5.3 MORE WORK APPLICATIONS

In Section 4.7 we introduced the problem of calculating the **work** done in lifting an object using a cable which had weight. This section continues that introduction and extends the process to handle situations in which the applied force or the distance or both may be variables. The method we used before is used again here. The first step is to divide the problem into small "slices" so that the force and distance vary only slightly on each slice. Then the work for each slice is calculated, the total work is approximated by adding together (a Riemann sum) the work for each slice, and , finally, a limit is taken to get a definite integral representing the total work. **There are so many possible variations in work problems that it is vital that you understand the process.**

The work done on an object by a constant force is the magnitude of the force applied to the object multiplied by the distance over which the force is applied: **work = (force)•(distance)**.

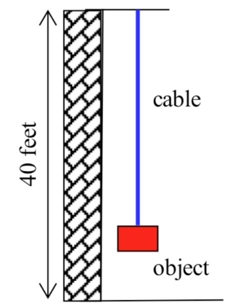

Fig. 1

Example 1: A 10 pound object is lifted 40 feet from the ground to the top of a building using a cable which weighs 1/2 pound per foot (Fig. 1). How much work is done?

Solution: This type of problem appeared in section 4.7, but it is a good example of the process of dividing the problem into pieces and analyzing each piece. We can partition the height of the building (Fig. 2). Then the work done to lift the object from the height x_i to the height x_{i+1} is the force applied times the distance moved:

force = (weight of the object) + (weight of the cable)
= (10 pounds) + (0.5 (length of hanging cable))
= $10 + 0.5(40 - x_i)$ pounds = $30 - 0.5x_i$ pounds

distance = $x_i - x_{i-1}$ feet = Δx_i feet

work = (force)(distance) = $\{ 30 - 0.5x_i \} \Delta x_i$ foot–pounds.

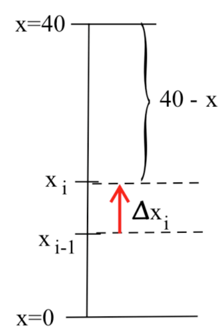

Fig. 2

$$\text{Total work} \approx \sum_{i=1}^{n} \{ \text{work on } i^{th} \text{ slice} \} = \sum_{i=1}^{n} \{ 30 - 0.5x_i \} \Delta x_i \text{ foot–pounds}$$

$$\longrightarrow \int_0^{40} \{ 30 - 0.5x \} \, dx = (30x - 0.25x^2) \Big|_0^{40} = 800 \text{ foot–pounds.}$$

Practice 1: How much work is done lifting a 130 pound injured person to the top of a 30 foot cliff using a stretcher weighing 10 pounds and a cable weighing 2 pounds per foot?

In the previous Example and Practice problem the distance moved on each part of the partition was always Δx, and the force was more complicated. In some of the following examples, the Δx is part of the force calculation. Analyze each problem.

Example 2: A cola glass in Fig. 3 has the dimensions given in Table 1. Approximately how much work do you do when you drink a cola glass full of water

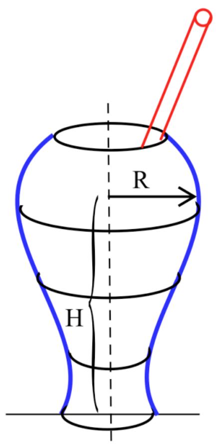

Fig. 3

(weight density = 62.5 pounds/ft^3 = 0.5787 ounces/in^3) by sucking it through a straw to a point 3 inches above the top edge of the glass?

Height above bottom of the glass (inches)	Inside radius (inches)
4	1.4
3	1.6
2	1.5
1	1.0
0	1.1

Table 1: Inside radius of a cola glass

Solution: The Table naturally partitions the water into 1 inch thick "slices" (Fig. 4). The work to move each slice is approximately the weight of the slice times the distance it is moved. We can use the radius at the bottom of each slice to approximate the volume and then the weight of the slice, and a point half way up each slice to calculate the distance the slice is moved.

top slice: force = weight = (volume)(density) $\approx \pi(1.6 \text{ in})^2(1 \text{ in})(0.5787 \text{ oz/in}^3) \approx 4.7$ oz.

distance $\approx$ (distance from middle of slice to lips) = 3.5 in.

work $\approx$ (force)(distance) = (4.7 oz)(3.5 in) = 16.4 oz–in.

next slice: force = weight = (volume)(density) $\approx \pi(1.5 \text{ in})^2(1 \text{ in})(0.5787 \text{ oz/in}^3) \approx 4.1$ oz.

distance $\approx$ (distance from middle of slice to lips) = 4.5 in

work $\approx$ (force)(distance) = (4.1 oz)(4.5 in) = 18.4 oz–in.

The work for the last two slices is (1.8 oz)(5.5 in) = 9.9 oz–in and (2.2 oz)(6.5 in) = 14.3 oz–in.

