6.0 INTRODUCTION TO DIFFERENTIAL EQUATIONS

This chapter is an introduction to differential equations, a major field in applied and theoretical mathematics and a very useful one for engineers, scientists, and others who study changing phenomena. The physical laws of motion and heat and electricity can be written as differential equations. The growth of a population, the changing gene frequencies in that population, and the spread of a disease can be described by differential equations. Economic and social models use differential equations, and the earliest examples of "chaos" came from studying differential equations used for modeling atmospheric behavior. Some scientists even say that the main purpose of a calculus course should be to teach people to understand and solve differential equations.

The purpose of this chapter is to introduce some basic ideas, vocabulary, and techniques for differential equations and to explore additional applications. Applications in this chapter include

exponential population growth and calculating how long it takes a population to double in size,

radioactive decay and its use for dating ancient objects and detecting fraud,

describing the motion of an object through a resisting medium, and

chemical mixtures and rates of reaction.

More complicated differential equations and ways of using and solving them are discussed in Chapter 17 and in later courses.

Differential Equations

Algebraic equations contain constants and variables, and the solutions of an algebraic equation are typically numbers. For example, $x = 3$ and $x = -2$ are solutions of the algebraic equation $x^2 = x + 6$. Differential equations contain derivatives or differentials of functions. Solutions of differential equations are functions. The differential equation $y' = 3x^2$ has infinitely many solutions, and two of those solutions are the functions $y = x^3 + 2$ and $y = x^3 - 4$ (Fig. 1).

Three solutions of $y' = 3x^2$

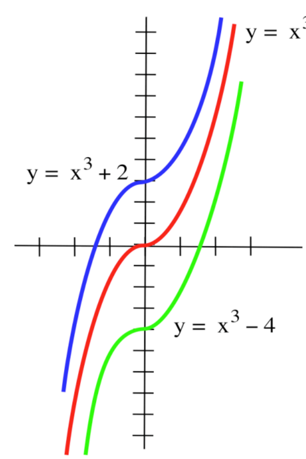

Fig. 1

You have already solved lots of differential equations: every time you found an antiderivative of a function $f(x)$, you solved the differential equation $y' = f(x)$ to get a solution y. You have also used differential equations in applications. Areas, volumes, work, and motion problems all involved integration and finding antiderivatives so they all used differential equations. The differential equation $y' = f(x)$, however, is just the beginning. Other applications generate different differential equations.

Checking Solutions of Differential Equations

Whether a differential equation is easy or difficult to solve, it is important to be able to check that a possible solution really satisfies the differential equation.

A possible solution of an algebraic equation can be checked by putting the solution into the equation to see if the result is true: $x = 3$ is a solution of $5x + 1 = 16$ since $5(3) + 1 = 16$ is true. Similarly, a solution of a differential equation can be checked by substituting the function and the appropriate derivatives into the equation to see if the result is true: $y = x^2$ is a solution of $xy' = 2y$ since $y' = 2x$ and $x(2x) = 2(x^2)$ is true.

Example 1: Check (a) that $y = x^2 + 5$ is a solution of $y'' + y = x^2 + 7$ and (b) that $y = x + 5/x$ is a solution of $y' + \frac{y}{x} = 2$.

Solution: (a) $y = x^2 + 5$ so $y' = 2x$ and $y'' = 2$. Substituting these functions for y and y'' into the differential equation $y'' + y = x^2 + 7$, we have $y'' + y = (2) + (x^2 + 5) = x^2 + 7$, so $y = x^2 + 5$ is a solution of the differential equation.

(b) $y = x + 5/x$ so $y' = 1 - 5/x^2$. Substituting these functions for y and y' in the differential equation $y' + \frac{y}{x} = 2$, we have

$$y' + \frac{y}{x} = (1 - 5/x^2) + \frac{1}{x}(x + 5/x) = 1 - 5/x^2 + 1 + 5/x^2 = 2,$$ the result we wanted to verify.

