5.2 LENGTHS OF CURVES & AREAS OF SURFACES OF REVOLUTION

This section introduces two additional geometric applications of integration: finding the length of a curve and finding the area of a surface generated when a curve is revolved about a line. The general strategy is the same as before: partition the problem into small pieces, approximate the solution on each small piece, add the small solutions together in the form of a Riemann sum, and finally, take the limit of the Riemann sum to get a definite integral.

ARC LENGTH: How Long Is A Curve?

In order to understand an object or an animal, we often need to know how it moves about its environment and how far it travels. We need to know the length of the path it moves along. If we know the object's location at successive times, then it is straightforward to calculate the distances between those locations and add them together to get a total distance.

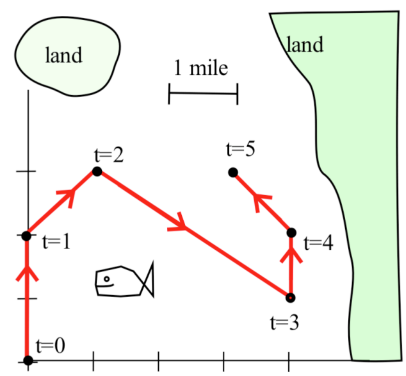

Fig. 1

Example 1: In order to study the movement of whales, a scientist attached a small radio transmitter to the fin of a whale and tracked the location of the whale at 1 hour time intervals over a period of several weeks. The data for a 5 hour period is shown in Fig. 1. How far did the whale swim during the first 3 hours?

Solution: In moving from the point (0,0) to the point (0,2), the whale traveled at least 2 miles. Similarly, the whale traveled at least $\sqrt{(1-0)^2 + (3-2)^2} \approx 1.4$ miles during the second hour and at least $\sqrt{(4-1)^2 + (1-3)^2} \approx 3.6$ miles during the third hour. The scientist concluded that the whale swam **at least** $2 + 1.4 + 3.6 = 7$ miles during the 3 hours.

Practice 1: How far did the whale swim during the entire 5 hour period?

The scientist noted that the whale did not swim in a straight line from location to location so its actual swimming distance was more than 7 miles for the first 3 hours. The scientist hoped to get better distance estimates in the future by determining the whale's position over shorter, five–minute time intervals.

Our strategy for finding the length of a curve will be similar to the one the scientist used, and if the locations are given by a formula, then we can calculate the successive locations over very short intervals and get very good approximations of the total path length. In fact, we can get the exact length of the path by evaluating a definite integral.

Suppose C is a curve, and we pick some points (x_i, y_i) along C (Fig. 2) and connect the points with straight line segments. Then the sum of the lengths of the line segments will approximate the length of C. We can think of this as pinning a string to the curve at the selected points, and then measuring the length of the string as an approximation of the length of the curve. Of course, if we only pick a few points as in Fig. 2, then the total length approximation will probably be rather poor, so eventually we want lots of points (x_i, y_i), close together all along C.

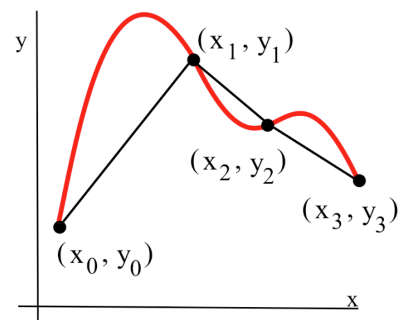

Fig. 2

Suppose the points are labeled so (x_0, y_0) is one endpoint of C and (x_n, y_n) is the other endpoint and that the subscripts increase as we move along C. Then the distance between the successive points (x_{i-1}, y_{i-1}) and (x_i, y_i) is $\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$, and the total length of the line segments is simply the sum of the successive lengths. This is an important approximation of the length of C, and all of the integral representations for the length of C come from it.

The length of the curve C is approximately $\sum \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$.

Example 2: Use the points (0,0), (1,1), and (3,9) to approximate the length of $y = x^2$ for $0 \le x \le 3$.

