5.4 MOMENTS & CENTERS OF MASS

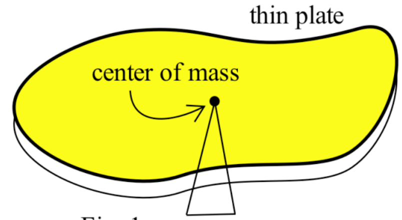
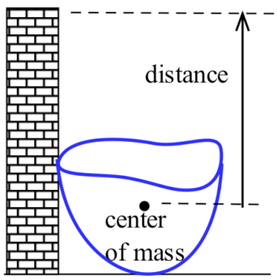


Fig. 1

This section develops a method for finding the center of mass of a thin, flat shape — the point at which the shape will balance without tilting (Fig. 1). Centers of mass are important because in many applied situations an object behaves as though its entire mass is located at its center of mass. For example, the work done to pump the water in a tank to a higher point is the same as the work to move the center of mass of the water to the higher point (Fig. 2), a much easier problem, if we know the mass and the center of mass of the water. Also, volumes and surface areas of solids of revolution can be easy to calculate, if we know the center of mass of the region being revolved.



work = (force)(distance)
Fig. 2

POINT MASSES

Before looking for the centers of mass of complicated regions, we consider point masses and systems of point masses, first in one dimension and then in two dimensions.

Point Masses Along A Line


Fig. 3

Two people with different masses can position themselves on a seesaw so that the seesaw balances (Fig. 3). The person on the right causes the seesaw to "want to turn" clockwise about the fulcrum, and the person on the left causes it to "want to turn" counterclockwise. If these two "tendencies" are equal, the seesaw will balance. A measure of this tendency to turn about the fulcrum is called the **moment** about the fulcrum of the system, and its magnitude is the mass multiplied by the distance from fulcrum.

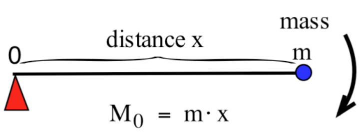


Fig. 4

In general, the **moment about the origin, M_0 ,** produced by a mass m at a location x is $m \cdot x$, the product of the mass and the "signed distance" of the mass from the origin (Fig. 4). For a system of masses $m_1, m_2, \ldots, m_n$ at locations $x_1, x_2, \ldots, x_n$, respectively,

$$M = \text{total mass of the system} = \sum_{i=1}^{n} m_i \text{ , and}$$

$$M_0 = \text{moment about the origin} = x_1 \cdot m_1 + x_2 \cdot m_2 + x_3 \cdot m_3 + \ldots + x_n \cdot m_n = \sum_{i=1}^{n} x_i \cdot m_i \text{ .}$$

If the moment about the origin is positive then the system tends to rotate clockwise about the origin. If the moment about the origin is negative then the system tends to rotate counterclockwise about the origin. If the moment about the origin is zero, then the system does not tend to rotate in either direction about the origin; it balances on a fulcrum at the origin.

The **moment about the point p, M_p** , produced by a mass m at the location x is the signed distance of x from p times the mass m: $(x-p)\cdot m$. The moment about the point p produced by masses $m_1, m_2, \ldots, m_n$ at locations $x_1, x_2, \ldots, x_n$, respectively, is

$$M_p = \text{moment about the point } p = (x_1-p)\cdot m_1 + (x_2-p)\cdot m_2 + \ldots + (x_n-p)\cdot m_n = \sum_{i=1}^{n} (x_i-p)\cdot m_i \,.$$

The point at which the system balances is called the **center of mass** of the system and is written $\bar{x}$ (pronounced "x bar") . Since the system balances at $\bar{x}$, the moment about $\bar{x}$ must be zero. Using this fact and properties of summation, we can find a formula for $\bar{x}$.

$$0 = M_{\bar{x}} = \text{moment about } \bar{x} = \sum_{i=1}^{n} (x_i - \bar{x})\cdot m_i = \sum_{i=1}^{n} (x_i m_i - \bar{x}\, m_i)$$

$$= \sum_{i=1}^{n} x_i m_i - \sum_{i=1}^{n} \bar{x}\, m_i = \Big(\sum_{i=1}^{n} x_i m_i\Big) - \bar{x}\Big(\sum_{i=1}^{n} m_i\Big) \quad \text{, since } \bar{x} \text{ is a constant.}$$

So $\left(\sum_{i=1}^{n} x_i m_i\right) = \bar{x}\left(\sum_{i=1}^{n} m_i\right)$, and $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} = \dfrac{M_0}{M} = \dfrac{\text{moment about the origin}}{\text{total mass}}$.

The **center of mass** of a system of masses $m_1, m_2, \ldots, m_n$ at locations $x_1, x_2, \ldots, x_n$ is the point $\bar{x}$ at which the system balances. The moment of the system about $\bar{x}$ is zero.

$$\bar{x} = (\text{moment about the origin}) \,/\, (\text{total mass}) = M_0 \,/\, M = \sum_{i=1}^{n} x_i\cdot m_i \Big/ \sum_{i=1}^{n} m_i \,.$$

The single point mass with mass M located at $\bar{x}$, the center of mass of the system, produces the same moment about any point on the line as the whole system. For many purposes, the mass of the entire system can be thought of as "concentrated at $\bar{x}$."

Example 1: Find the center of mass of the first three point–masses given in Table 1.

Solution: $M = 2 + 3 + 1 = 6$. $M_0 = (2)(-3) + (3)(4) + (1)(6) = 12$.

$\bar{x} = M_0 / M = 12/6 = 2$.

The first three point–masses will balance on a fulcrum located at 2.

i	m_i	x_i
1	2	–3
2	3	4
3	1	6
4	5	–2
5	3	4

Table 1

Practice 1: Find the center of mass of the last three point–masses given in Table 1.

