6.1 THE DIFFERENTIAL EQUATION $y' = f(x)$

This section introduces the basic concepts and vocabulary of differential equations as they apply to the familiar problem, $y' = f(x)$. The fundamental notions of a general solution of a differential equation, the particular solution of an initial value problem, and a direction field are introduced here and are used in later sections as we examine more complicated differential equations and their applications.

Solving $y' = f(x)$

The solution of the differential equation $y' = f(x)$ is the collection of all antiderivatives of f, $y = \int f(x)\, dx$. If $y = F(x)$ is one antiderivative of f, then we have essentially found all of the antiderivatives of f since every antiderivative of f has the form $F(x) + C$ for some value of the constant C. If F is an antiderivative of f, the collection of functions $F(x) + C$ is called the **general solution** of $y' = f(x)$. The general solution is a whole family of functions.

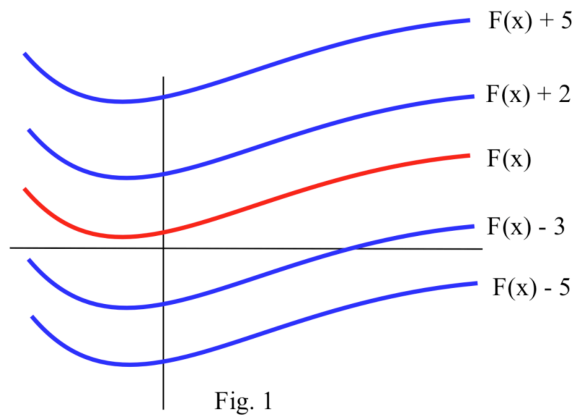


Fig. 1

Example 1: Find the general solution of the differential equation $y' = 2x + e^{3x}$.

Solution: $y = \int 2x + e^{3x}\, dx = \mathbf{x^2 + \frac{1}{3} e^{3x} + C}$.

Practice 1: Find the general solutions of the differential equations $y' = x + \dfrac{3}{x+2}$ and $y' = \dfrac{6}{x^2 + 1}$.

Direction Fields

Geometrically, a derivative gives the slope of the tangent line to a curve, so the differential equation $y' = f(x)$ can be interpreted as a geometric condition: at each point (x,y) on the graph of y, the slope of the tangent line is $f(x)$. The differential equation $y' = 2x$ says that at each point (x,y) on the graph of y, the **slope** of the line tangent to the graph is $2x$: if the point $(5,3)$ is on the graph of y then the slope of the tangent line there is $2 \cdot 5 = 10$. This information can be represented graphically as a **direction field** for $y' = 2x$. A direction field for $y' = 2x$ is a a collection of short line segments through a number of sample points (x, y) in the plane (Fig. 2), and the slope of the segment through (x,y) is $2x$. Figures 3a and 3b show direction fields for the differential equations $y' = 3x^2$ and $y' = \cos(x)$. For a differential equation of the form $y' = f(x)$, the values of y' depend only on x so along any vertical line (for a fixed value of x) all the line segments have the same y', the same slope, and they are parallel (Fig. 4).

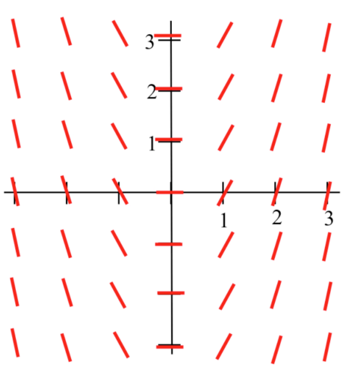
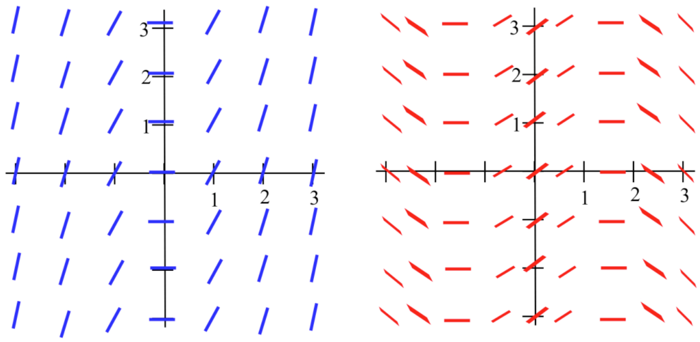


