattern firmly in mind, you can reason your way to the definite integral. Try to make cuts so the resulting faces have regular shapes (rectangles, triangles, circles) whose areas you can calculate. Try not to let the complexity of the whole solid confuse you. Sketch the shape of **one** face and label its dimensions. If you can find the area of **one** face in the middle of the solid, you can usually find the pattern for all of the faces and then you can easily set up the integral.

PROBLEMS

In problems 1 – 6, use the values given in the tables to calculate the volumes of the solids. (Fig. 28 – 33)

Table 1:	box	base	height	thickness
(Fig. 28)	1	5	6	1
	2	4	4	2
	3	3	3	1

Table 2:	box	base	height	thickness
(Fig. 29)	1	5	6	2
	2	5	4	1
	3	3	3	1
	4	2	2	1

Table 3:	disk	radius	thickness
(Fig. 30)	1	4	0.5
	2	3	1
	3	1	2

Table 4:	disk	height	thickness
(Fig. 31)	1	8	0.5
	2	6	1
	3	2	2

Table 5:	slice	face area	thickness
(Fig. 32)	1	9	0.2
	2	6	0.2
	3	2	0.2

Table 6:	slice	rock area	min. area	thickness
(Fig. 33)	1	4	1	0.6
	2	12	1	0.6
	3	20	4	0.6
	4	10	3	0.6
	5	8	2	0.6

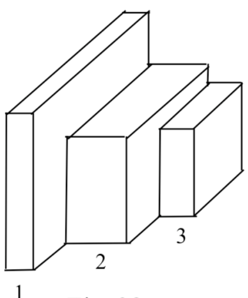


Fig. 28

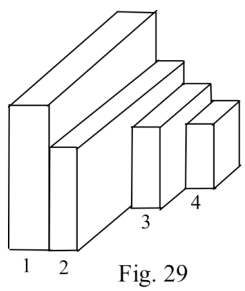


Fig. 29

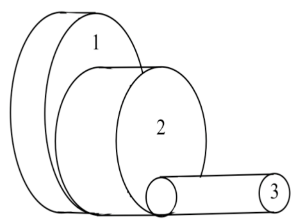


Fig. 30

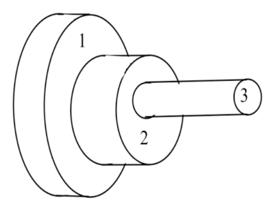


Fig. 31

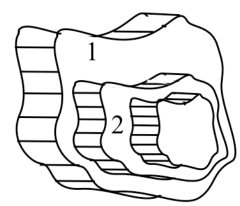


Fig. 32

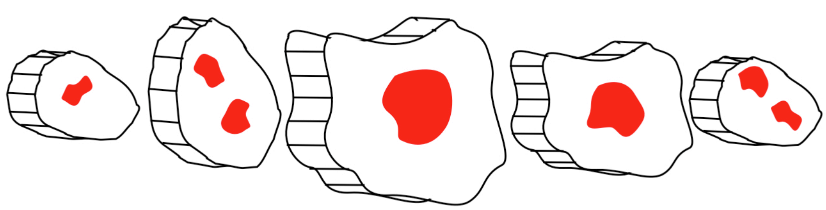


Fig. 33

In problems 7 – 12, represent each volume as an integral and evaluate the integral.