Point Masses In The Plane

The ideas of moments and centers of mass extend nicely from one dimension to a system of masses located at points in the plane. For a "knife edge" fulcrum located along the y–axis (Fig. 5), the moment of a mass m at the point (x, y) is the mass times the signed distance of the mass from the y–axis:

(mass)(signed distance from the y–axis) = $m \cdot x$. This "tendency to rotate about the y–axis" is called the **moment about the y–axis**, written M_y : $\mathbf{M_y = m \cdot x}$. Similarly, the mass M at the point (x, y) has a **moment about the x–axis** (Fig. 6): $\mathbf{M_x = m \cdot y}$. For a system of masses m_i located at the points (x_i, y_i) ,

$$M = \text{total mass} = \sum_{i=1}^{n} m_i \qquad M_y = \text{moment about y–axis} = \sum_{i=1}^{n} m_i \cdot x_i$$

$$M_x = \text{moment about x–axis} = \sum_{i=1}^{n} m_i \cdot y_i$$

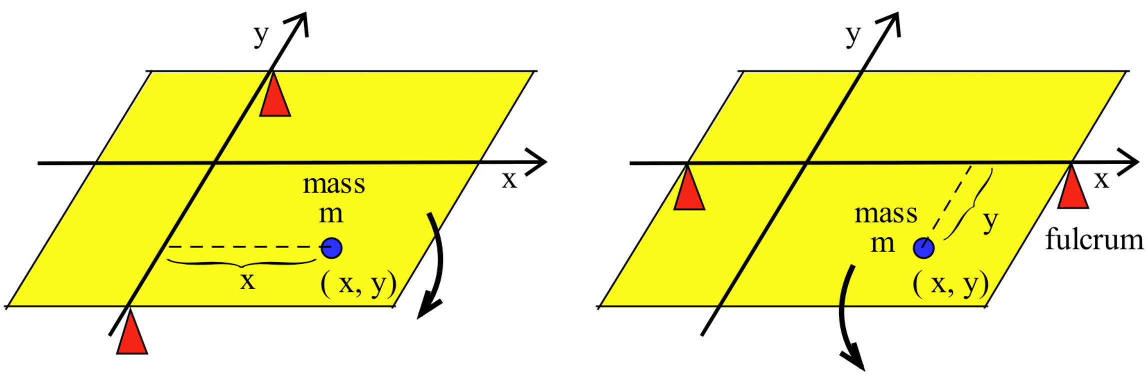


Fig. 5 Fig. 6

The total mass M of the system, located at the center of mass $(\bar{x}, \bar{y})$, has the same moment about any line as the entire system has about that line. For the moment about the y–axis, $M \cdot \bar{x} = M_y$ so $\bar{x} = M_y / M$. Similarly, for the moment about the x–axis, $M \cdot \bar{y} = M_x$ so $\bar{y} = M_x/M$.

Point Masses:	**Along a Line**	**In the Plane**
masses:	$m_1, m_2, \ldots, m_n$	$m_1, m_2, \ldots, m_n$
locations:	$x_1, x_2, \ldots, x_n$	$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$
total mass:	$M = \sum_{i=1}^{n} m_i$	$M = \sum_{i=1}^{n} m_i$
moments:	$M_0 = \sum_{i=1}^{n} m_i \cdot x_i$	$M_y = \sum_{i=1}^{n} m_i \cdot x_i$, $M_x = \sum_{i=1}^{n} m_i \cdot y_i$
center of mass:	$\bar{x} = M_0 / M$	$\bar{x} = M_y / M$, $\bar{y} = M_x / M$

Example 2: Find the center of mass of the first three point–masses in Table 2.

Solution: $M = 2 + 3 + 1 = 6$. $M_y = (2)(-3) + (3)(4) + (1)(6) = 12$.

$M_x = (2)(4) + (3)(-7) + (1)(-2) = -15$.

$\bar{x} = M_y / M = 12/6 = 2$. $\bar{y} = M_x / M = -15/6 = -2.5$.

The first three point–masses will balance at the point $(2, -2.5)$.

i	m_i	x_i	y_i
1	2	–3	4
2	3	4	–7
3	1	6	–2
4	5	–2	1
5	3	4	–6

Table 2

Practice 2: Find the center of mass of all five point–masses in Table 2.

CENTER OF MASS OF A REGION

When we move from discrete point masses to whole, continuous regions in the plane, we move from finite sums and arithmetic to limits of Riemann sums, definite integrals, and calculus. The following material extends the ideas and calculations from point masses to uniformly thin, flat plates that have a constant density given as mass per area such as "grams/cm^2" (Fig. 7). The center of mass of one of these plates is the point $(\bar{x}, \bar{y})$ at which the plate balances without tilting. It turns out that the center of mass $(\bar{x}, \bar{y})$ of such a plate depends only on the region of the plane covered by the plate and not on its density.

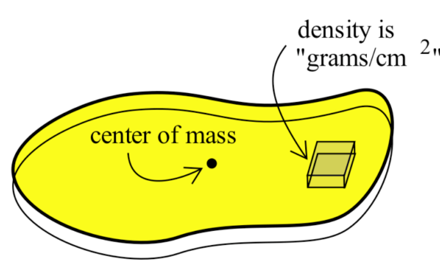


Fig. 7

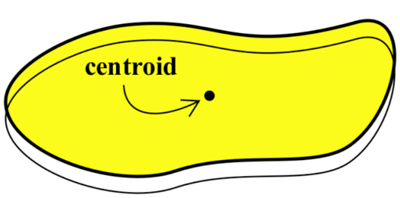


Fig. 8

The point $(\bar{x}, \bar{y})$ is also called the **centroid of the region**. (Fig. 8) In the following discussion, you should notice that each finite sum that appeared in the discussion of point masses has a counterpart for these thin plates in terms of integrals.