Solution: The lengths of the two linear pieces in Fig. 3 are

$\sqrt{1^2 + 1^2} = \sqrt{2} \approx 1.41$ and

$\sqrt{2^2 + 8^2} = \sqrt{68} \approx 8.25$ so the length of the curve

is approximately $1.41 + 8.25 = 9.66$.

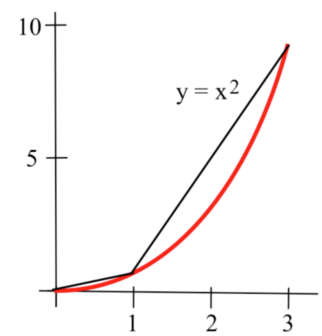

Fig. 3

Practice 2: Get a better approximation of the length of $y = x^2$ for $0 \le x \le 3$ by using the points (0,0), (1,1), (2,4), and (3,9). Is your approximation longer or shorter than the actual length?

The summation does not have the form $\sum f(c_i)\cdot\Delta x_i$ so it is not a Riemann sum. It is, however, algebraically equivalent to several Riemann sums, and each one leads to a definite integral representation for the length of C.

(a) **y = f(x)**: When y is a function of x, we can factor $(\Delta x_i)^2$ from inside the radical and simplify.

$$\text{Length of C} \approx \sum \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sum \sqrt{(\Delta x_i)^2 \cdot \left\{ 1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2 \right\}}$$

$$= \sum \Delta x_i \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \quad \text{(Riemann sum)} \longrightarrow \boxed{\int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

(b) **x = g(y)**: When x is a function of y, we can factor $(\Delta y_i)^2$ from inside the radical and simplify.

$$\text{Length of C} \approx \sum \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sum \sqrt{(\Delta y_i)^2 \cdot \left\{ \left(\frac{\Delta x_i}{\Delta y_i}\right)^2 + 1 \right\}}$$

$$= \sum \Delta y_i \sqrt{\left(\frac{\Delta x_i}{\Delta y_i}\right)^2 + 1} \quad \text{(Riemann sum)} \longrightarrow \boxed{\int_{y=c}^{y=d} \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy}$$

(c) **Parametric equations:** When x and y are functions of t, $x = x(t)$ and $y = y(t)$, for $\alpha \le t \le \beta$ we can factor $(\Delta t_i)^2$ from inside the radical and simplify.

$$\text{Length of C} \approx \sum \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sum \sqrt{(\Delta t_i)^2 \cdot \left\{ \left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2 \right\}}$$

$$= \sum \Delta t_i \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2} \quad \text{(Riemann sum)} \longrightarrow \boxed{\int_{t=\alpha}^{t=\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

The integrals in (a), (b), and (c) each represent the length of C, and we can use whichever one is more convenient. Unfortunately, for most functions these integrands do not have easy antiderivatives, and most arc length integrals must be approximated using one of our approximate integration methods or a calculator.

Example 3: Represent the length of each curve as a definite integral.

(a) The length of $y = x^2$ between (1,1) and (4,16).

(b) The length of $x = \sqrt{y}$ between (1,1) and (4,16).

(c) The length of the parametric curve $x(t) = \cos(t)$ and $y(t) = \sin(t)$ for $0 \le t \le 2\pi$.

Solution: (a) Length = $\int_{x=1}^{x=4} \sqrt{1 + (\,dy/dx\,)^2}\; dx = \int_{x=1}^{x=4} \sqrt{1 + 4x^2}\; dx \approx 15.34$.

(b) Length = $\int_{y=1}^{y=16} \sqrt{(\,dx/dy\,)^2 + 1}\; dy = \int_{y=1}^{y=16} \sqrt{\frac{1}{4y} + 1}\; dy \approx 15.34$.

The values of the integrals in (a) and (b) were approximated using Simpson's rule with n = 10. It is not an accident that the lengths in (a) and (b) are equal. Why not?

(c) Length = $\int_{t=0}^{t=2\pi} \sqrt{\left(-\sin(t)\right)^2 + \left(\cos(t)\right)^2}\; dt = \int_{t=0}^{t=2\pi} \sqrt{\sin^2(t) + \cos^2(t)}\; dt = \int_{t=0}^{t=2\pi} 1\, dt = 2\pi$.