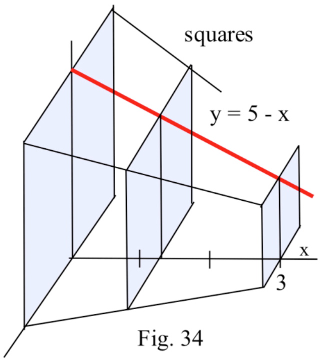


Fig. 34

7. Fig. 34. For $0 \le x \le 3$, each face is a rectangle with base 2 inches and height $5-x$ inches.

8. Fig. 35. For $0 \le x \le 3$, each face is a rectangle with base x inches and height x^2 inches.

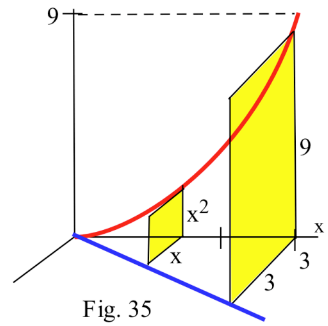


Fig. 35

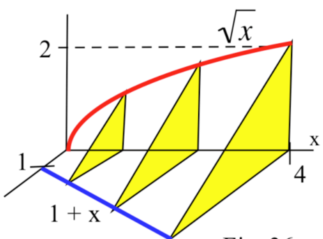


Fig. 36

9. Fig. 36. For $1 \le x \le 4$, each face is a triangle with base x + 1 meters and height $\sqrt{x}$ meters.

10. Fig. 37. For $0 \le x \le 3$, each face is a circle with height (diameter) $4-x$ meters.

11. Fig. 38. For $2 \le x \le 4$, each face is a circle with height (diameter) $4-x$ meters.

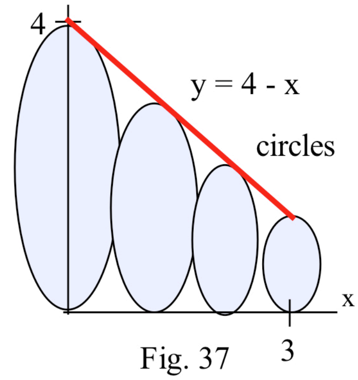


Fig. 37

12. Fig. 39. For $0 \le x \le 2$, each face is a square with a side extending from $y = 1$ to $y = x + 2$.

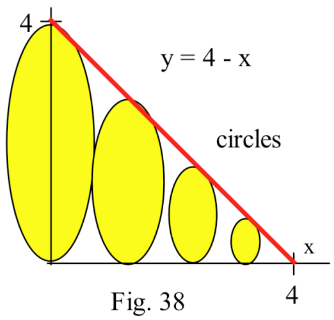


Fig. 38

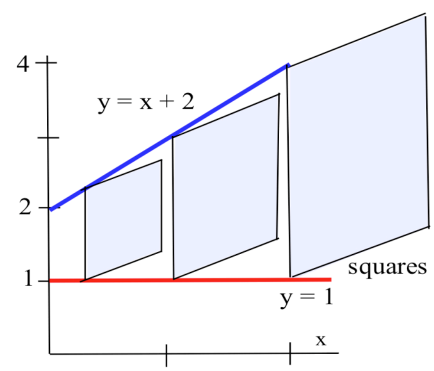


Fig. 39

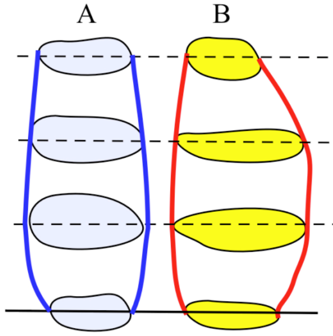
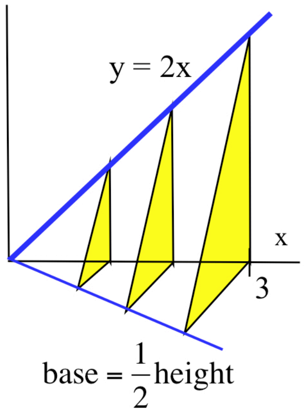


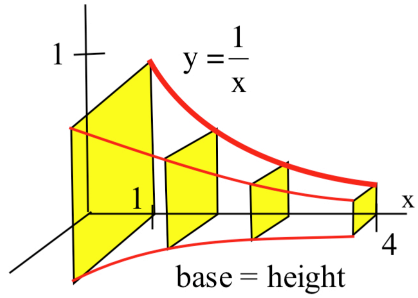
Fig. 40

13. Suppose A and B are solids (Fig. 40) so that every horizontal cut produces faces of A and B that have equal areas. What can we conclude about the volumes of A and B? Justify your answer.

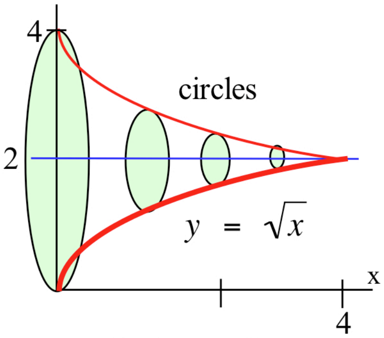
In problems 14 – 22, represent each volume as an integral and evaluate the integral.