Practice 1: Check (a) that $y = 2x + 6$ is a solution of $y - 3y' = 2x$ and (b) that $y = e^{3x}$, $y = 5e^{3x}$, and $y = Ae^{3x}$ (A is a constant) are all solutions of $y'' - 2y' - 3y = 0$.

A solution of a differential equation with the initial condition $y(x_0) = y_0$ is a function that satisfies the differential equation and the initial condition. To check the solution of an initial value problem we must check that both the equation and the initial condition are satisfied.

Example 2: Which of the given functions is a solution of the initial value problem $y' = 3y$, $y(0) = 5$? (a) $y = e^{3x}$ (b) $y = 5e^{3x}$ (c) $y = -2e^{3x}$

Solution: All three functions satisfy the differential equation $y' = 3y$, but only one of the functions satisfies the initial condition that $y(0) = 5$. If $y = e^{3x}$, then $y(0) = e^{3(0)} = 1 \neq 5$ so $y = e^{3x}$ does not satisfy the initial condition (Fig. 2).
If $y = 5e^{3x}$, then $y(0) = 5e^{3(0)} = 5$ satisfies the initial condition.
If $y = -2e^{3x}$, then $y(0) = -2e^{3(0)} = -2 \neq 5$ so $y = e^{3x}$ does not satisfy the initial condition.

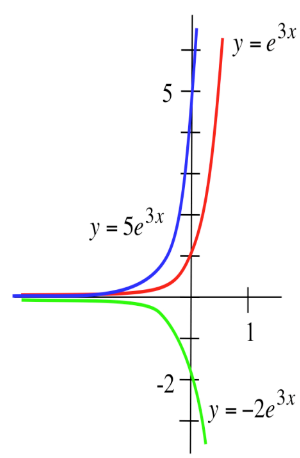

Fig. 2

Practice 2: Which function is a solution of the initial value problem $y'' = -9y$, $y(0) = 2$?

(a) $y = \sin(3x)$ (b) $y = 2\sin(3x)$ (c) $y = 2\cos(3x)$.

Finding the Value of the Constant

Differential equations usually have lots of solutions, a whole "family" of them, and each solution satisfies a different initial condition. To find which solution of a differential equation also satisfies a given initial condition $y(x_0) = y_0$, we can replace x and y in the solution function with the values x_0 and y_0 and algebraically solve for the value of the unknown constant.

Example 3: For every value of C, the function $y = Cx^2$ is a solution of $xy' = 2y$ (Fig. 3). Find the value of C so that $y(5) = 50$.

Solution: Substituting the initial condition $x = 5$ and $y = 50$ into the solution $y = Cx^2$, we have that $50 = C(5)^2$ so $C = 50/25 = 2$. The function $y = 2x^2$ satisfies both the differential equation and the initial condition.

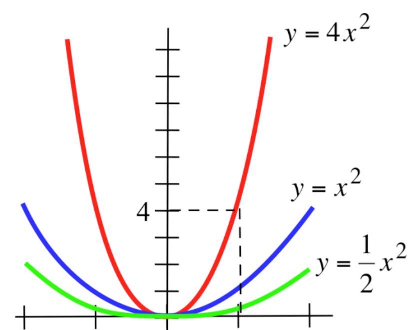

Fig. 3

Practice 3: For every value of C, the function $y = e^{2x} + C$ is a solution of $y' = 2e^{2x}$. Find the value of C so that $y(0) = 7$. (Be careful, $C \neq 7$.)

PROBLEMS

In problems 1 – 10, check that the function y is a solution of the given differential equation.