The graph of $(x(t), y(t))$ for $0 \le t \le 2\pi$ is a circle of radius 1 so we know that its length is exactly 2π .

Practice 3: Represent the length of each curve as a definite integral.

(a) The length of one period of y = sin(x).

(b) The length of the parametric path $x(t) = 1 + 3t$ and $y(t) = 4t$ for $1 \le t \le 3$.

AREAS OF SURFACES OF REVOLUTION

Rotated Line Segments

Just as all of the integral formulas for arc length came from the simple distance formula, all of the integral formulas for the area of a revolved surface come from the formula for revolving a single straight line segment. If a line segment of length L, parallel to a line P, (Fig. 4) is revolved about the line P, then the resulting surface can be unrolled and laid flat. The flattened surface is a rectangle with area $A = 2\pi \cdot r \cdot L$.

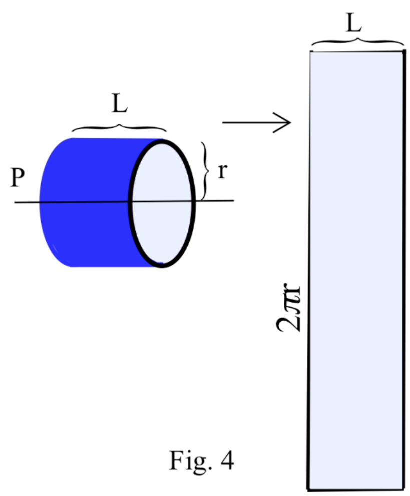

Fig. 4

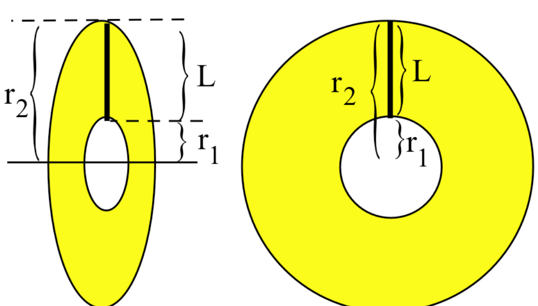

Fig. 5

If a line segment of length L, perpendicular to a line P and not intersecting P, (Fig. 5) is revolved about the line P, then the resulting surface is the region between two concentric circles and its area is

A = (area of large cirle) – (area of small circle) = $\pi(r_2)^2 - \pi(r_1)^2$

$$= \pi\{ (r_2)^2 - (r_1)^2 \} = 2\pi \left\{ \frac{r_2+r_1}{2} \right\} (r_2 - r_1) = 2\pi \cdot \left(\frac{r_1 + r_2}{2} \right) \cdot L$$

In general, if a line segment of length L which does not intersect a line P (Fig. 6) is revolved about P, then the resulting surface has area

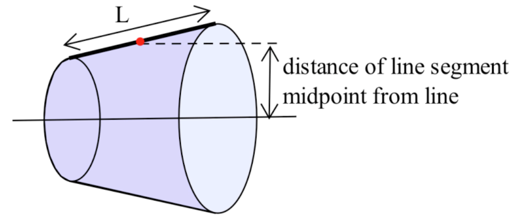

Fig. 6

A = surface area

= (distance traveled by midpoint of the line segment)•(length of the line segment)

= **2π•{ distance of segment midpoint from the line P }• L**

There are several integral formulas for the surface area of a curve rotated about a line, but all of the formulas come rather easily and quickly from this one fundamental formula for a surface area of a rotated line segment.

Example 4: Find the surface area generated when each line segment in Fig. 7 is rotated about the x–axis and the y–axis.

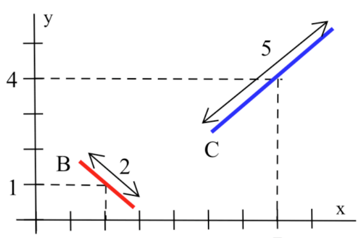

Fig. 7

Solution: Line segment B has length L=2 and its midpoint is at (2,1).