The total work is the sum of the work needed to raise each slice of water:

Total work $\approx$ (16.4 oz–in) + (18.4 oz–in) + (9.9 oz–in) + (14.3 oz–in) = 59 oz–in.

Practice 2: Approximate the total work needed to raise the water in Example 2 by using the **top** radius of the slice to approximate the weight and the midpoint of each slice to approximate the distance the slice is raised.

If we knew the radius of the glass at every height, then we could improve our approximation by taking thinner slices. In fact, if we knew the radius at every height we could have formed a Riemann sum, taken the limit of the Riemann sum as the thickness of the slices approached 0, and obtained a definite integral. In the next Example we do know the radius of the container at every height.

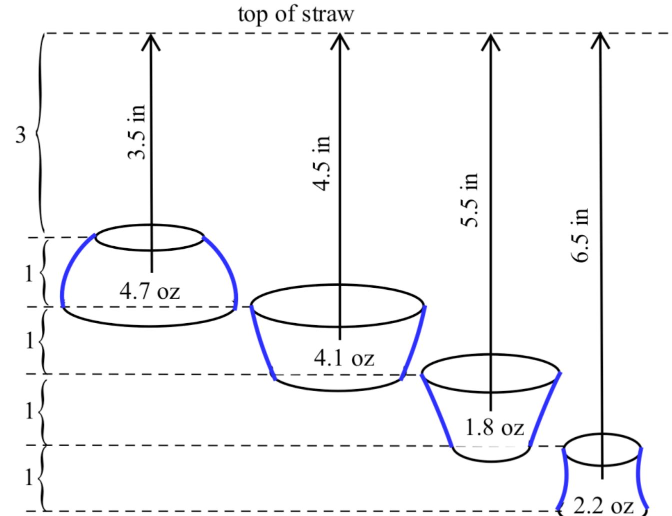

Fig. 4

Example 3: Find the work needed to raise the water in the cone in Fig. 5a to the top of the straw.

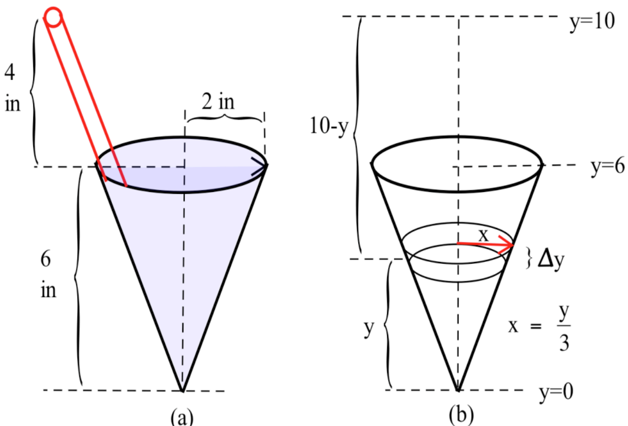

Fig. 5

Solution: We can label the cone (Fig. 5b), and partition the height of the cone to get slices of water. The work done raising the i^{th} slice is the distance the slice is raised times the force needed to move it, the weight of the slice. For any c_i in the subinterval $[y_{i-1}, y_i]$, the slice is raised a distance of approximately $(10 - c_i)$ inches.

Each slice is approximately a right circular cylinder so its volume is $\pi(\text{radius})^2 \Delta y$. At the height y, the radius of the cylinder is $x = y/3$ so at the height c_i the radius is $c_i/3$. Then the force is

$$\text{force} = (\text{volume})(\text{density}) \approx \pi(\text{radius})^2(\Delta y_i)(0.5787 \text{ oz/in}^3) = \pi\,(\,c_i / 3\,)^2\,(\Delta y_i)\,(0.5787) \text{ ounces.}$$

The work to raise the i^{th} slice $\approx \pi(c_i / 3)^2(\Delta y_i)(0.5787)(10 - c_i)$ ounce–inches, and the total work is approximately $\sum_{i=1}^{n} \pi(c_i / 3)^2(\Delta y_i)(0.5787)(10 - c_i)$. As the mesh of the partition approaches 0, the Riemann sum approaches the definite integral:

$$\text{total work} \approx \sum_{i=1}^{n} \pi(c_i / 3)^2(\Delta y_i)(0.5787)(10 - c_i) \longrightarrow \int_0^6 \pi(y/3)^2(0.5787)(10 - y)\,dy .$$

$$\text{Total work} = \int_0^6 \pi(\,y/3\,)^2(0.5787)(10 - y)\,dy = \frac{0.5787\pi}{9}\int_0^6 10\,y^2 - y^3\,dy$$

$$= \frac{0.5787\pi}{9}\left\{ \frac{10}{3}(6)^3 - \frac{1}{4}(6)^4 \right\} = 79.99 \text{ oz–in}$$

.

In this example, both the force and the distance were variables and both depended on the height of the slice above the bottom of the cone.

Practice 3: How much work is done in drinking just the top 3 inches of the water in Example 3?