Fig. 2: Direction field for $y' = 2x$



(a) $y' = 3x^2$ (b) $y' = \cos(x)$

Fig. 3: Direction fields

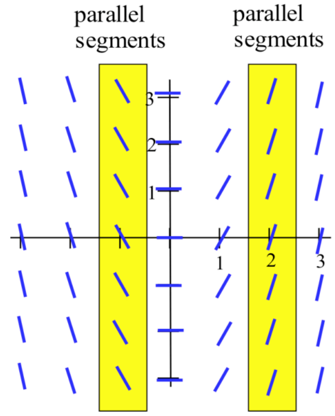


Fig. 4: Direction field for $y' = f(x)$

If y' depends on both x and y, then the slopes of the line segments will depend on both x and y, and the slopes along a vertical line can vary. Fig. 5 shows a direction field for the differential equation $y' = x - y$ in which y' is a function of both x and y.

A **direction field** of a differential equation $y' = g(x,y)$ is a collection of short line segments with slopes $g(x,y)$ at the points (x, y).

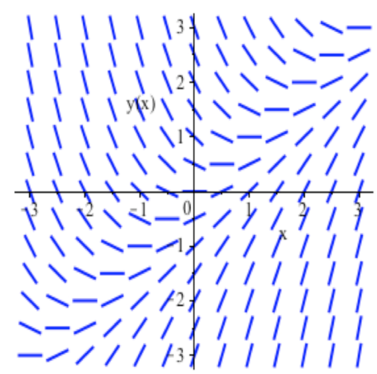


Fig. 5: Diretion field for $y' = x - y$

Practice 2: Construct direction fields for (a) $y' = x + 1$ and (b) $y' = x + y$ by sketching a short line segment with slope y' at each point (x, y) with integer coordinates between -3 and 3.

If the function f in the differential equation $y' = f(x)$ is given graphically, an approximate direction field can be constructed.

Example 2: Construct a direction field for the differential equation $y' = f(x)$ for the f shown in Fig. 6.

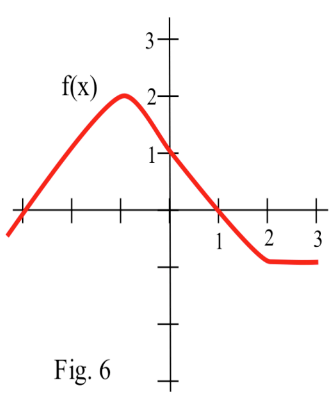


Fig. 6

Solution: If $x = 0$, then $y' = f(0) = 1$ so at every point on the vertical line $x = 0$ (the y–axis) the line segments of the direction field have slope $y' = 1$. Several short segments with slope 1 are shown along the y–axis in Fig. 7. Similarly, if $x = 1$, then $y' = f(1) = 0$ so the line segments of the direction field have slope $y' = 0$ at every point on the vertical line $x = 1$. The line segments along each vertical line are parallel.

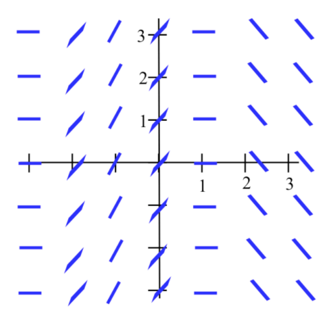


Fig. 7: Direction field for $y' = f(x)$

Practice 3: Construct a direction field for the differential equation $y' = f(x)$ for the f shown in Fig. 8.