1. $y' + 3y = 6$. $y = e^{-3x} + 2$.
2. $y' - 2y = 8$. $y = e^{2x} - 4$.
3. $y'' - y' + y = x^2$. $y = x^2 + 2x$
4. $3y'' + y' + y = x^2 - 4x$. $y = x^2 - 6x$
5. $xy' - 3y = x^2$. $y = 7x^3 - x^2$
6. $xy'' - y' = 3$. $y = x^2 - 3x + 5$
7. $y' + y = e^x$. $y = \frac{1}{2} e^x + 2e^{-x}$
8. $y'' + 25y = 0$. $y = \sin(5x) + 2\cos(5x)$
9. $y' = -x/y$. $y = \sqrt{7 - x^2}$.
10. $y' = x - y$. $y = x - 1 + 2e^{-x}$.

In problems 11 – 20, check that the function y is a solution of the given initial value problem.

11. $y' = 6x^2 - 3$ and $y(1) = 2$. $y = 2x^3 - 3x + 3$.
12. $y' = 6x + 4$ and $y(2) = 3$. $y = 3x^2 + 4x - 17$.
13. $y' = 2\cos(2x)$ and $y(0) = 1$. $y = \sin(2x) + 1$.
14. $y' = 1 + 6\sin(2x)$ and $y(0) = 2$. $y = x - 3\cos(2x) + 5$.

15. $y' = 5y$ and $y(0) = 7$. $y = 7e^{5x}$.

16. $y' = -2y$ and $y(0) = 3$. $y = 3e^{-2x}$.

17. $xy' = -y$ and $y(1) = -4$. $y = -4/x$.

18. $yy' = -x$ and $y(0) = 3$. $y = \sqrt{9 - x^2}$.

19. $y' = 5/x$ and $y(e) = 3$. $y = 5\ln(x) - 2$.

20. $y' + y = e^x$ and $y(0) = 5$. $y = \frac{1}{2} e^x + \frac{9}{2} e^{-x}$.

In problems 21 – 30, a family of solutions of a differential equation is given. Find the value of the constant C so the solution satisfies the initial value condition.

21. $y' = 2x$ and $y(3) = 7$. $y = x^2 + C$.

22. $y' = 3x^2 - 5$ and $y(1) = 2$. $y = x^3 - 5x + C$.

23. $y' = 3y$ and $y(0) = 5$. $y = Ce^{3x}$.

24. $y' = -2y$ and $y(0) = 3$. $y = Ce^{-2x}$.

25. $y' = 6\cos(3x)$ and $y(0) = 4$. $y = 2\sin(3x) + C$.

26. $y' = 3 - 2\sin(2x)$ and $y(0) = 1$. $y = 3x + \cos(2x) + C$

27. $y' = 1/x$ and $y(e) = 2$. $y = \ln(x) + C$.

28. $y' = 1/x^2$ and $y(1) = 3$. $y = -1/x + C$.

29. $y' = -y/x$ and $y(2) = 10$. $y = -C/x$.

30. $y' = -x/y$ and $y(3) = 4$. $y = \sqrt{C - x^2}$.

In problems 31 – 40, find the function y which satisfies each initial value problem.

31. $y' = 4x^2 - x$ and $y(1) = 7$.

32. $y' = x - e^x$ and $y(0) = 3$.

33. $y' = 3/x$ and $y(1) = 2$.

34. $xy' = 1$ and $y(e) = 7$.

35. $y' = 6e^{2x}$ and $y(0) = 1$.

36. $y' = 36(3x - 2)^2$ and $y(1) = 8$.

37. $y' = x \cdot \sin(x^2)$ and $y(0) = 3$.

38. $y' = 6/x^2$ and $y(1) = 2$.

39. $xy' = 6x^3 - 10x^2$ and $y(2) = 5$.

40. $x^2y' = 6x^3 - 1$ and $y(1) = 10$.

41. Show that if $y = f(x)$ and $y = g(x)$ are solutions to $y' + 5y = 0$, then so are $y = 3 \cdot f(x)$, $y = 7 \cdot g(x)$, $y = f(x) + g(x)$, and $y = A \cdot f(x) + B \cdot g(x)$ for any constants A and B.