Example 4: The trough in Fig. 6 is filled with a liquid weighing 70 pounds per cubic foot. How much work is done pumping the liquid over the wall next to the trough?

Solution: As before, we can partition the height of the trough to get slices of liquid. In order to form a Riemann sum for the total work, we need the weight of a typical slice (Fig. 7) and the distance it is raised.

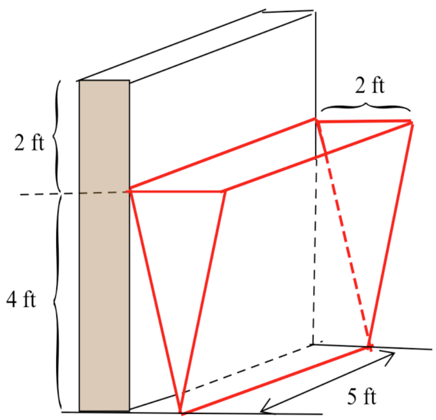

Fig. 6

slice at height y_i : weight $= \text{(volume)(density)}$

$= \text{(length)(width)(height)}(70 \text{ pounds/ft}^3)$

$= (5 \text{ ft})(\frac{1}{2}\, y_i \text{ ft})(\Delta y_i \text{ ft})(70 \text{ pounds/ft}^3)$

$= 175\, y_i\, \Delta y_i$ pounds.

distance raised $= 6 - y_i$ ft

work $= (175\, y_i\, \Delta y_i \text{ pounds})(6 - y_i \text{ feet})$

$= 175\, y_i\, (6 - y_i)\, \Delta y_i$ foot–pounds.

The rest of the solution is straightforward and follows the pattern of the previous problems:

$$\text{Total work} \approx \sum_{i=1}^{n} \{ \text{work to raise } i^{\text{th}} \text{ slice} \}$$

$$= \sum_{i=1}^{n} \{ 175\, y_i\, (6 - y_i)\, \Delta y_i \text{ foot–pounds} \}$$

$$\longrightarrow \int_0^4 175\, y(6 - y)\, dy$$

$$= 175(3y^2 - \frac{y^3}{3}) \Big|_0^4 = 4666.7 \text{ foot–pounds.}$$

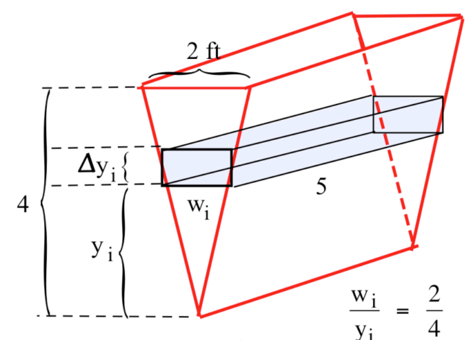

Fig. 7

"Raise the liquid" problems can be handled by partitioning the height of the container and then focusing your attention on **one typical slice**. If you can calculate the **weight** of that slice and the **distance** it is raised, the rest of the steps are straightforward: form a Riemann sum, form a definite integral, and evaluate the integral to get the total work.

Work Moving An Object In A Straight Line

Suppose we are pushing a box along a flat surface (Fig. 8a) which is smooth in places and rough in other places so at some places we only have to push lightly and in other places we have to push hard. If f(x) is the amount of force we need to use at location x, and we want to push the box along a straight line from x=a to x=b, then we can partition the interval [a, b] into pieces, $[a,x_1], [x_1,x_2], \ldots, [x_{n-1},b]$ (Fig. 8b). The work to move the box along the i^{th} piece from x_{i-1} to x_i is approximately

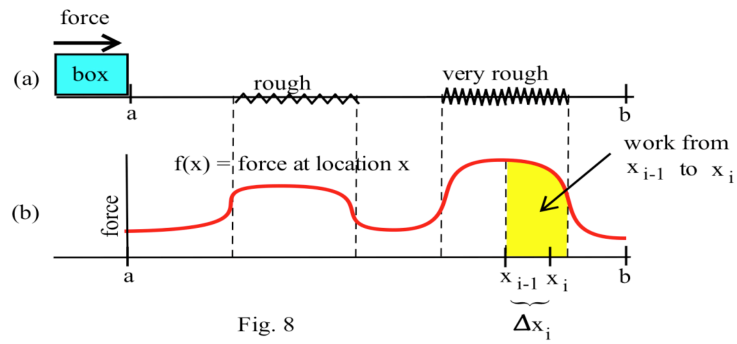

Fig. 8

$$\text{(force)}\cdot\text{(distance)} \approx f(c_i)\cdot(x_i - x_{i-1}) = f(c_i)\,\Delta x_i \text{ for any } c_i \text{ in the subinterval } [x_{i-1}, x_i].$$

The total work is the sum of the work along each piece, $\sum_{k=1}^{n} f(c_i)\,\Delta x_i$, a Riemann sum. As we take smaller and smaller subintervals (as the mesh of the partition approaches 0), the Riemann sum approaches the definite integral:

$$\sum_{k=1}^{n} f(c_i)\,\Delta x_i \longrightarrow \int_a^b f(x)\,dx = \text{total work}.$$

> **If** an object starts at $x = a$ and is moved in a straight line to the location $x = b > a$ by applying a force of $f(x)$ at every location x between a and b,
>
> **then** the total work done on the object is $\int_a^b f(x)\,dx$.