Rectangles

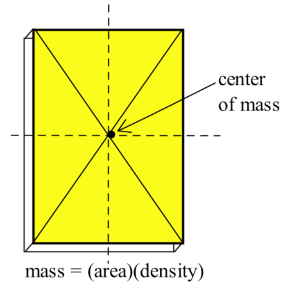


Fig. 9

The rectangle is the basic shape used to extend the point mass ideas to regions. The total mass of a rectangular plate is the area of the plate multiplied by the density constant: mass M = {area}{density}.

We assume that the center of mass of a thin, rectangular plate is located half way up and half way across the rectangle, at the point where the diagonals of the rectangle cross (Fig. 9). Then the moments of the rectangle can be found by treating the rectangle as a point with mass M located at the center of mass of the rectangle.

Example 3: Find the moments about the x–axis, y–axis, and the line x = 5 of the rectangular plate in Fig. 10.

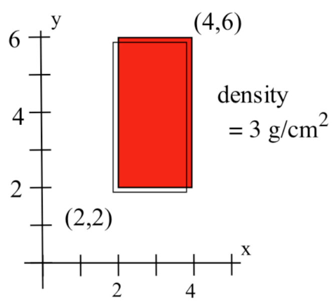


Fig. 10

Solution: The density of the plate is 3 g/cm^2 and the area of the plate is (2 cm)(4 cm) = 8 cm^2 so the total mass is M = (8 cm^2)(3 g/cm^2) = 24 g.

The center of mass of the rectangular plate is $(\bar{x}, \bar{y}) = (3, 4)$.

The moment about the x–axis is the mass multiplied by the signed distance of the mass from the x–axis:

M_x = (24 g)•{signed distance of (3,4) from the x–axis} = (24 g)(4 cm) = 96 g–cm.

Similarly,

M_y = (24 g)•{signed distance of (3,4) from the y–axis} = (24 g)(3 cm) = 72 g–cm.

The moment about the line x = 5 is

(24 g)•{signed distance of (3,4) from the line x = 5} = (24 g)(2 cm) = 48 g–cm.

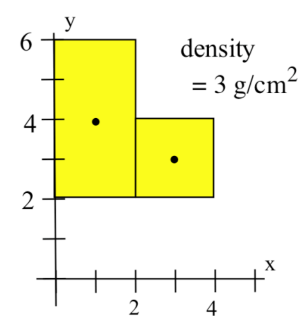


Fig. 11

To find the moments and center of mass of a plate made up of several rectangular regions, just treat each of the rectangular pieces as a point mass concentrated at its center of mass. Then the plate is treated as a system of discrete point masses.. The plate in Fig. 11 can be divided into two rectangular plates, one with mass 24 g and center of mass (1,4), and one with mass 12 g and center of mass (3,3). The total mass of the pair is M = 36 g, and the moments about the axes are

M_x = (24 g)(4 cm) + (12 g)(3 cm) = 132 g–cm, and

M_y = (24 g)(1 cm) + (12 g)(3 cm) = 60 g–cm.

Then $\bar{x} = M_y/M$ = (60 g–cm)/(36 g) = 5/3 cm and $\bar{y} = M_x/M$ = (132 g–cm)/(36 g) = 11/3 cm so the center of mass of the plate is at $(\bar{x}, \bar{y})$ = (5/3, 11/3).

Practice 3: Find the center of mass of the region in Fig. 12 .

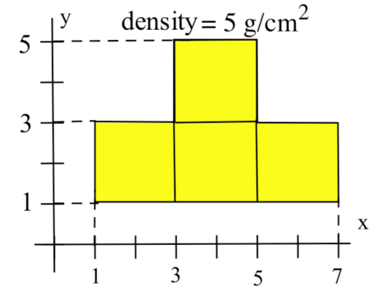


Fig. 12

To find the center of mass of a thin plate, we will "slice" the plate into narrow rectangular plates and treat the collection of rectangular plates as a system of point masses located at the centers of mass of the rectangles. The total mass and moments about the axes for the system of point masses will be Riemann sums. Then, by taking limits as the widths of the rectangles approach 0, we will obtain exact values for the mass and moments as definite integrals

$\bar{x}$ For A Region

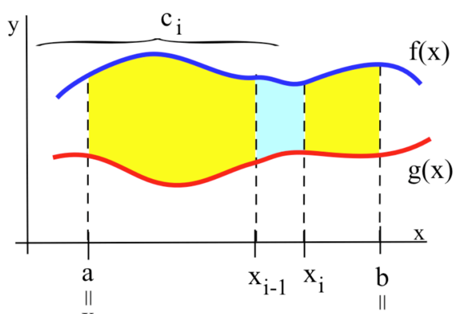


Fig.13

Suppose $f(x) \ge g(x)$ on $[a,b]$ and R is a plate on the region between the graphs of f and g for $a \le x \le b$ (Fig. 13). If the interval $[a, b]$ is partitioned into subintervals $[x_{i-1}, x_i]$ and the point c_i is the midpoint of each subinterval, then the slice between vertical cuts at x_{i-1} and x_i is approximately rectangular and has mass approximately equal to

$$\text{(area)(density)} = \text{(height)(width)(density)} \approx \{ f(c_i) - g(c_i) \}(x_{i-1} - x_i) k = \{ f(c_i) - g(c_i) \} (\Delta x_i) k .$$

The mass of the whole plate is approximately

$$M = \sum \{f(c_i) - g(c_i)\} (\Delta x_i) k \longrightarrow k \int_a^b \{ f(x) - g(x) \} dx = k \cdot \{ \text{area of the region between f and g} \}.$$

The moment about the y–axis of each rectangular piece is

$$\text{(distance from the y–axis to the center of mass of the piece)} \cdot \text{(mass)} = c_i \cdot \{f(c_i) - g(c_i)\} (\Delta x_i) k$$

so $$M_y = \sum c_i \{ f(c_i) - g(c_i) \} (\Delta x_i) k \longrightarrow k \int_a^b x \cdot \{ f(x) - g(x) \} dx .$$

Then the center of mass of the plate is $$\bar{x} = \frac{M_y}{M} = \frac{\int_a^b x \{ f(x) - g(x) \} dx}{\int_a^b f(x) - g(x) dx} .$$

The density constant k is a factor of M_y and of M, so it has no effect on the value of $\bar{x}$. The value of $\bar{x}$ depends only on the shape and location of the region, and $\bar{x}$ is the x coordinate of the centroid of the region between the graphs of $y = f(x)$ and $y = g(x)$ for $a \le x \le b$.