When B is rotated about the x–axis, the surface area is

2π•(distance of midpoint from x–axis)•2 = 2π(1)2 = 4π.

When B is rotated about the y–axis, the surface area is

2π•(distance of midpoint from y–axis)•2 = 8π.

Line segment C has length 5 and its midpoint is at (7,4). When C is rotated about the x–axis the resulting surface area is

2π•(distance of midpoint from x–axis)•5 = 2π(4)5 = 40π.

When C is rotated about the y–axis, the surface area is

2π•(distance of midpoint from y–axis)•5 = 70π.

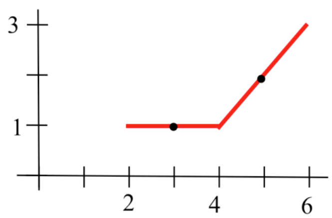

Fig. 8

Practice 4: Find the surface areas generated when the graph in Fig. 8 is rotated about each axis.

Rotated Curves

When a curve is rotated about a line P, we can use our old strategy again (Fig. 9). Select some points (x_i, y_i) along the curve, connect the points with line segments, calculate the surface area of each rotated line segment, and add together the surface areas of the rotated line segments. This final sum can be converted to a Riemann sum, and the limit of the Riemann sum is a definite integral for the surface area of the rotated curve.

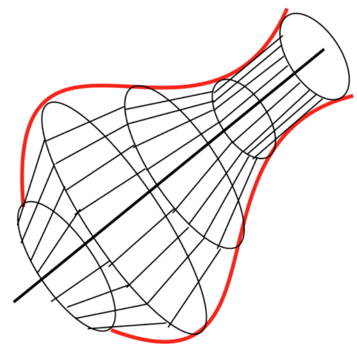

Fig. 9

Suppose we select points (x_i, y_i) along C and number them so (x_0, y_0) is one endpoint of C, (x_n, y_n) is the other endpoint, and the subscripts increase as we move along C. Then each pair of successive points $(x_{i-1}, y_{i-1}), (x_i, y_i)$ are the endpoints of a line segment with

$$\text{length} = \sqrt{(\Delta x_i)^2+(\Delta y_i)^2} \qquad \text{and} \qquad \text{midpoint} = \left(\frac{x_{i-1}+x_i}{2}, \frac{y_{i-1}+y_i}{2}\right).$$

The midpoint is $(y_{i-1}+y_i)/2$ units from the x–axis and $(x_{i-1}+x_i)/2$ units from the y–axis.

About the x–axis: If C is rotated about the x–axis, then each segment generates an area equal to

$$2\pi\cdot(\text{distance of midpoint from the x–axis})\cdot(\text{length of the segment}) = 2\pi\cdot\left\{\frac{y_{i-1}+y_i}{2}\right\}\sqrt{(\Delta x_i)^2+(\Delta y_i)^2}\ .$$

The sum of these surface areas is

$$\sum 2\pi\left\{\frac{y_{i-1}+y_i}{2}\right\}\sqrt{(\Delta x_i)^2+(\Delta y_i)^2} = \sum 2\pi\left\{\frac{y_{i-1}+y_i}{2}\right\}\sqrt{(\Delta x_i/\Delta x_i)^2+(\Delta y_i/\Delta x_i)^2}\ \Delta x_i$$

$$\longrightarrow \boxed{\int_a^b 2\pi\, y\sqrt{1+(dy/dx)^2}\ dx = \text{area of the surface of revolution of curve C}}\ .$$

About the y–axis: If C is rotated about the y–axis, then each segment generates an area equal to

$$2\pi\cdot(\text{distance of midpoint from the y–axis})\cdot(\text{length of the segment}) = 2\pi\left\{\frac{x_{i-1}+x_i}{2}\right\}\sqrt{(\Delta x_i)^2+(\Delta y_i)^2}\ .$$

The sum of these surface areas is

$$\sum 2\pi\left\{\frac{x_{i-1}+x_i}{2}\right\}\sqrt{(\Delta x_i)^2+(\Delta y_i)^2} = \sum 2\pi\left\{\frac{x_{i-1}+x_i}{2}\right\}\sqrt{(\Delta x_i/\Delta x_i)^2+(\Delta y_i/\Delta x_i)^2}\ \Delta x_i$$

$$\longrightarrow \boxed{\int_a^b 2\pi\, x\sqrt{1+(dy/dx)^2}\ dx = \text{area of the surface of revolution of curve C}}\ .$$

Example 5: Use definite integrals to represent the areas of the surfaces generated when the curve $y = 2 + x^2$, $0 \le x \le 3$, is rotated about each axis.