This has a simple geometric interpretation. If $f(x)$ is the force applied at the position x (Fig. 9), then the work done to move from position $x = a$ to position $x = b$ is the area under the graph of f from $x = a$ to $x = b$.

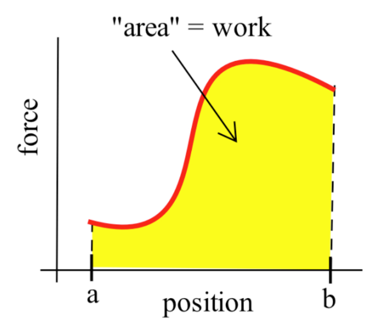

Fig. 9

Example 5: Suppose a force of $7x$ pounds is required to stretch a spring (Fig. 10) x inches past its natural length. How much work will be done stretching the spring from its natural length ($x = 0$) to 5 inches beyond its natural length ($x = 5$ inches)?

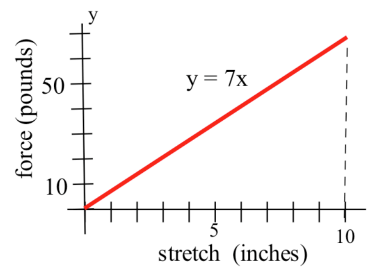

Fig. 10

Solution: $\text{Work} = \int_a^b f(x)\,dx = \int_0^5 7x\,dx = \frac{7x^2}{2}\Big|_0^5 = 87.5$ inch–pounds.

This can also be don graphically (Fig. 11).

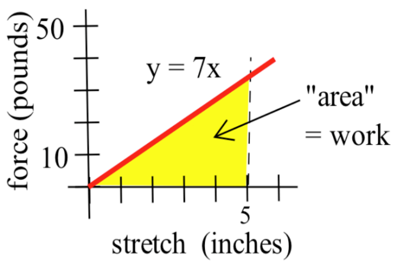

Fig. 11

Practice 4: How much work is done to stretch the spring in the previous example from 5 inches past its natural length to 10 inches past its natural length?

The spring example is an application of a physical principle discovered by the English physicist Robert Hooke (1635–1703), a contemporary of Newton.

Hooke's Law: The force needed to stretch or compress a spring x units from its natural length is proportional to the distance x : force $f(x) = kx$ for some constant k. (Fig. 12)

The "k" in Hooke's Law is called the "spring constant". It varies from spring to spring (depending on the materials and dimensions of the spring and even on the temperature), but is constant for each spring as long as the spring is not overextended or overcompressed. In fact, Hooke's law holds for most solid objects, at least for limited ranges of force:

"Nor is it observable in those bodies only, but in all other springy bodies whatsoever, whether metal, wood, stones, baked earth, hair, horns, silk, bones, sinews, glass and the like." *(Hooke).*

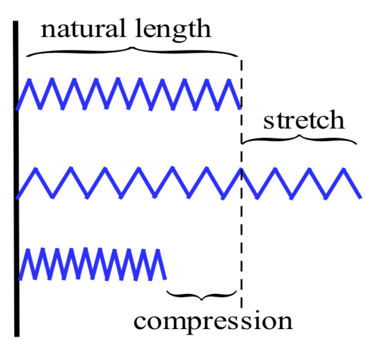

Fig. 12

Most bathroom scales use springs and are based on Hooke's Law for compressing a spring.

Example 6: A spring has a natural length of 43 centimeters, and a weight of 4 grams stretches it to a total length of 75 centimeters. How much work is done stretching the spring from a total length of 63 cm to a total length of 93 cm?

Solution: First we need to use the given information to find the value of k, the spring constant. A force of 4 g produces a stretch of 32 cm (total length of 75 cm minus the rest length of 43 cm). Substituting $x = 32$ cm and $f(x) = 4$ g into Hooke's Law, $f(x) = kx$, we have $4\text{ g} = k\,(32\text{ cm})$ so

$$k = \frac{4\text{ g}}{32\text{ cm}} = \frac{1\text{ g}}{8\text{ cm}} = .125\ \text{g/cm}.$$

The total length of 63 cm represents a stretch of 20 cm beyond the natural length, and the total length of 93 cm represents a 50 cm stretch. Then the work done is

$$\text{work} = \int_a^b f(x)\,dx = \int_{20}^{50} (.125)\,x\,dx = (.125)\cdot\frac{x^2}{2}\Bigg|_{20}^{50} = 131.25 \text{ g–cm.}$$

Practice 5: A spring has a natural length of 3 inches, and a force of 2 pounds stretches it to a total length of 8 inches. How much work is done stretching the spring from a total length of 5 inches to a total length of 10 inches?