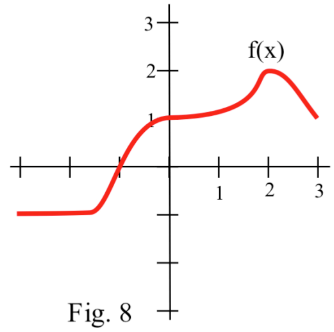


Fig. 8

Once we have a direction field for a differential equation, we can sketch several curves which have the appropriate tangent line slopes (Fig. 9). In that way we can see the shapes of the solutions even if we do not have formulas for them. These shapes can be useful for estimating which initial conditions lead to linear solutions or periodic solutions or solutions with other properties, and they can help us understand the behavior of machines and organisms.

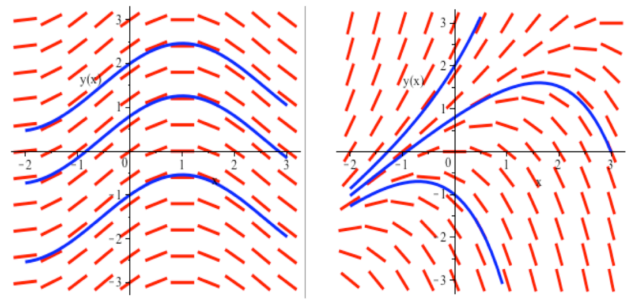


Fig. 9: Functions with the appropriate slopes for the two direction fields

Direction fields are tedious to plot by hand, but computers and calculators can plot them quickly. Programs are available for plotting direction fields on graphic calculators.

Initial Conditions and Particular Solutions

An **initial condition** $y(x_0) = y_0$ specifies that the solution y of the differential equation should go through the point (x_0, y_0) in the plane. To solve a differential equation with an initial condition, we typically use integration to find the general solution (the family of solutions which contains an arbitrary constant), and then we use algebra to find the value of the constant so the solution satisfies the initial condition. The member of the family that satisfies the initial condition is called a **particular solution**.

Example 3: Solve the differential equation $y' = 2x$ with the initial condition $y(2) = 1$.

Solution: The general solution is $y = \int 2x\,dx = x^2 + C$. Substituting the values $x_0 = 2$ and $y_0 = 1$ into the general solution, $1 = (2)^2 + C$ so $C = -3$. Then the solution we want is $y = x^2 - 3$. (A quick check verifies that $y' = 2x$ and $y = 1$ when $x = 2$.) Fig. 10 shows the direction field $y' = 2x$ and the particular solution which goes through the point $(2,1)$, $y = x^2 - 3$. The solution of the differential equation which satisfies the initial condition $y(0) = -1$ is also shown.

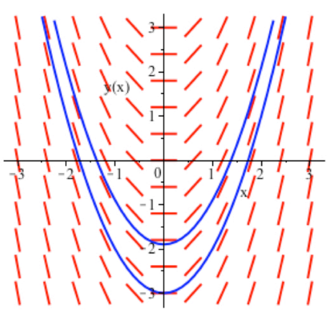


Fig. 10: Direction field $y'=2x$ and solutions y for $y(2)=1$ and $y(0)= -1$

Example 4: If a ball is tossed upward with a velocity of 100 feet/second, its height y at time t satisfies the differential equation $y' = 100 - 32t$. Sketch the direction field for y $(0 \le t \le 4)$ and then sketch the solution that satisfies the initial condition that the ball is 200 feet high after 3 seconds.

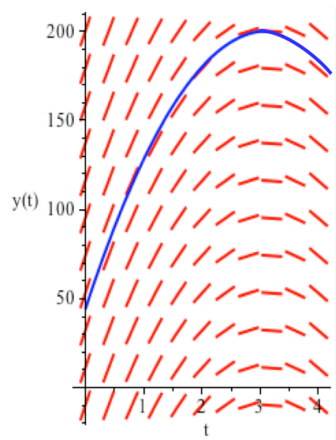


Fig. 11: Direction field $y' = 100 - 32t$ and solution with $y(3) = 200$

Solution: $y = \int 100 - 32t \, dt = 100t - 16t^2 + C$.