Solution: Surface area about the x–axis is $\int_a^b 2\pi \cdot f(x)\sqrt{1+(dy/dx)^2}\ dx = \int_0^3 2\pi(2 + x^2)\sqrt{1+4x^2}\ dx \approx 383.8$.

Surface area about the y–axis is $\int_a^b 2\pi \cdot x\sqrt{1+(dy/dx)^2}\ dx = \int_0^3 2\pi \cdot x\sqrt{1+4x^2}\ dx \approx 117.32$.

Parametric Form For Surface Area of Revolution

If the curve C is described by parametric equations, $x = x(t)$ and $y = y(t)$, then the forms of the surface area integrals are somewhat different, but they still follow from the fundamental surface area formula for the surface area of a line segment rotated about a line P:

Surface area **= 2π•{ distance of midpoint from the line P }• L** .

About x–axis: Starting with the previous equation for the area of a segment rotated about the **x–axis**,

$$2\pi \cdot \left\{\frac{y_{i-1}+y_i}{2}\right\}\sqrt{(\Delta x_i)^2+(\Delta y_i)^2} ,$$

we can factor Δt_i from the radical, sum the pieces and take the limit, as the mesh apporaches 0, to get

$$\int_{t=\alpha}^{t=\beta} 2\pi\, y\sqrt{(dx/dt)^2 + (dy/dt)^2}\ dt = \text{area of the surface of revolution of curve C} .$$

About y–axis: Starting with the previous equation for the area of a segment rotated about the **y–axis**,

$$2\pi \left\{\frac{x_{i-1}+x_i}{2}\right\}\sqrt{(\Delta x_i)^2+(\Delta y_i)^2}$$

we can factor Δt_i from the radical, sum the pieces and take the limit, as the mesh apporaches 0, to get

$$\int_{t=\alpha}^{t=\beta} 2\pi\, x\sqrt{(dx/dt)^2 + (dy/dt)^2}\ dt = \text{area of the surface of revolution of curve C} .$$

WRAP UP

One purpose of this section was to obtain a variety of integral formulas for two geometric quantities, the length of a curve and the area of the surface generated when a curve is rotated about a line. The integral formulas are useful, but a more basic and fundamental point was to illustate again how relatively simple approximation formulas can lead us, via Riemann sums, to integral formulas. We will see it again.

PROBLEMS

For arc length

1. A squirrel was spotted at the backyard locations in Table 1 at the given times. The squirrel traveled at least how far during the first 15 minutes?

time (min)	location relative to oak tree north	east (feet)
0	10	7
5	25	27
10	1	45
15	13	33
20	24	40
25	10	23
30	0	14

Table 1: Locations of Squirrel

2. The squirrel in Problem 1 traveled at least how far during the first 30 minutes?

3. Use the partition $\{0, 1, 2\}$ to estimate the length of $y = 2^x$ between the points (0, 1) and (2, 4).

4. Use the partition $\{1, 2, 3, 4\}$ to estimate the length of $y = 1/x$ between the points (1, 1) and (4, 1/4).

The graphs of the functions in problems 5 – 8 are straight lines. Calculate each length (a) using the distance formula between 2 points and (b) by setting up and evaluating the arc length integrals.

5. $y = 1 + 2x$ for $0 \le x \le 2$.

6. $y = 5 - x$ for $1 \le x \le 4$.

7. $x = 2 + t$, $y = 1 - 2t$ for $0 \le t \le 3$.

8. $x = -1 - 4t$, $y = 2 + t$ for $1 \le t \le 4$.