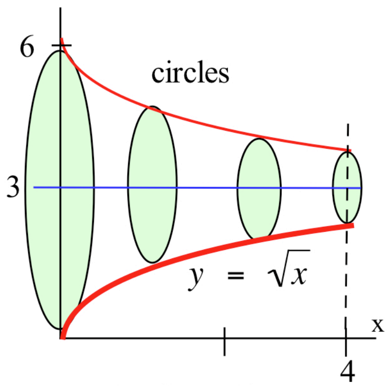
Problem 14



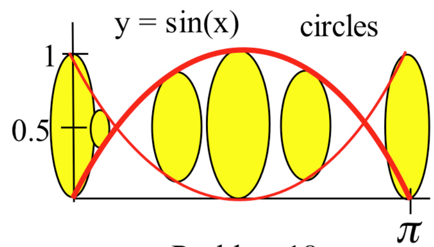
Problem 15



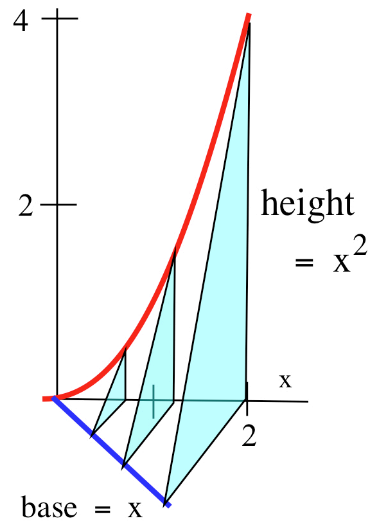
Problem 16



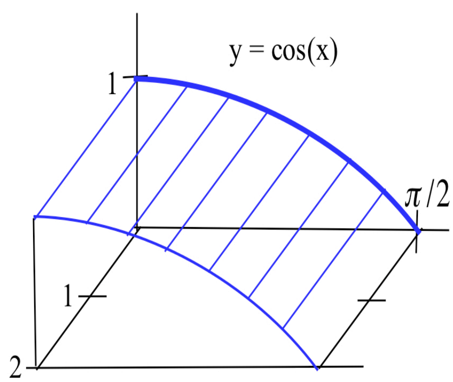
Problem 17



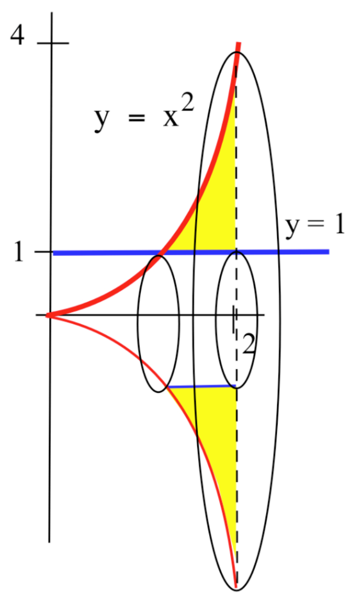
Problem 18



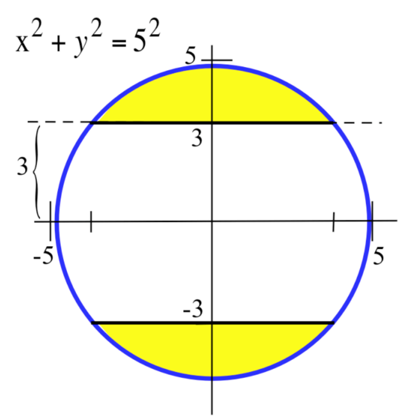
Problem 19



Problem 20



Problem 21



A sphere with radius 5 has a hole of radius 3 drilled through its center

Problem 22

23. Calculate the volume of a sphere of radius 2. (A sphere is formed when the region bounded by the x–axis and the top half of the circle $x^2 + y^2 = 2^2$ is revolved about the x–axis.)

24. Determine the volume of a sphere of radius r. (A sphere is swept out when the region bounded by the x–axis and the top half of the circle $x^2 + y^2 = r^2$ is revolved about the x–axis.)