If the bottom curve is the x–axis, then $g(x) = 0$, and the previous results simplify to

$$M = k\int_a^b f(x)\,dx\,, \quad M_y = k\int_a^b x\cdot f(x)\,dx\,, \quad \text{and } \bar{x} = \frac{M_y}{M}\,.$$

Practice 4: Find the x–coordinate of the center of mass of the region between $f(x) = x^2$ and the x–axis for $0 \le x \le 2$. (In this case, $g(x) = 0$.)

$\bar{y}$ For A Region

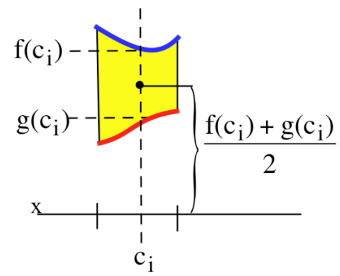


Fig. 14

To find $\bar{y}$, the y–coordinate of the center of mass of a plate R, we need to find M_x , the moment of the plate about the x–axis. When R is partitioned vertically (Fig. 14), the moment of each (very narrow) strip about the x–axis, M_x , is

(the signed distance from the x–axis to the center of mass of the strip)•(mass of strip).

Since each thin strip is approximately rectangular, the y–coordinate of the center of mass of each strip is approximately **half way** up the strip: $\bar{y}_i \approx \{ \mathbf{f(c_i)+g(c_i)} \}/2$. Then

M_x for the strip = (signed distance from the x–axis to the center of mass of the strip)•(mass of strip)

= (signed distance from x–axis)•(height of strip)•(width of strip)•(density constant)

$$= \frac{f(c_i)+g(c_i)}{2} \cdot (\, f(c_i) - g(c_i)\,)\cdot(\Delta x_i\,)\cdot k\,.$$

The total moment about the x–axis is $M_x = \sum_{i=1}^{n} \frac{f(c_i)+g(c_i)}{2} \cdot \{\, f(c_i) - g(c_i) \,\}\, (\Delta x_i\,)\, k$

$$\longrightarrow \quad k\int_a^b \frac{f(x)+g(x)}{2}\cdot\{f(x) - g(x)\}\,dx = \frac{k}{2}\int_a^b \{f^2(x) - g^2(x)\}\,dx\,.$$

If the lower curve is the x–axis, then $g(x) = 0$ and the formulas simplify to

$$M = k\int_a^b f(x)\,dx\,, \quad M_x = k\int_a^b \frac{f(x)}{2}\cdot f(x)\,dx = \frac{k}{2}\int_a^b f^2(x)\,dx\,, \quad \text{and } \bar{y} = \frac{M_x}{M}\,.$$

Example 4: Find the y–coordinate of the centroid of the region between the x–axis and the top half of a circle of radius r (Fig. 15).

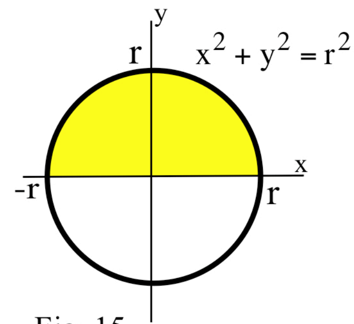


Fig. 15

Solution: The equation of the circle is $x^2 + y^2 = r^2$ so $f(x) = y = \sqrt{r^2 - x^2}$.

$$M = k \int_a^b f(x)\,dx = k \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx$$

$$= k\,\frac{1}{2}(\text{area of the circle of radius r}) = k\,\frac{1}{2}(\pi r^2) = \frac{1}{2}\,k\pi r^2 \ .$$

The region is symmetric about the y–axis so $\bar{x} = 0$. The moment of the region about the x–axis is

$$M_x = \frac{k}{2}\int_a^b f^2(x)\,dx = \frac{k}{2}\int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx = \frac{k}{2}\int_{-r}^{r} r^2 - x^2\,dx = \frac{k}{2}\left\{ r^2 x - \frac{x^3}{3} \right\}\Bigg|_{-r}^{r} = \frac{2}{3}\,k\,r^3 \ .$$

$$\text{Finally, } \bar{y} = \frac{M_x}{M} = \frac{\frac{2}{3}k\,r^3}{\frac{1}{2}k\pi r^2} = \frac{4}{3}\,\frac{r}{\pi} \approx 0.4244\,r \ .$$

Practice 5: Show that the centroid of a triangular region with vertices (0,0), (0,h) and (b,0) is $(\bar{x}, \bar{y}) = (b/3, h/3)$.

The integral formulas for moments are given below in a form useful for actually calculating moments of regions between the graphs of two functions, but it is also important that you understand the process used to derive the formulas.