When $t=3$, $y=200$ so $200 = 100(3) - 16(3)^2 + C = 156 + C$ and $C = 200 - 156 = 44$.

The particular function we want is $y = 100t - 16t^2 + 44$. The direction field and the particular solution are shown in Fig. 11.

Practice 4: Find the solution of $y' = 9x^2 - 6\sin(2x) + e^x$ which goes through the point $(0,6)$.

Example 5: A direction field for $y' = x - y$ is shown in Fig. 12. Sketch the three particular solutions of the differential equation $y' = x - y$ which satisfy the initial conditions $y(0) = 2$, $y(0) = -1$, and $y(1) = -2$.

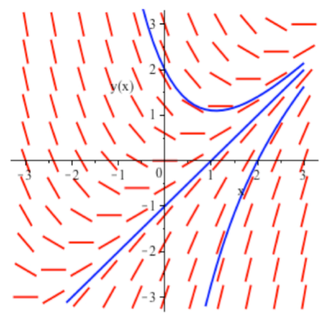


Fig. 12: Particular solutions of $y' = x - y$

Solution: The three particular solutions are shown in Fig. 12.

PROBLEMS

In problems 1 – 6, the direction field of a differential equation is shown. Sketch the solutions which satisfy the given initial conditions.

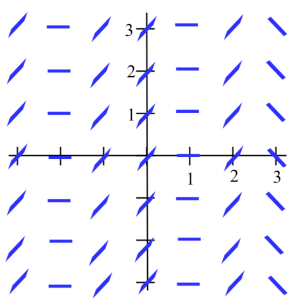


Fig. 13: Direction field for $y' = f(x)$

1. Fig. 13. The initial conditions are (a) $y(0) = 1$, (b) $y(1) = -2$, and (c) $y(1) = 3$.

2. Fig. 13. The initial conditions are (a) $y(0) = 2$, (b) $y(1) = -1$, and (c) $y(0) = -2$.

3. Fig. 14. The initial conditions are (a) $y(-2) = 1$, (b) $y(0) = 1$, and (c) $y(2) = 1$.

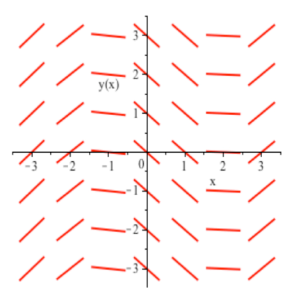


Fig. 14: Direction field for $y' = g(x)$

4. Fig. 14. The initial conditions are (a) $y(-2) = -1$, (b) $y(0) = -1$, and (c) $y(2) = -1$.

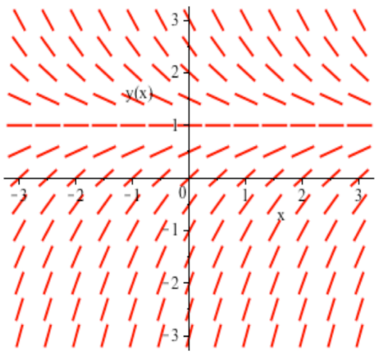


Fig. 15: Direction field for y'=f(x)

5. Fig. 15. The initial conditions are (a) $y(0) = -2$, (b) $y(0) = 0$, and (c) $y(0) = 2$. What happens to these three solutions for large values of x ?

6. Fig. 15. The initial conditions are (a) $y(2) = -2$, (b) $y(2) = 0$, and (c) $y(2) = 2$. What happens to these three solutions for large values of x ?

In problems 7 – 12, (i) sketch the direction field for each differential equation and (ii) without solving the differential equation, sketch the solutions that go through the points (0,1) and (2,0) .

7. $y' = 2x$.

8. $y' = 2 - x$.

9. $y' = 2 + \sin(x)$.

10. $y' = e^x$.

11. $y' = 2x + y$.

12. $y' = 2x - y$.

In problems 13 – 18, (a) find the family of functions which solve each differential equation, and (b) find the particular member of the family that goes through the given point.

13. $y' = 2x - 3$ and $y(1) = 4$.

