6.2 SEPARABLE DIFFERENTIAL EQUATIONS

In the previous section, y' was a function of x alone and the slopes of the line segments of the direction field did not depend on the y–coordinate of the location of the line segment. In many situations, however, y' depends on both x and y, for example, $y' = xy$ (Fig. 1) or $y' = x + y$ (Fig. 2). This section emphasizes how to solve differential equations in which the variables can be "separated," and the next section examines several applications of these "separable" differential equations.

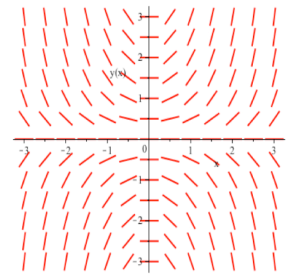


Fig. 1: $y' = x*y$

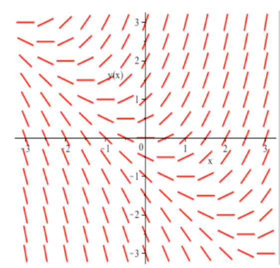


Fig. 2: $y' = x + y$

Definition: A differential equation is called **separable** if the variables can be separated algebraically so that the equation has the form $\{\text{function of } \mathbf{y} \text{ alone}\}\cdot \mathbf{y'} = \{\text{function of } \mathbf{x} \text{ alone}\}$: $g(\mathbf{y})\cdot y' = f(\mathbf{x})$.

Example 1: "Separate the variables" by writing each differential equation in the form $g(y)\cdot y' = f(x)$ $(x, y > 0)$.

(a) $y' = xy$ (b) $x\,y' = \frac{y+1}{x}$ (c) $y' = \frac{1 + \sin(x)}{y^2 + y}$ (d) $y' = y$

Solution: (a) Dividing each side of $y' = xy$ by y, we have $\frac{1}{y}\cdot y' = x$ so $g(y) = \frac{1}{y}$ and $f(x) = x$.

(b) Dividing each side by $x(y+1)$, we have $\frac{1}{y+1}\cdot y' = \frac{1}{x^2}$ so $g(y) = \frac{1}{y+1}$ and $f(x) = \frac{1}{x^2}$.

(c) Multiplying each side by $y^2 + y$, $(y^2 + y)\cdot y' = 1 + \sin(x)$ so $g(y) = y^2 + y$ and $f(x) = 1+\sin(x)$.

(d) Dividing each side by y, $\frac{1}{y}\cdot y' = 1$ so $g(y) = \frac{1}{y}$ and $f(x) = 1$.

The differential equations (a) – (d) are separable.

Practice 1: Show these differential equations are separable by rewriting them in the form $g(y)\cdot y' = f(x)$.

(a) $y' = x^3(y - 5)$ $(y > 5)$ (b) $y' = \frac{3}{2x + x\cdot\sin(y+2)}$ $(x > 0)$

Many differential equations can not be written in the form $g(y)\cdot y' = f(x)$; they are not separable. For example, $y' = x + y$ and $y' = \sin(xy) + x$ are not separable. Techniques for solving some of these nonseparable equations are discussed in Chapter 17.

Solving Separable Differential Equations

The steps to solve a separable differential equation are straightforward:

- use algebra to separate the variables,
- put the equation into an equivalent form with differentials, and
- integrate each side of the equation.

Example 2: Find the general solution of $\frac{1}{x}\ y' = \frac{x}{2y}\quad (x, y > 0)$.

Solution: By multiplying each side by $2xy$, this differential equation can be written as $2y\ y' = x^2$, so it is separable and can be put into differential form:

$$2y\ y' = x^2 \quad \text{so} \quad 2y\frac{dy}{dx} = x^2 \quad \text{and} \quad 2y\ dy = x^2\ dx.$$

Integrating each side, $\int 2y\ dy = \int x^2\ dx$, so $y^2 = \frac{1}{3}\ x^3 + C$, an implicit form of the general solution.

(Each antiderivative has an integration constant, $y^2 + C_1 = \frac{1}{3}\ x^3 + C_2$, but C_1 can be moved to the right side of the equation, combined with C_2 , and the final result expressed using only a single constant, $C = C_2 - C_1$. Then $y^2 = x^3/3 + C$.)

Finally, solving for y, we have $\mathbf{y = \pm\sqrt{\frac{1}{3}x^3 + C}}$, the explicit form of the general solution.

Steps for solving a separable equation $g(y)\cdot y' = f(x)$:

(a) Rewrite in differential form: $g(y)\ dy = f(x)\ dx$

(b) Integrate each side: $\int g(y)\ dy = \int f(x)\ dx$

(c) Find antiderivatives of g and f: $G(y) = F(x) + C \quad (G' = g \text{ and } F' = f)$

(d) If an initial value (x_0, y_0) is given, put the values for x_0 and y_0 into F and G and solve for C.