	Point masses in the plane	**Region between f and g for a ≤ x ≤ b (f ≥ g)**
total mass:	$M = \sum_{i=1}^{n} m_i$	$M = \int_a^b \{\text{area}\}\{\text{density}\} = k \cdot \int_a^b f(x) - g(x)\, dx$
moments:	$M_y = \sum_{i=1}^{n} x_i \cdot m_i$	$M_y = \int_a^b \{\text{dist. of c.m. of slice to y–axis}\}\{\text{mass}\}$ $= k \cdot \int_a^b x \cdot \{ f(x) - g(x) \}\, dx$
	$M_x = \sum_{i=1}^{n} y_i \cdot m_i$	$M_x = \int_a^b \{\text{dist. of c.m. of slice to x–axis}\}\{\text{mass}\}$ $= \frac{k}{2} \cdot \int_a^b \{ f^2(x) - g^2(x)\}\, dx$
center of mass:	$\bar{x} = \frac{M_y}{M}$, $\bar{y} = \frac{M_x}{M}$	$\bar{x} = \frac{M_y}{M}$, $\bar{y} = \frac{M_x}{M}$

Example 5: Find the centroid of the region bounded between the graphs of $y = x$ and $y = x^2$ for $0 \le x \le 1$.

Solution: $M = k \cdot \int_0^1 (x - x^2)\, dx = \frac{k}{6}$, $M_y = k \cdot \int_0^1 x(x - x^2)\, dx = \frac{k}{12}$ and

$M_x = \frac{k}{2} \cdot \int_0^1 (x)^2 - (x^2)^2\, dx = \frac{k}{15}$. Then $\bar{x} = M_y / M = 1/2$ and $\bar{y} = \frac{M_x}{M} = 2/5$.

Symmetry

Symmetry is a very powerful geometric concept that can simplify many mathematical and physical problems, including the task of finding centroids of regions. For some regions, we can use symmetry alone to determine the centroid.

Geometrically, a region R is **symmetric about a line L** if, when R is folded along the line L, each point of R on one side of the fold matches up with one point of R on the other side of the fold (Fig. 16).

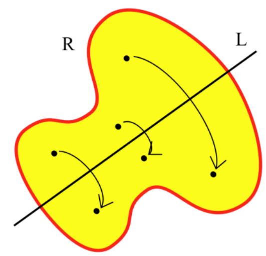


Fig. 16

Example 6: Sketch two lines of symmetry for each region in Fig. 17.

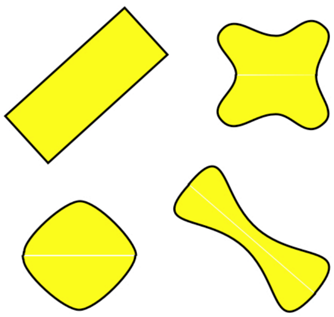


Fig. 17

Solution: The lines of symmetry are shown in Fig. 18. Every line through the center of the circular region is a line of symmetry.

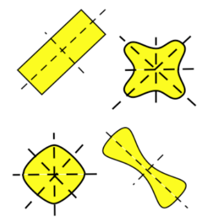


Fig. 18

A very useful fact about symmetric regions is that the centroid $(\bar{x}, \bar{y})$ of a symmetric region must lie on **every** line of symmetry of the region. If a region has two different lines of symmetry, then the centroid must lie on each of them, so the centroid is located at the point where the lines of symmetry intersect.

Practice 6: Locate the centroid of each region in Fig. 17.

WRAP UP

The purpose of this section is to illustrate again the process of going from an applied problem, to an approximate solution as a Riemann sum, to an exact solution represented as a definite integral. The emphasis is on using calculus to solve an applied problem. However, centroids and centers of mass can themselves be used to solve other applied problems. If we know the center of mass of a region, then some work, volume of revolution, and surface area of revolution problems become simple. These applications of centroids and centers of mass are discussed very briefly in the "optional" section of the problems, and a physical method for determining the centroid of a region is described. Centers of mass of regions with variable density are discussed in a later chapter.

PROBLEMS

1. (a) Find the total mass and the center of mass for a system consisting of the 3 masses in Table 3.
   (b) Where should you locate a new object with mass 8 so the new system has its center of mass at x = 5?
   (c) How much mass should be located at x = 10 so the original system plus the new mass at x = 10 has is center of mass at x = 6?

m	x
2	4
5	2
5	6

Table 3

2. (a) Find the total mass and the center of mass for a system consisting of the 4 masses in Table 4.
   (b) Where should you locate a new object with mass 10 so the new system has its center of mass at x = 6?
   (c) How much mass should be located at x = 14 so the original system plus the new mass at x = 14 has is center of mass at x = 6?

m	x
5	1
3	7
2	5
6	2

Table 4

3. (a) Find the total mass and the center of mass for a system consisting of the 3 masses in Table 5.
   (b) Where should you locate a new object with mass 10 so the new system has its center of mass at (5, 2)?

m	x	y
2	4	3
5	2	4
5	6	2

Table 5

4. (a) Find the total mass and the center of mass for a system consisting of the 4 masses in Table 6.
   (b) Where should you locate a new object with mass 12 so the new system has its center of mass at (3, 5)?

m	x	y
1	5	4
2	2	7
3	1	0
5	3	8

Table 6

In problems 5 – 10, divide the flat plate in each Figure into rectangles and semicircles, calculate the mass, moments and centers of mass of each piece, and use those results to find the center of mass of the plate. Assume that the density of the plate is 1 g/cm^2 . Plot the location of the center of mass for each shape. (**See Example 4 for the centroid of a semicircular region.**)

5. Fig. 19. 6. Fig. 20. 7. Fig. 21. 8. Fig. 22. 9. Fig. 23. 10. Fig. 24.

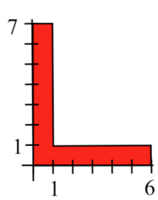


Fig. 19

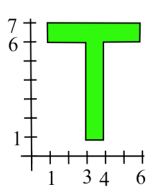


Fig. 20

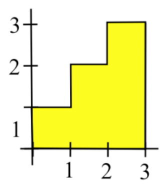


Fig. 21

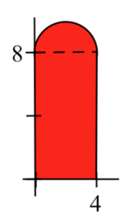


Fig. 22

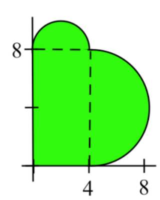


Fig. 23

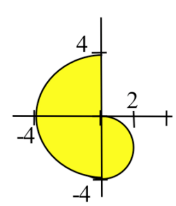


Fig. 24

In problems 11 – 26, sketch the region bounded between the given functions on the interval and calculate the centroid of each region (use Simpson's rule with n = 20 if necessary). Plot the location of the centroid on your sketch of the region.