Lifting a Payload: The problem of finding the work done lifting a payload from the surface of a moon (or any body with no atmosphere) is very similar. Suppose the moon has a radius of R miles and the payload weighs P pounds at the surface of the moon (at a distance of R miles from the center of the moon). When the payload is x miles from the center of the moon (x ≥ R), the gravitational attraction between the moon and the payload is proportional to the reciprocal of the square of the distance x between the centers of the moon and the payload:

$$\text{required force} = f(x) = \frac{R^2P}{x^2} \text{ pounds (Fig. 13).}$$

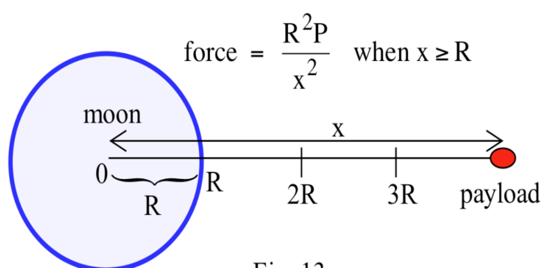

Fig. 13

The total amount of work done raising the payload from the surface (altitude is 0, so x = R) to an altitude of R (x = R+R= 2R) is

$$\text{work} = \int_a^b f(x)\,dx = \int_R^{2R} \frac{R^2P}{x^2}\,dx = R^2P\left(-\frac{1}{x}\right)\Bigg|_R^{2R} = R^2P\left(-\frac{1}{2R}\right) - R^2P\left(-\frac{1}{R}\right) = \frac{RP}{2} \text{ mile–pounds.}$$

Practice 6: How much work will be needed to raise the payload from the altitude R above the surface (x = 2R) to an altitude of 2R?

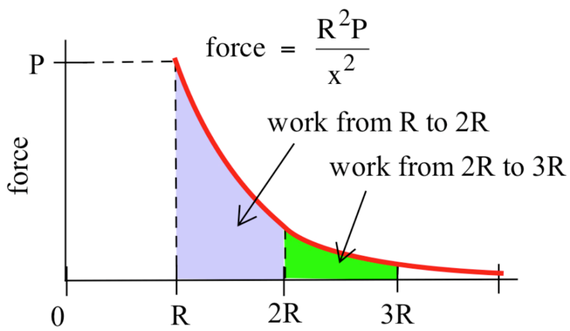

Fig. 14

The appropriate areas under the force graph (Fig. 14) illustrate why the work to raise the payload from x = R to x = 2R was so much larger than the work to raise it from x = 2R to x = 3R. In fact, the work to raise the payload from x = 2R to x = 100R is 0.49RP which is still less that the 0.5RP needed to raise the payload from x = R to x = 2R.

The real problem of lifting a payload is much more difficult because the rocket doing the lifting must also lift itself (more work) and the mass of the rocket will keep changing as it burns up fuel. Lifting a payload from a body with an atmosphere is even harder: there is friction from the atmosphere, and the frictional force depends on the density of the atmosphere (which varies with height), the speed of the rocket and the shape of the rocket. Life can get complicated.

PROBLEMS

1. A tank 4 feet long, 3 feet wide and 7 feet tall (Fig. 15) is filled with water which weighs 62.5 pounds per cubic foot. How much work is done pumping the water out over the top of the tank?

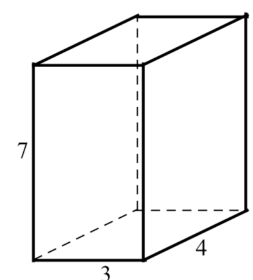

Fig. 15

2. A tank 4 feet long, 3 feet wide and 6 feet tall is filled with a oil which weighs 60 pounds per cubic foot.
   (a) How much work is done pumping the oil over the top edge of the tank?
   (b) How much work is done pumping the 3 feet of oil of the top edge of the tank?

3. A tank 5 feet long, 2 feet wide and 4 feet tall is filled with a oil which weighs 60 pounds per cubic foot.
   (a) How much work is done pumping all of the oil out over the top edge of the tank?
   (b) How much work is done pumping the top 36 cubic feet of oil out over the top edge of the tank?
   (c) How long does a 1 horsepower pump take to empty the tank over the top edge of the tank? (A 1 horsepower pump works at a rate of 33,000 foot–pounds per minute.) A 1/2 horsepower pump? Which pump does more work?

4. A cylindrical aquarium with radius 2 feet and height 5 feet (Fig. 16) is filled with salt water (65 pounds/ft^3).
   (a) How much work is done pumping all of the water over the top edge of the tank?
   (b) How much work is done pumping the water to a point 3 feet above the top edge of the tank?
   (c) How long does a 1 horsepower pump take to empty the tank over the top edge of the tank? A 1/2 horsepower pump? Which pump does more work?