9 Calculate the length of $y = \frac{2}{3} x^{3/2}$ for $0 \le x \le 4$.

10. Calculate the length of $y = 4x^{3/2}$ for $1 \le x \le 9$.

Very few functions $y = f(x)$ lead to integrands of the form $\sqrt{1 + (dy/dx)^2}$ which have elementary antiderivatives. In problems 11 – 14, $1 + (dy/dx)^2$ is a perfect square and the resulting arc length integrals can be evaluated using antiderivatives. Do so.

11. $y = \frac{x^3}{3} + \frac{1}{4x}$ for $1 \le x \le 5$.

12. $y = \frac{x^4}{4} + \frac{1}{8x^2}$ for $1 \le x \le 9$.

13. $y = \frac{x^5}{5} + \frac{1}{12x^3}$ for $1 \le x \le 5$.

14. $y = \frac{x^6}{6} + \frac{1}{16x^4}$ for $4 \le x \le 25$.

In problems 15 – 23, (a) represent each length as a definite integral, and (b) evaluate the integral using your calculator's integral command.

15. The length of $y = x^2$ from (0,0) to (1,1).

16. The length of $y = x^3$ from (0,0) to (1,1).

17. The length of $y = \sqrt{x}$ from (1,1) to (9,3).

18. The length of $y = \ln(x)$ from (1,0) to (e,1).

19. The length of $y = \sin(x)$ from (0,0) to $(\pi/4, \sqrt{2}/2)$ and from $(\pi/4, \sqrt{2}/2)$ to $(\pi/2, 1)$.

20. The length of the ellipse $x(t) = 3\cos(t), y(t) = 4\sin(t)$ for $0 \le t \le 2\pi$.

21. The length of the ellipse $x(t) = 5\cos(t), y(t) = 2\sin(t)$ for $0 \le t \le 2\pi$.

22. A robot was programmed to follow a spiral path and be at location $x(t) = t\cos(t), y(t) = t\sin(t)$ at time t . How far did the robot travel between $t = 0$ and $t = 2\pi$?

23. A robot was programmed to follow a spiral path and be at location $x(t) = t\cos(t), y(t) = t\sin(t)$ at time t . How far did the robot travel between $t = 10$ and $t = 20$?

24. As a tire of radius R rolls, a small stone stuck in the tread will travel a "cycloid" path, $x(t) = R\cdot(t - \sin(t))$, $y(t) = R\cdot(1 - \cos(t))$. As t goes from 0 to 2π, the tire makes one complete revolution and travels forward $2\pi R$ units. How far does the small stone travel?

25. As a tire with a 1 foot radius rolls forward 1 mile, how far does a pebble stuck in the tire tread travel? ($x(t) = 1(t - \sin(t))$ and $y(t) = 1(1 - \cos(t))$)

26. (Calculator) Graph $y = x^n$ for n = 1, 3, 10, and 20. As the value of n gets large, what happens to the graph of $y = x^n$? Estimate the value of $\lim_{n\to\infty}\left(\int_{x=0}^{x=1}\sqrt{1+(n\cdot x^{n-1})^2}\,dx\right)$.

27. (Calculator) Find the point on the curve segment $f(x) = x^2$ for $0 \le x \le 4$ which will divide the segment into two equally long pieces. Find the points which will divide the segment into 3 equally long pieces.

28. Find the pattern for the functions in problems 11 – 14. If $y = \frac{x^n}{n} + \frac{1}{Ax^B}$, then how are A and B related to n? ($A = 4(n-2)$ and $B = n-2$)

29. Use the formulas for A and B from the previous problem with $n = 3/2$ and write a new function $y = \frac{2}{3}x^{3/2} + \frac{1}{Ax^B}$ so that $1 + (dy/dx)^2$ is a perfect square.