11. $y = x$ and the x–axis for $0 \le x \le 3$.
12. $y = x^2$ and the x–axis for $-2 \le x \le 2$.
13. $y = x^2$ and the line $y = 4$ for $-2 \le x \le 2$.
14. $y = \sin(x)$ and the x–axis for $0 \le x \le \pi$.
15. $y = 4 - x^2$ and the x–axis for $-2 \le x \le 2$.
16. $y = x^2$ and $y = x$ for $0 \le x \le 1$.
17. $y = 9 - x$ and $y = 3$ for $0 \le x \le 3$.
18. $y = \sqrt{1 - x^2}$ and the x–axis for $0 \le x \le 1$.
19. $y = \sqrt{x}$ and the x–axis for $0 \le x \le 9$.
20. $y = \ln(x)$ and the x–axis for $1 \le x \le e$.
21. $y = e^x$ and the line $y = e$ for $0 \le x \le 1$.
22. $y = x^2$ and the line $y = 2x$ for $0 \le x \le 2$.
23. An empty one foot square tin box (Fig. 25) weighs 10 pounds and its center of mass is 6 inches above the bottom of the box. When the box is **full** with 60 pounds of liquid, the center of mass of the box–liquid system is again 6 inches above the bottom. (a) Write the height of the center of mass of the box–liquid system as a function of the x, the height of liquid in the box.
(b) What height of liquid in the bottom of the box results in the box–liquid system having the lowest center of mass (and the greatest stability)?

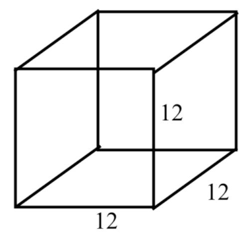


Fig. 25

24. The empty can in Fig. 26 weighs 1 ounce when empty and 13 ounces when full. Write the height of the center of mass of the can–liquid system as a function of the height of the liquid in the can.
25. The empty glass in Fig. 27 weighs 4 ounces when empty and 20 ounces when full. Write the height of the center of mass of the glass–liquid system as a function of the height of the liquid in the glass.

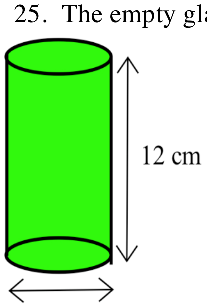


Fig. 26

26. Give a **practical** set of directions someone could actually use to find the **height** of the center of gravity of their body with their arms at their sides (Fig. 28). How will the height of the center of gravity change if they lift their arms? (In a uniform gravitational field such as the surface of the earth, the center of mass and center of gravity are at the same point.)

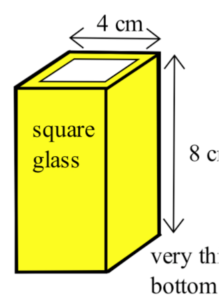


Fig. 27

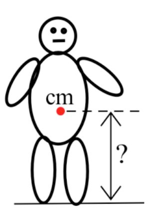


Fig. 28

27. Try the following experiment. Stand straight with your back and heels against a wall. Slowly raise one leg, keeping it straight, in front of you. What happened? Why?

28. Why can't two dancers stand in the position shown in Fig. 29?

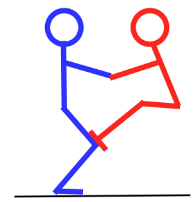


Fig. 29

29. Determine the centroid of your state.

30. (a) Sketch regions with exactly 2 lines of symmetry, exactly 3 lines of symmetry, and exactly 4 lines of symmetry.

(b) If a shape has exactly two lines of symmetry, the lines can meet at right angles. Do they have to meet at right angles?

Work

In a uniform gravitational field, the center of gravity of an object is at the same point as the center of mass of the object, and the work done to lift an object is the weight of the object multiplied by the distance that the center of gravity of the object is raised:

work = (total weight of object)•(distance the center of gravity of the object is raised).

In the high jump, this explains the effectiveness of the "Fosbury Flop", a technique in which the jumper assumes an inverted U position while going over the bar (Fig. 30). In this way, the jumper's body goes over the bar while the jumper's center of gravity goes under it, so a given amount of upward thrust produces a higher bar cleared.

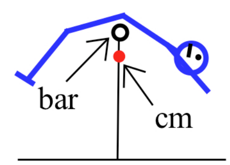


Fig. 30

If the center of gravity of an object is known, some work problems become easy.

31. A rectangular box is filled to a depth of 4 feet with 300 pounds of liquid. How much work is done pumping the liquid to a point 10 feet high? (How high is the center of gravity of the liquid, and how much must it be raised?)

32. A cylinder is filled to a depth of 2 feet with 40 pounds of liquid. How much work is done pumping the liquid to a point 7 feet high? (How high is the center of gravity of the liquid, and how much must it be raised?)

33. A sphere of radius 1 foot is filled with 250 pounds of liquid. How much work is done pumping the liquid to a point 3 feet above the top of the sphere? (Draw a picture.)