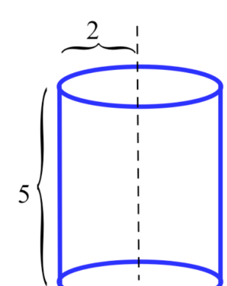

Fig. 16

5. A cylindrical barrel with a radius of 1 foot and a height of 6 feet is filled with oil (60 pounds/ft^3).
   (a) How much work is done pumping all of the oil over the top edge of the barrel?
   (b) How much work is done pumping the top 1 foot of oil to a point 2 feet above the top of the barrel?
   (c) How long will it take a 1 horsepower pump to empty the top 3 feet of oil over the top edge of the barrel?

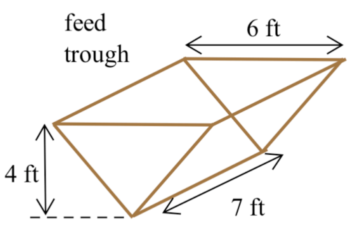

Fig. 17

6. An animal feed trough (Fig. 17) is filled with food weighing 80 pounds/ft^3 . How much work is done lifting all of the food over the top of the trough?

7. How much work is done lifting the top 1 foot of food over the top of the trough in Problem 6?

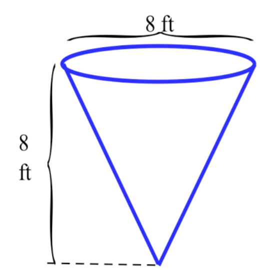

Fig. 18

8. The conical container in Fig. 18 is filled with loose grain which weighs 40 pounds/ft^3 .
   (a) How much work is done lifting all of the grain over the top of the cone?
   (b) lifting the top 2 feet of grain over the top of the cone?

9. If you and a friend share the work equally in emptying the conical container in Problem 8, what depth of grain should the first person leave for the second person to empty?

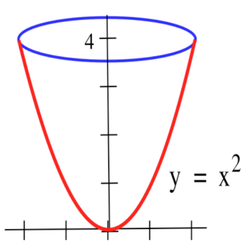

Fig. 19

10. The parabolic container in Fig. 19 is filled with water. (a) How much work is done pumping the water over the top of the tank? (b) to a point 3 feet above the top of the tank?

11. The parabolic container in Fig. 20 is filled with water.
    (a) How much work is done pumping the water over the top of the tank?
    (b) to a point 3 feet above the top of the tank?

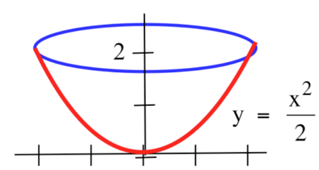

Fig. 20

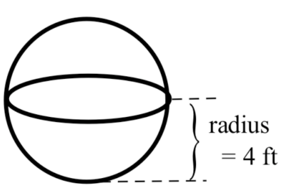

Fig. 21

12. The spherical tank in Fig. 21 is full of water. How much work is done lifting the water to the top of the tank?

13. There are two feet of water in the bottom of spherical tank in Fig. 21. How much work is done lifting the water to the top of the tank?

14. The student said, "I've got a shortcut for these tank problems, but it doesn't always work. I figure the weight of the liquid and multiply that by the distance I have to move the middle point in the water. It worked for the first 5 problems and then it didn't."

(a) Does it really give the right answer for the first 5 problems?

(b) How are the containers in the first 5 problems different from the others?

(c) For which of the containers in Fig. 22 will the "shortcut" work?

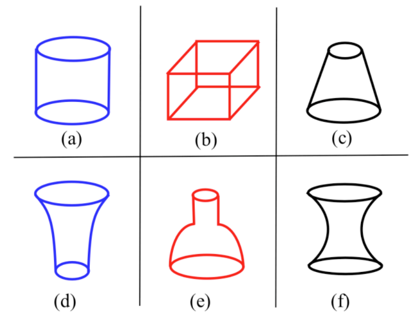

Fig. 22

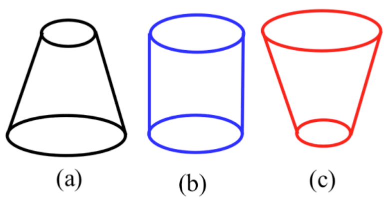

Fig. 23

15. All of the containers in Fig. 23 have the same height and hold the same volume of water. Which requires

(a) the most work to empty?

(b) the least work to empty?

Explain how you reached your conclusions.

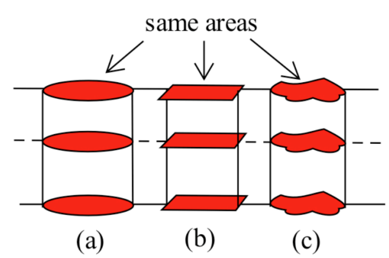

Fig. 24

16. All of the containers in Fig. 24 have the same height and at each height x they all have the same cross sectional area. Which requires

(a) the most work to empty? (b) the least work to empty?