14. $y' = 1 - 2x$ and $y(2) = -3$.

15. $y' = e^x + \cos(x)$ and $y(0) = 7$.

16. $y' = \sin(2x) - \cos(x)$ and $y(0) = -5$.

17. $y' = \dfrac{6}{2x+1} + \sqrt{x}$ and $y(1) = 4$.

18. $y' = e^x/(1 + e^x)$ and $y(0) = 0$.

Problems 19 and 20 concern the direction field (Fig. 16) that comes from a differential equation called the logistic equation ($y' = y(1 - y)$) , that is used to model the growth of a population in an enviroment with renewable but limited resources. It is also used to describe the spread of a rumor or disease through a population.

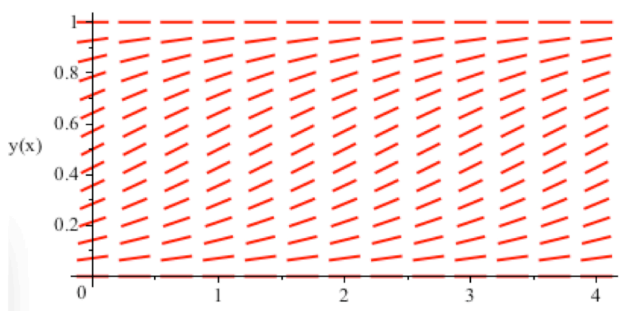


Fig. 16: Direction field for a Logistic Equation $y' = y*(1 - y)$

19. Sketch the solution that satisfies the initial condition $P(0) = 0.1$. What letter of the alphabet does this solution look like?

20. Sketch several solutions that have different initial values for P(0). What appears to happen to all of these solutions after a "long time" (for large values of x)?

In problems 21 and 22, the figures show the direction of surface flow at different locations along a river. Sketch the paths small corks will follow if they are put into the river at the dots in each figure. (Since the magnitude and the direction of flow are indicated, each diagram is called a **vector field**.) Notice that corks that start close to each other can drift far apart and corks that start far apart can drift close.

21. Fig. 17 .   22. Fig. 18 .

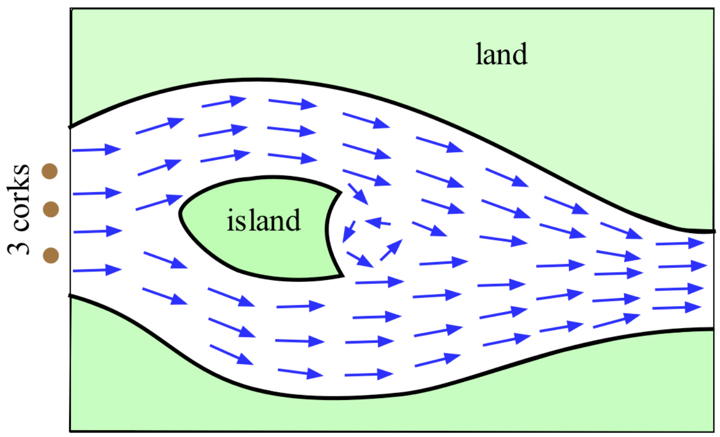
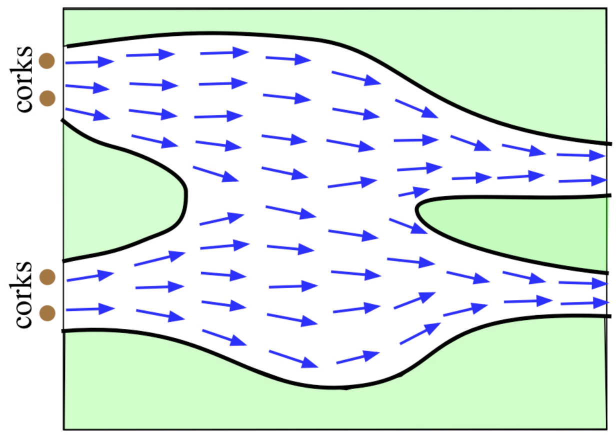