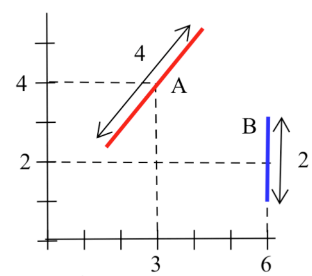

Fig. 10

For surface area of revolution

30. Find the surface area when each line segment in Fig. 10 is rotated about the (a) x–axis and (b) y–axis.

31. Find the surface area when each line segment in Fig. 11 is rotated about the (a) x–axis and (b) y–axis.

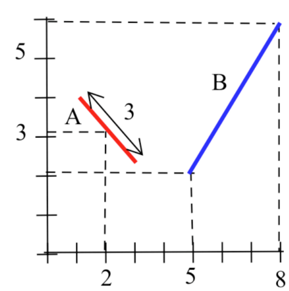

Fig. 11

32. Find the surface area when each line segment in Fig. 10 is rotated about the lines (a) y = 1 and (b) x = –2 .

33. Find the surface area when each line segment in Fig. 11 is rotated about the lines (a) y = 1 and (b) x = –2.

34. A line segment of length 2 has its center at the point (2,5) and makes an angle of θ with horizontal. What value of θ will result in the largest surface area when the line segment is rotated about the y–axis? Explain your reasoning.

35. A line segment of length 2 has its one end at the point (2,5) and makes an angle of θ with horizontal. What value of θ will result in the largest surface area when the line segment is rotated about the x–axis? Explain your reasoning.

In problems 36 – 44, (a) represent each surface area as a definite integral, and (b) evaluate the integral using your calculator's integral command.

36. Find the area of the surface when the graph of $y = x^3$ for $0 \le x \le 2$ is rotated about the y–axis.

37. Find the area of the surface when the graph of $y = 2x^3$ for $0 \le x \le 1$ is rotated about the y–axis.

38. Find the area of the surface when the graph of $y = x^2$ for $0 \le x \le 2$ is rotated about the x–axis.

39. Find the area of the surface when the graph of $y = 2x^2$ for $0 \le x \le 1$ is rotated about the x–axis.

40. Find the area of the surface when the graph of $y = \sin(x)$ for $0 \le x \le \pi$ is rotated about the x–axis.

41. Find the area of the surface when the graph of $y = x^3$ for $0 \le x \le 2$ is rotated about the x–axis.

42. Find the area of the surface when the graph of $y = \sin(x)$ for $0 \le x \le \pi/2$ is rotated about the y–axis.

43. Find the area of the surface when the graph of $y = x^2$ for $0 \le x \le 2$ is rotated about the y–axis.

44. Find the area of the surface when the graph of $y = \sqrt{4 - x^2}$ is rotated about the x–axis (a) for $0 \le x \le 1$, (b) for $1 \le x \le 2$, and (c) for $2 \le x \le 3$.

45. (a) Show that if a thin hollow sphere is sliced into pieces by equally–spaced parallel cuts (Fig. 12), then each piece has the same weight. (Show that each piece has the same surface area).
(b) What does the result of part (a) mean for an orange cut into slices with equally–spaced parallel cuts?
(c) Suppose a hemispherical cake with uniformly thick layer of frosting is sliced with equally–spaced parallel cuts. Does everyone get the same amount of cake? Does everyone get the same amount of frosting?

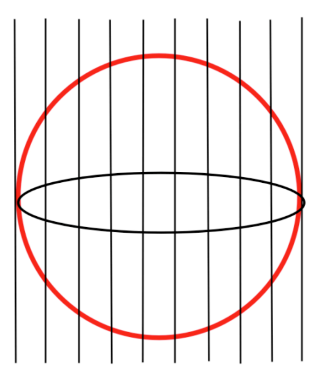

Fig. 12

3–D Arc Length Problems (Optional)

The parametric equation form of arc length extends very nicely to 3 dimensions. If a curve C in 3 dimensions (Fig. 13) is given parametrically by $x = x(t)$, $y = y(t)$, and $z = z(t)$ for $a \le t \le b$, then the distance between the successive points $(x_{i-1}, y_{i-1}, z_{i-1})$ and (x_i, y_i, z_i) is

$$\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2 + (z_{i-1} - z_i)^2}$$

$$= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2} \quad .$$

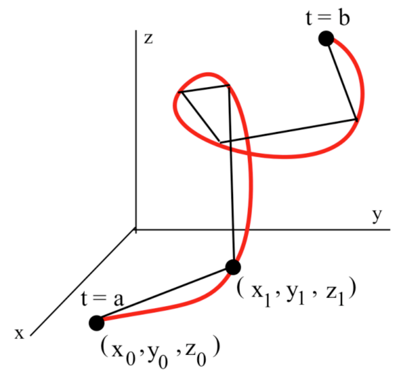

Fig. 13

We can, as before, factor $(\Delta t_i)^2$ from each term under the radical, sum the pieces to get a Riemann sum, and take a limit of the Riemann sum to get a definite integral representing the length of the curve C.