Justify your conclusions.

17. Fig. 25 shows the force required to move a box along a rough surface. How much work is done pushing the box (a) from x = 0 to x = 5 feet? (b) from x = 3 to x = 5 feet?

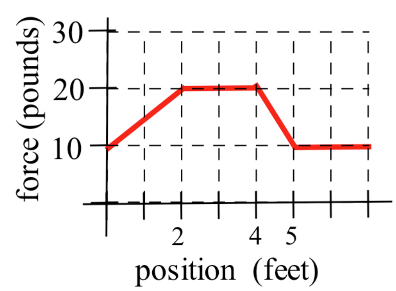

Fig. 25

18. How much work is done pushing the box in Fig. 25

(a) from x = 3 to x = 7 feet? (b) from x = 0 to x = 7 feet?

19. A spring requires a force of $6x$ ounces to stretch it x inches past its natural length. How much work is done stretching the spring

(a) from its natural length ($x = 0$) to 3 inches beyond its natural length?

(b) from its natural length to 6 inches beyond its natural length?

20. A spring requires a force of $5x$ grams to compress it x cm. How much work is done compressing the spring (a) 7 cm from its natural length? (b) 10 cm from its natural length?

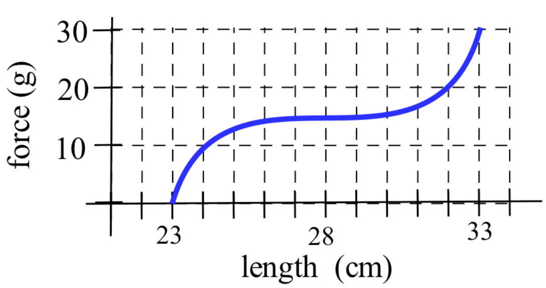

Fig. 26

21. Fig. 26 shows the force needed to stretch a material that does not obey Hooke's Law. Approximately how much work is done stretching it

(a) from a total length of 23 cm to 33 cm?

(b) from a total length of 28 cm to 33 cm?

22. Approximately how much work is done stretching the defective spring in the previous problem (a) from a total length of 23 cm to 26 cm? (b) from a total length of 30 cm to 35 cm?

23. A 3 pound object stretches a spring 5 inches. How much work is done stretching it 4 more inches?

24. A 2 pound fish stretches a spring 3 inches. How much work is done stretching it 3 more inches?

25. A payload weighs 100 pounds at the surface of an asteroid which has a radius of 300 miles. How much work is done lifting the payload from the asteroid's surface to an altitude of (a) 100 miles? (b) 200 miles? (c) 300 miles?

26. Calculate the amount of work required to lift **you** from the surface of the moon where your weight is approximately 1/6 what it is on earth to an altitude of 200 miles? (The moon's radius is approximately 1,080 miles.)

27. Calculate the amount of work required to lift **you** from the surface of the earth to an altitude (a) of 200 miles? (b) of 400 miles? (c) of 1,000,000 miles? (The earth's radius is approximately 4,000 miles.)

28. An object located at the origin repels **you** with a force inversely proportional to your distance from the object ($f(x) = -\frac{1}{kx}$). When you are 10 feet from the origin the repelling force is 0.1 pound. How much work is done as you move (a) from $x = 20$ to $x = 10$? (b) from $x = 10$ to $x = 1$? (c) from $x = 1$ to $x = 0.1$?

29. An object located at the origin repels you with a force inversely proportional to the square of your distance from the object ($f(x) = -\frac{1}{kx^2}$). When you are 10 feet from the origin the repelling force is 0.1 pound. How much work is done as you move (a) from $x = 20$ to $x = 10$? (b) from $x = 10$ to $x = 1$? (c) from $x = 1$ to $x = 0.1$?

30. The student said "I've got a 'work in a line' shortcut that always seems to work. I figure the average force and then multiply by the total distance. Will it always work?" (a) Will it? Justify your answer. (Hint: What is the formula for 'average force'?) (b) Is it a shortcut?

Work Along A Curved Path

Suppose the location of a moving object is defined parametrically as $x = x(t)$ and $y = y(t)$ for $a \le t \le b$, and the force is a function of t, $f = f(t)$. Then we can represent the work done moving along the path as a definite integral. Partition the time interval $[a, b]$ into short subintervals. For the interval $[t_{i-1}, t_i]$:

force $\approx f(c_i)$ for any c_i in $[t_{i-1}, t_i]$

distance moved $\approx \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{(\Delta x_i/\Delta t_i)^2 + (\Delta y_i/\Delta t_i)^2}\ \Delta t_i$

work $\approx f(c_i)\sqrt{(\Delta x_i/\Delta t_i)^2 + (\Delta y_i/\Delta t_i)^2}\ \Delta t_i$

Total work $\approx \sum$ {work along each subinterval} $\approx \sum f(c_i)\sqrt{(\Delta x/\Delta t)^2 + (\Delta y/\Delta t)^2}\ \Delta t$

$$\longrightarrow \boxed{\int_{t=a}^{t=b} f(t)\sqrt{(dx/dt)^2 + (dy/dt)^2}\ dt = \text{total work along the path } (x(t), y(t)).}$$

In problems 31 – 35, find the total work along the given parametric path. If necessary, approximate the value of the integral using your calculator. f is in pounds, x and y are in feet, t is in minutes.