42. Show that if $y = f(x)$ and $y = g(x)$ are solutions to $y'' + 2y' - 3y = 0$, then so are $y = 3 \cdot f(x)$, $y = 7 \cdot g(x)$, $y = f(x) + g(x)$, and $y = A \cdot f(x) + B \cdot g(x)$ for any constants A and B.

43. Show that $y = \sin(x) + x$ and $y = \cos(x) + x$ are both solutions of $y'' + y = x$.
Are $y = 3\{\sin(x) + x\}$ and $y = \{\sin(x) + x\} + \{\cos(x) + x\}$ solutions of $y'' + y = x$?

44. Show that $y = e^{3x} - 2$ and $y = 5e^{3x} - 2$ are both solutions of $y' - 3y = 6$.
Are $y = 7\{e^{3x} - 2\}$ and $y = \{e^{3x} - 2\} + \{5e^{3x} - 2\}$ solutions of $y' - 3y = 6$?

45. $\frac{dy}{dt} = A - By$ (A and B positive constants) describes the concentration y of glucose in the blood at time t. Check that $y = \frac{A}{B} - C \cdot e^{-Bt}$ is a solution of the differential equation for every value of the constant C.

46. $\frac{dy}{dt} = Ay$ (A a constant) is used to model "exponential" growth and decay. Check that $y = C \cdot e^{At}$ is a solution of the differential equation for every value of the constant C.

47. $L\frac{dI}{dt} + RI = E$ (L, R, and E are positive constants) describes the current I in an electrical circuit. Show that $I = \frac{E}{R}(1 - e^{-Rt/L})$ is a solution of the differential equation.

48. $m \cdot y'' = -C \cdot y$ (C a positive constant) describes the position y of an object on a spring as it moves up and down. Show that $y = A \cdot \sin(wt) + B \cdot \cos(wt)$ with $w = \sqrt{C/m}$ is a solution of the differential equation for all values of the constants A and B.

Section 6.0 **PRACTICE Answers**

Practice 1: (a) $y = 2x + 6$. $y' = 2$. $y - 3y' = (2x+6) - 3(2) = 2x$ (OK)

(b) $y = e^{3x}$. $y' = 3e^{3x}$. $y'' = 9e^{3x}$. $y'' - 2y' - 3y = 9e^{3x} - 2(3e^{3x}) - 3(e^{3x}) = 0$ (OK)

$y = 5e^{3x}$. $y' = 15e^{3x}$. $y'' = 45e^{3x}$. $y'' - 2y' - 3y = 45e^{3x} - 2(15e^{3x}) - 3(5e^{3x}) = 0$ (OK)

$y = Ae^{3x}$. $y' = 3Ae^{3x}$. $y'' = 9Ae^{3x}$. $y'' - 2y' - 3y = 9Ae^{3x} - 2(3Ae^{3x}) - 3(Ae^{3x}) = 0$ (OK)

Practice 2: Want $y'' = -9y$, $y(0) = 2$.

(a) $y = \sin(3x)$, $y' = 3\cos(3x)$, $y'' = -9\sin(3x) = 9y$ (OK) but $y(0) = 2\sin(0) = 0 \neq 2$.

(b) $y = 2\sin(3x)$, $y' = 6\cos(3x)$, $y'' = -18\sin(3x) = 9y$ (OK) but $y(0) = 2\sin(0) = 0 \neq 2$.

(c) $y = 2\cos(3x)$. $y' = -6\sin(3x)$. $y'' = -18\cos(3x) = 9y$ (OK) and $y(0) = 2\cos(0) = 2$ (OK).

$y = 2\cos(3x)$ satisfies both the differential equation and the initial condition.

Practice 3: $y = e^{2x} + C$. $y' = 2e^{2x}$ (OK)

$7 = y(0) = e^{2 \cdot 0} + C = 1 + C$ so $C = 6$. Then $y = e^{2x} + 6$.