34. A sphere of radius 2 foot is filled with 2000 pounds of liquid. How much work is done pumping the liquid to a point 5 feet above the top of the sphere?

If the amount of work is already known, it can be used to find the height of the center of gravity.

Theorems of Pappus

When location of the center of mass of an object is known, the theorems of Pappus make some volume and surface area calculations very easy.

Volume of Revolution:

If a plane region with **area** A and centroid $(\bar{x}, \bar{y})$ is revolved around a line in the plane which does not go through the region (touching the boundary is alright),

then the **volume** swept out by one revolution is the area of the region times the distance traveled by the centroid (Fig. 31):

$$\text{Volume about line L} = A \cdot 2\pi \cdot \{ \text{distance of } (\bar{x}, \bar{y}) \text{ from the line L} \}.$$

$$\text{Volume about x–axis} = A \cdot 2\pi \cdot \bar{y} . \qquad \text{Volume about y–axis} = A \cdot 2\pi \cdot \bar{x} .$$

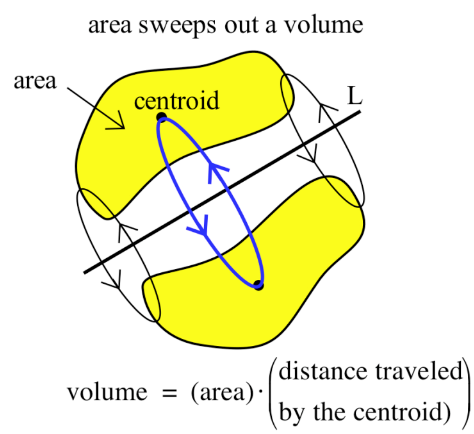
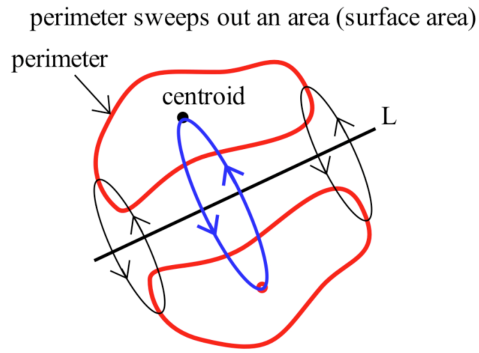


Fig. 31

Surface Area of Revolution



$$\text{surface area} = (\text{perimeter})\cdot\begin{pmatrix}\text{distance traveled}\\ \text{by the centroid}\end{pmatrix}$$

Fig. 32

If a plane region with **perimeter** P and centroid of the edge $(\bar{x}, \bar{y})$ is revolved around a line in the plane which does not go through the region (touching the boundary is alright),

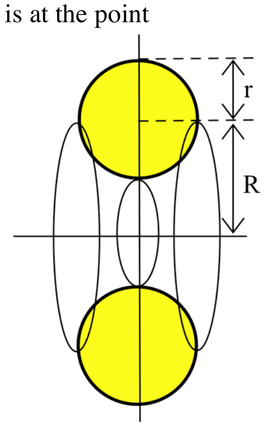
then the **surface area** swept out by one revolution is the perimeter of the region times the distance traveled by the centroid (Fig. 32):

Surface area about line L = P•2π•{ distance of $(\bar{x}, \bar{y})$ from the line L }.

Surface area about x–axis = P• 2π• $\bar{y}$. Surface area about y–axis = P• 2π• $\bar{x}$.

35. The center of a square region with 2 foot sides is at the point (3,4). Use the Theorems of Pappus to find the volume and surface area swept out when the square is rotated (a) about the x–axis, (b) about the y–axis, and (c) about the horizontal line y = 6.

36. The lower left corner of a rectangular region with an 8 inch base and a 4 inch height is at the point (3,5). Use the Theorems of Pappus to find the volume and surface area swept out when the rectangle is rotated (a) about the x–axis, (b) about the y–axis, and (c) about the line y = x + 5.



torus ("doughnut")
Fig. 33

37. The center of a circle with radius 2 feet is at the point (3,5). Use the Theorems of Pappus to find the volume and surface area swept out when the circular region is rotated (a) about the x–axis, (b) about the y–axis, and (c) about the vertical line x = 6.

38. The center of a circle (Fig. 33) with radius r is at the point (0, R). Use the Theorems of Pappus to find the volume and surface area swept out when the circular region is rotated about the x–axis.

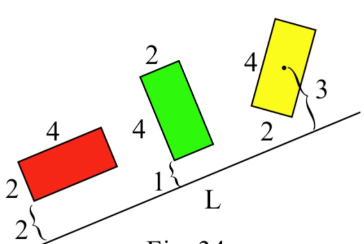


Fig. 34

39. Find the volumes and surface areas swept out when the rectangles in Fig. 34 are rotated about the line L. (Measurements are in feet.)

Physically Approximating Centoids of Regions

The centroid of a region can be approximated experimentally, even if the region, such as a state or country, is not described by a formula.

Cut the shape out of a piece of some uniformly thick material such as paper and pin an edge to a wall. The shape will pivot about the pin until its center of mass is directly below the pin (Fig. 35) so the center of mass of the shape must lie directly below the pin, on the line connecting the pin with the center of mass of the earth. Repeat the process using a different point near the edge of the shape and a different line can be found. The center of mass also lies on the new line, so we can conclude that the centroid of the shape is located where the two lines intersect, the only point located on both lines (Fig. 36). It is a good idea to pick a third point near the edge and plot a third line. This line should also pass through the point of intersection of the other two lines.