31. $f(t) = t$. $x(t) = \cos(t)$, $y(t) = \sin(t)$, $0 \le t \le 2\pi$.

32. $f(t) = t$. $x(t) = t$, $y(t) = t^2$, $0 \le t \le 1$.

33. $f(t) = t$. $x(t) = t^2$, $y(t) = t$, $0 \le t \le 1$.

34. $f(t) = \sin(t)$. $x(t) = 2t$, $y(t) = 3t$, $0 \le t \le \pi$. (Fig. 27)

35. $f(t) = t$. $x(t) = \cos(t)$, $y(t) = \sin(t)$, $0 \le t \le 2\pi$ (Fig. 28). (Can you find a geometric way to calculate the shaded area?)

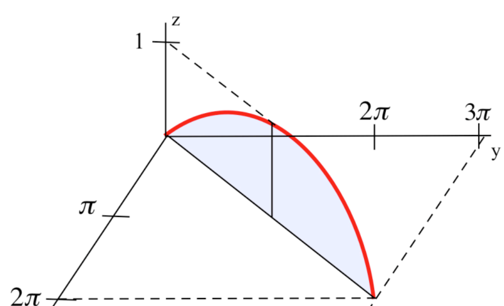

Fig. 27

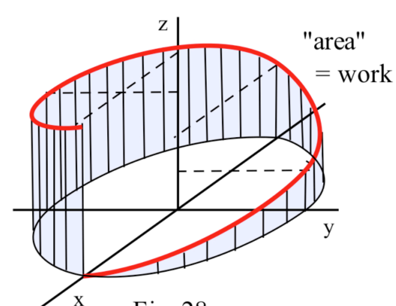

Fig. 28

Section 5.3 **PRACTICE Answers**

Practice 1: x_i = distance to top of cliff, Δx_i = distance lifted (length of "slice")

$w_i = (130 + 10 + 2x_i)\,\Delta x_i = (140 + 2x_i)\,\Delta x_i$.

$$\text{Total work} = \int_{x=0}^{x=30} (140 + 2x)\ dx = 140x + x^2 \Big|_0^{30}$$

$$= 4{,}200 + 900 = \mathbf{5{,}100\ \ foot{\cdot}pounds}.$$

Practice 2: Total work $\approx \{ \pi(\mathbf{1.4})^2(3.5) + \pi(\mathbf{1.6})^2(4.5) + \pi(\mathbf{1.5})^2(5.5) + \pi(\mathbf{1.0})^2(6.5) \}\cdot(0.5787)$

$\approx$ **67.73 inch·ounces**.

Practice 3: Total work to drink top 3 inches in Example 3 is

$$\int_{\mathbf{y=3}}^{y=6} \pi(y/3)^2(0.5787)(10 - y)\ dy = \frac{0.5787\pi}{9}\left\{ \frac{10}{3}x^3 - \frac{1}{4}x^4 \right\}\Big|_3^6$$

$$= \frac{0.5787\pi}{9}\{ 396 - 69.75 \} \approx \mathbf{65.904\ inch{\cdot}ounces}.$$

Practice 4: From Example 5 we know $f(x) = 7x$ so the total work "5 inches past its natural length to 10 inches past its natural length" is

$$\int_a^b f(x)\,dx = \int_5^{10} 7x\ dx = \frac{7x^2}{2}\Big|_5^{10} = \mathbf{262.5\ \ inch\text{–}pounds}.$$

Graphically, this total work is the area of the trapezoidal region bounded by $y = 7x$, the x–axis, and vertical lines at $x = 5$ and $x = 10$.

Practice 5: Be careful to do all calculations based on the amount of **stretch**, not just on the length. $f(x) = kx$ and we are told that $f(5) = 2$ so $2 = 5k$ and $k = 2/5$. Then the total work is

$$\int_a^b f(x)\,dx = \int_2^7 \frac{2}{5}x\ dx = \frac{x^2}{5}\Big|_2^7 = \frac{45}{5} = \mathbf{9\ \ inch\text{–}pounds}.$$

Practice 6: $$\text{work} = \int_a^b f(x)\,dx = \int_{\mathbf{2R}}^{\mathbf{3R}} \frac{R^2P}{x^2}\ dx = R^2P\left(-\frac{1}{x}\right)\Big|_{2R}^{3R}$$

$$= R^2P\left(-\frac{1}{3R}\right) - R^2P\left(-\frac{1}{2R}\right) = \mathbf{\frac{RP}{6}\ \ mile\text{–}pounds}.$$