Fig. 17: Surface flow along a river



Fg. 18: Surface flow along a river

MAPLE

Many of the direction fields in this section were created using a computer program called MAPLE.
The commands below were used to create Fig. 12

with(DEtools):   *This loads a library needed for the DEplott command.*

DEplot(diff(y(x),x) = x - y(x), y(x), x=-3..3, y=3..3, dirgrid=[11,11], color=red, arrows=line, thickness=2,
  [[y(0)=-1], [y(0)=2],[y(2)=0]], linecolor=blue);

Plots the direction field for y ‘ = x – y
for -3 ≤ x ≤ 3 and -3 ≤ y ≤ 3
using an 11 by 11 grid
of red arrows with thickness 2.
 (arrows can be set equal to “small”, “medium”, “large” or “line,”

It includes the solution curves for the initial conditions y(0)=-1, y(0)=2 and y(2)=0 in blue.

Section 6.1 **PRACTICE Answers**

Practice 1: (a) $y' = x + \frac{3}{x+2}$. $y = \int x + \frac{3}{x+2}\, dx = \frac{1}{2} x^2 + 3 \cdot \ln|x+2| + C$.

(b) $y' = \frac{6}{x^2+1}$. $y = \int \frac{6}{x^2+1}\, dx = 6 \cdot \arctan(x) + C$.

Practice 2: (a) (See Fig. 19)

x	y'= x + 1
–3	–2
–2	–1
–1	0
0	1
1	2
2	3
3	4

y ' does not depend on the value of y.

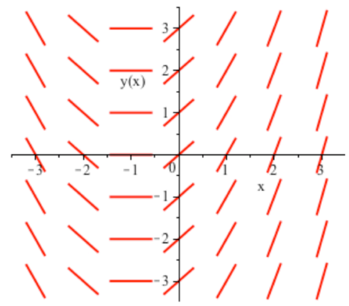


Fig. 19: Direction field for y' = x + 1

(b) y ' = x + y (See Fig. 20)

y = –3

x	y ' = x–3
–3	–6
–2	–5
–1	–4
0	–3
1	–2
2	–1
3	0

y = –2

x	y ' = x–2
–3	–5
–2	–4
–1	–3
0	–2
1	–1
2	0
3	1

y = –1

x	y ' = x–1
–3	–4
–2	–3
–1	–2
0	–1
1	0
2	1
3	2

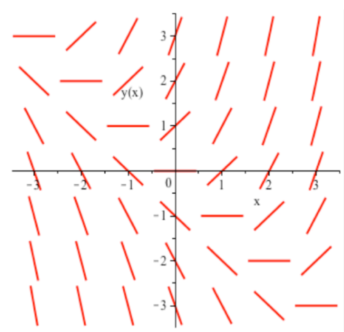


Fig. 20: Direction field for y' = x + y

Practice 3: The direction field for the differential equation y' = f(x) for f (given in Fig. 8) is shown in Fig. 21.

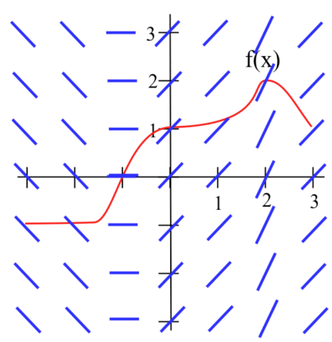


Fig. 21: Directio field for Practice 3

Practice 4: $y' = 9x^2 - 6\sin(2x) + e^x$ and $y(0) = 6$.

$y = \int 9x^2 - 6\sin(2x) + e^x\, dx$

$= 3x^3 + 3\cos(2x) + e^x + C$.

$6 = y(0) = 3(0)^3 + 3\cos(2 \cdot 0) + e^0 + C = 0 + 3 + 1 + C$

so $C = 6 - 3 - 1 = 2$ and

$\mathbf{y = 3x^3 + 3\cos(2x) + e^x + 2.}$