$$\sum \sqrt{(\Delta x_i/\Delta t_i)^2 + (\Delta y_i/\Delta t_i)^2 + (\Delta z_i/\Delta t_i)^2} \quad \Delta t_i \longrightarrow \int_{t=a}^{t=b} \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} \quad dt \,.$$

The length of the curve C is $\displaystyle\int_{t=a}^{t=b} \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} \quad dt$.

In problems 46 – 50, (a) represent each length as a definite integral, and (b) evaluate the integral using your calculator's integral command.

46. Find the length of the helix $x = \cos(t)$, $y = \sin(t)$, $z = t$ for $0 \le t \le 4\pi$. (Fig. 14)

47. Find the length of the straight line segment $x = t$, $y = t$, $z = t$ for $0 \le t \le 1$.

48. Find the length of the curve $x = t$, $y = t^2$, $z = t^3$ for $0 \le t \le 1$.

49. Find the length of the "stretched helix" $x = \cos(t)$, $y = \sin(t)$, $z = t^2$ for $0 \le t \le 2\pi$.

50. Find the length of the curve $x = 3\cos(t)$, $y = 2\sin(t)$, $z = \sin(7t)$ for $0 \le t \le 2\pi$.

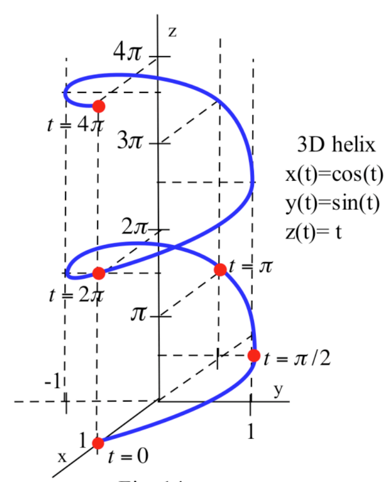

Fig. 14

Section 5.2 **PRACTICE Answers**

Practice 1: At least $2 + \sqrt{2} + \sqrt{13} + 1 + \sqrt{2} \approx 9.43$ miles.

Practice 2: Using the points (0,0), (1,1), (2,4), and (3,9),

$L \approx \sqrt{2} + \sqrt{10} + \sqrt{26} \approx 9.68 <$ actual length.

Practice 3: (a) Using $\displaystyle\int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ with $y = \sin(x)$, $L = \displaystyle\int_{x=0}^{x=2\pi} \sqrt{1 + \cos^2(x)}\, dx$.

(b) Using $\displaystyle\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ with $x(t) = 1 + 3t$ and $y(t) = 4t$ for $1 \le t \le 3$,

$$L = \int_{t=1}^{t=3} \sqrt{3^2 + 4^2}\, dt = \int_{t=1}^{t=3} 5\, dt = 10 .$$

Practice 4: surface area of revolved segment = $2\pi \cdot \{$ distance of segment midpoint from the line P $\} \cdot L$

surface area of horizontal segment revolved about **x–axis** = $2\pi(1)(2) = 4\pi \approx 12.57$.

surface area of other segment revolved about **x–axis** = $2\pi(2)(\sqrt{8}) \approx 35.54$.

The total surface area about the x–axis is approximately $12.57 + 35.54 = 48.11$ square units.

surface area of horizontal segment revolved about **y–axis** = $2\pi(3)(2) = 12\pi \approx 37.70$.

surface area of other segment revolved about **y–axis** = $2\pi(5)(\sqrt{8}) \approx 88.86$.

The total surface area about the y–axis is approximately $37.70 + 88.86 = 126.56$ square units.