25. Calculate the volume swept out when the top half of the elliptical region bounded by $\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$ is revolved around the x–axis (Fig. 41). ($y = +3\sqrt{1 - (x^2/25)}$)

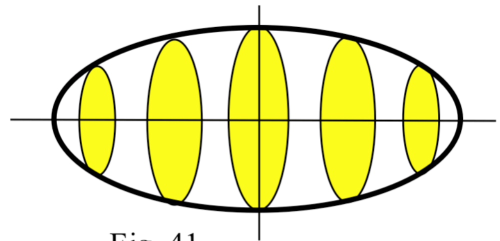
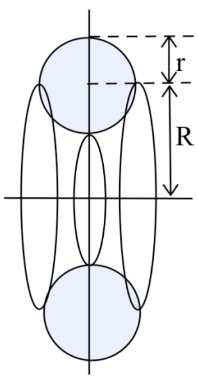


Fig. 41



torus ("doughnut")
Fig. 42

26. Calculate the volume swept out when the top half of the elliptical region bounded by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is revolved around the x–axis. ($y = +b\sqrt{1 - (x^2/a^2)}$)

27. Determine the volume of the "doughnut" in Fig. 42. (The top half of the circle is given by $f(x) = R + \sqrt{r^2 - x^2}$ and the bottom half is given by $g(x) = R - \sqrt{r^2 - x^2}$. (It is easier to use a single integral for this problem.)

28. (a) Find the **area** between $f(x) = 1/x$ and the x–axis for $1 \le x \le 10$, $1 \le x \le 100$, and $1 \le x \le A$. What is the limit of the area for $1 \le x \le A$ as $A \to \infty$?

(b) Find the **volume** swept out when the region in part (a) is revolved about the x–axis for $1 \le x \le 10$, $1 \le x \le 100$, and $1 \le x \le A$. What is the limit of the volumes for $1 \le x \le A$ as $A \to \infty$?

29. Personal Calculus:" Describe a **practical** way to determine the volume of your hand and arm up to the elbow.

30. Personal Calculus:" Most people have a body density between .95 and 1.05 times the density of water which is 62.5 pounds per cubic foot. Use your weight to estimate the volume of your body. (If you float in fresh water, your body density is less than 1.)

Volumes of "right cones"

31. Calculate (a) the volume of the right solid in Fig. 43a, (b) the volume of the "right cone" in Fig. 43b, and (c) the ratio of the "right cone" volume to the right solid volume. (square cross sections)

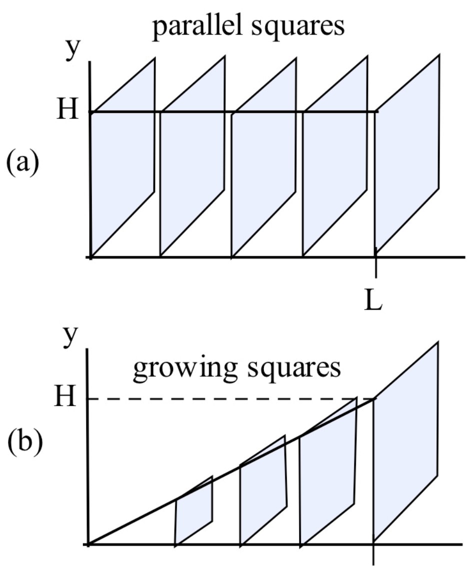


Fig. 43

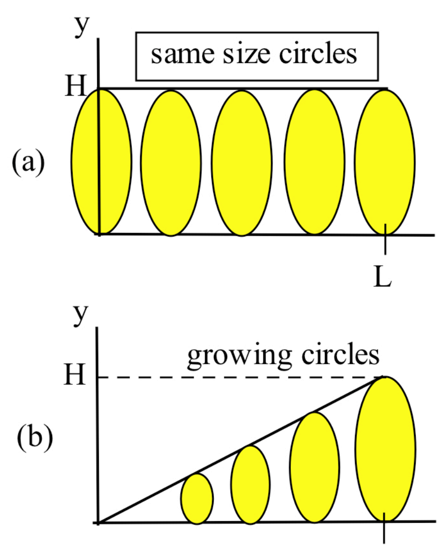


Fig. 44

32. Calculate (a) the volume of the right solid in Fig. 44a, (b) the volume of the "right cone" in Fig. 44b, and (c) the ratio of the "right cone" volume to the right solid volume. (circular cross sections)

33. The "blob" in Fig. 45 has area B.
   (a) Calculate the volume of the right solid in Fig. 45a.
   (b) If a "right cone" is formed (Fig. 45b), then the cross section area at x is $A(x) = (B/L^2)x^2$.
   Find the volume of the "right cone".
   (c) Find the ratio of the "right cone" volume to the right solid volume.