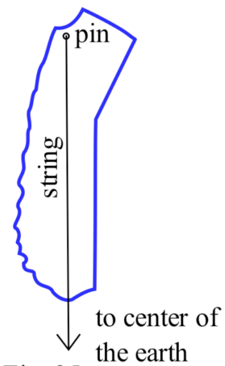


Fig. 35

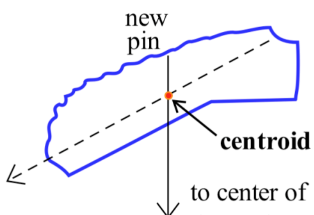


Fig. 36

The "population center" of a region can be physically approximated by attaching weights proportional to the populations of the different areas and then repeating the "pin" process with this weighted model. The point on the new model where the lines intersect is the approximate "population center" of the region.

Section 5.4 **PRACTICE Answers**

Practice 1: $M = 1 + 5 + 3 = 9$. $M_0 = (1)(6) + (5)(-2) + (3)(4) = 8$.

$\bar{x} = M_0 / M = 8/9$

The last three point–masses will balance on a fulcrum located at $x = 8/9$.

i	m_i	x_i
1	2	–3
2	3	4
3	1	6
4	5	–2
5	3	4

Table 1

Practice 2: $M = 2 + 3 + 1 + 5 + 3 = 14$.

$M_y = (2)(-3) + (3)(4) + (1)(6) + (5)(-2) + (3)(4) = 14$.

$M_x = (2)(4) + (3)(-7) + (1)(-2) + (5)(1) + (3)(-6) = -28$.

$\bar{x} = M_y / M = 14/14 = 1$. $\bar{y} = M_x / M = -28/14 = -2$.

The five point–masses balance at the point $(1, -2)$.

i	m_i	x_i	y_i
1	2	–3	4
2	3	4	–7
3	1	6	–2
4	5	–2	1
5	3	4	–6

Table 2

Practice 3: There are several ways to break the region in Fig. 12 into "easy" pieces — one way is to consider the four 2–by–2 cm squares.

The cm of each square is located at the center of the square (at (2,2), (4,2), (6,2), and (4,4)), and each square has mass $(4\text{ cm}^2)(5\text{ g/cm}^2) = 20$ g.

$M = 4(20\text{ g}) = 80$ g.

$M_y = 2(20) + 4(20) + 6(20) + 4(20) = 320$ g•cm

$M_x = 2(20) + 2(20) + 2(20) + 4(20) = 200$ g•cm

Then $\bar{x} = M_y / M = \dfrac{320\text{ g•cm}}{80\text{ g}} = \mathbf{4\ cm}$ and

$\bar{y} = M_x / M = \dfrac{200\text{ g•cm}}{80\text{ g}} = \mathbf{2.5\ cm}$.

The center of mass is (4, 2.5) .

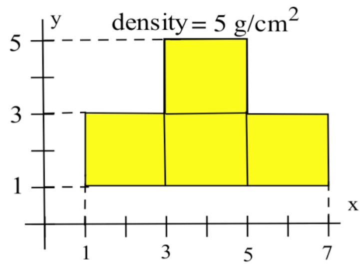


Fig. 12

Practice 4: $M = k\int_a^b f(x)\,dx == k\int_0^2 x^2\,dx = \frac{8}{3}k$, $M_y = k\int_a^b x\cdot f(x)\,dx = k\int_0^2 x\cdot x^2\,dx = 4k$.

Then $\bar{x} = M_y / M = \dfrac{4k}{\frac{8}{3}k} = \mathbf{1.5}$.

Practice 5: The triangular region is shown in Fig. 36 : $f(x) = h - \frac{h}{b}x$ for $0 \le x \le b$.

$$M = k\int_a^b f(x)\,dx = k\int_0^b \left(h - \frac{h}{b}x\right) dx = k\left\{ hx - \frac{h}{b}\frac{1}{2}x^2 \right\}\Big|_0^b$$

$$= k\left\{ hb - \frac{h}{b}\frac{b^2}{2} \right\} = \frac{k}{2}hb .$$

$$M_y = k\int_a^b x\cdot f(x)\,dx = k\int_0^b x\cdot\left(h - \frac{h}{b}x\right) dx = k\left\{ h\frac{x^2}{2} - \frac{h}{b}\frac{x^3}{3} \right)\Big|_0^b$$

$$= k\left\{ h\frac{b^2}{2} - \frac{h}{b}\frac{b^3}{3} \right\} = \frac{k}{6}hb^2 .$$

$$M_x = \frac{k}{2}\int_a^b f^2(x)\,dx = \frac{k}{2}\int_0^b \left(h - \frac{h}{b}x\right)^2 dx = \frac{kh^2}{2}\left\{ x - \frac{2}{b}\frac{x^2}{2} + \frac{1}{b^2}\frac{x^3}{3} \right\}\Big|_0^b$$

$$= \frac{kh^2}{2}\left\{ b - b + \frac{b}{3} \right\} = \frac{k}{6}h^2b^2 .$$

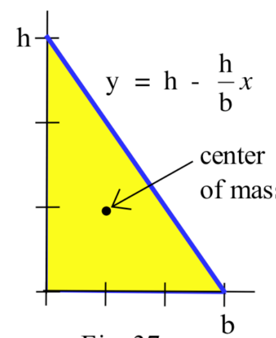


Fig. 37

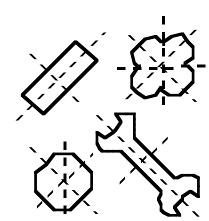


Fig. 18

Finally, $\bar{x} = \dfrac{M_y}{M} = \dfrac{\frac{k}{6}hb^2}{\frac{k}{2}hb} = \dfrac{b}{3}$ and $\bar{y} = \dfrac{M_x}{M} = \dfrac{\frac{k}{6}h^2b^2}{\frac{k}{2}hb} = \dfrac{h}{3}$ so **cm = (b/3, h/3)** .

Practice 6: The centroid of each region in Fig. 18 is located at the point where the lines of symmetry intersect.