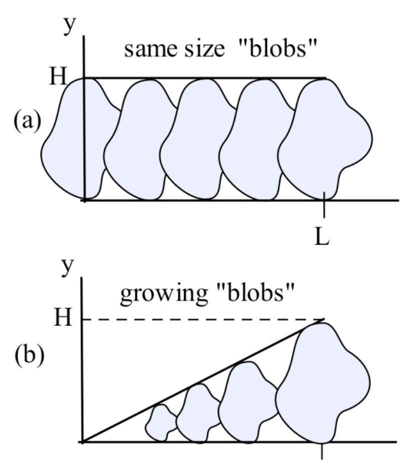


Fig. 45

Section 5.1 **PRACTICE Answers**

Practice 1: (a) Triangular base: $v = (\text{base area})\cdot(\text{height}) = (\frac{1}{2}\cdot 3\cdot 4)\cdot(6) = 36$.

(b) Semicircular base: $v = (\text{base area})\cdot(\text{height}) = \frac{1}{2}\cdot(\pi\cdot 3^2)\cdot(7) \approx 98.96$.

(c) Strangely shaped base: $v = (\text{base area})\cdot(\text{height}) = (8\text{ in}^2)\cdot(5\text{ in}) = 40\text{ in}^3$.

Practice 2: (a) $v_i \approx$ (area of face)(thickness) $\approx (\frac{1}{2} \cdot 4 \cdot x_i^2)\cdot(\Delta x_i) = 2x_i^2\,\Delta x_i$.

(b) $$v = \int_1^2 2x^2\,dx = \frac{2}{3}x^3 \Big|_1^2 = \frac{16}{3} - \frac{2}{3} = \mathbf{\frac{14}{3}}$$ **cubic inches**.

Practice 3: (a) $v_i \approx$ (area of face)(thickness) $\approx (\sqrt{x_i}\,)^2(\Delta x_i) = x_i\,\Delta x_i$.

(b) $$v = \int_1^4 x\,dx = \frac{1}{2}x^2 \Big|_1^4 = 8 - \frac{1}{2} = \mathbf{7.5} .$$

Practice 4: $v_i \approx$ (area of face)(thickness) $\approx (\pi \cdot r^2)$(thickness)

$$= (\pi\cdot((3-x_i) - 2)^2(\Delta x_i) = \pi(1 - 2x_i + x_i^2)\,\Delta x_i .$$

Then volume = $$\int_0^3 \pi(1 - 2x + x^2)\,dx = \pi(x - x^2 + \frac{1}{3}x^3)\Big|_0^3 = 3\pi \approx \mathbf{9.42} .$$

Practice 5: (a) $$v = \int_a^b \pi(\text{ radius })^2\,dx = \int_0^2 \pi(x^2)^2\,dx = \frac{\pi}{5}x^5\Big|_0^2 = \frac{32\pi}{5} = 20.1 .$$

(b) $$v = \int_a^b \pi(\text{ radius })^2\,dx = \int_0^2 \pi(2 - x^2)^2\,dx = \int_0^2 \pi(4 - 4x^2 + x^4)\,dx$$

$$= \pi\,(4x - \frac{4}{3}x^3 + \frac{1}{5}x^5)\Big|_0^2 = \frac{56\pi}{15} \approx \mathbf{11.73} .$$

Practice 6: (d) $$v = \int_a^b \pi(\text{ radius })^2\,dx = \int_1^5 \pi(3 - f(x))^2\,dx = \pi\int_1^5 9 - 6f(x) + f^2(x)\,dx$$

$$= \pi\int_1^5 (9 - 6f(x) + f^2(x))\,dx = \pi\int_1^5 9\,dx - 6\pi\int_1^5 f(x)\,dx + \pi\int_1^5 f^2(x)\,dx$$

$$= \pi(36) - 6\pi(4) + \pi(7) = 19\pi \approx \mathbf{59.69} .$$

(The values "7" and "4" are given in Example 9) .

Practice 7: The volume we want can be obtained by subtracting the volume of the "box" from the volume of the truncated cone generated by the rotated line segment.

volume of truncated cone = $$\int_a^b \pi(\text{ radius })^2\,dx = \int_0^2 \pi(x + 2)^2\,dx$$

$$= \pi\int_0^2 x^2 + 4x + 4\,dx = \pi\{\frac{1}{3}x^3 + 2x^2 + 4x\}\Big|_0^2 = \frac{56}{3}\pi \approx 58.64 .$$

volume of "box" = (length)(width)(height) = $2(\sqrt{2})(\sqrt{2}) = 4$.

The volume we want is $\frac{56}{3}\pi - 4 \approx 54.64$.