(e) If possible, explicitly solve for y .

Example 3: Find the solution of $y' = \frac{6x + 1}{2y}\quad (y > 0)$ which satisfies $y(2) = 3$.

Solution: This differential equation can be written as $2y\,y' = 6x + 1$ so it is separable and can be written using differentials:

$$2y\,y' = 6x + 1 \quad \text{so} \quad 2y\frac{dy}{dx} = 6x + 1 \quad \text{and} \quad 2y\,dy = (6x + 1)\,dx$$

Integrating each side, $\int 2y\,dy = \int 6x + 1\,dx$ so $y^2 = 3x^2 + x + C$.

In an initial value problem, it is usually safest to solve for C immediately after finding the antiderivatives. Putting x = 2 and y = 3 into the general solution $y^2 = 3x^2 + x + C$,

$$(3)^2 = 3(2)^2 + (2) + C \text{ so } 9 = 12 + 2 + C \text{ and } C = -5.$$

Then $y^2 = 3x^2 + x - 5$ or $y = \pm\sqrt{3x^2 + x - 5}$. Since the point (2, 3) is on the top half of the circle, we use only the + value of the square root for y: $y = +\sqrt{3x^2 + x - 5}$.

The general solution of $y' = \dfrac{6x + 1}{2y}$ is $y^2 = 3x^2 + x + C$ or $y = \sqrt{3x^2 + x + C}$.
The particular solution which satisfies the initial condition $y(2) = 3$ is $y^2 = 3x^2 + x - 5$ or $y = \sqrt{3x^2 + x - 5}$.

Practice 2: Find the general solution of $y' = \dfrac{1 - \sin(x)}{3y^2}$ and the particular solution through (0,2).

Sometimes algebra is the hardest part of the problem, and logarithms are often involved.

Example 4: Solve $x\,y' = y + 3$ assuming $x \neq 0$ and $y \neq -3$.

Solution: Putting the problem into the form $g(y)\cdot y' = f(x)$: $\dfrac{1}{y+3}\ y' = \dfrac{1}{x}$.

Rewriting this in differential form and integrating, we have that

$$\frac{1}{y+3}\,dy = \frac{1}{x}\,dx \quad \text{and} \quad \int \frac{1}{y+3}\,dy = \int \frac{1}{x}\,dx \quad \text{so} \quad \mathbf{\ln|\,y + 3\,| = \ln|\,x\,| + C}\ ,$$

an implicit form of the general solution. To explicitly solve for y, recall that $e^{\ln(a)} = a$. Then

$$e^{\ln|\,y+3\,|} = e^{\ln|\,x\,| + C} = e^{\ln|\,x\,|}\cdot e^{C} \quad \text{so} \quad |\,y + 3\,| = |\,x\,|\cdot e^{C} \quad \text{and} \quad y + 3 = \pm\, x \cdot e^{C}$$

Replacing the complicated constant $\pm e^{C}$ with A and subtracting 3 from each side, we have

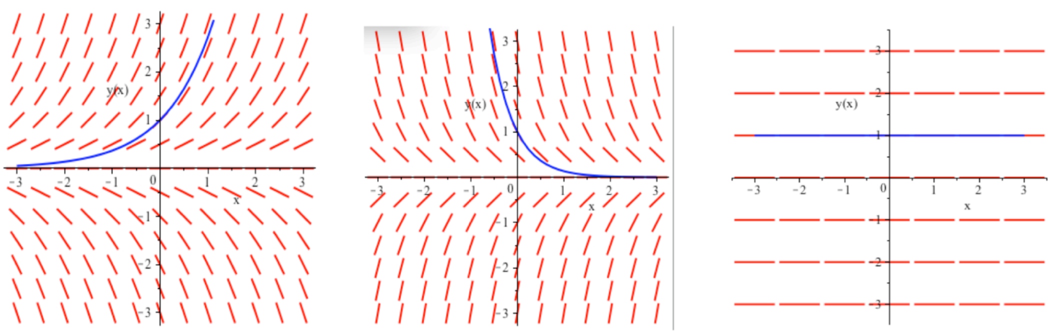
y = Ax – 3 , an explicit form of the general solution.

Two Special Cases: $y' = ky$ and $y' = k(y - a)$

The separable differential equations $y' = ky$ and $y' = k(y - a)$ are relatively simple, but they describe a wealth of important situations, including population growth, radioactive decay, drug testing, heating and cooling. The two differential equations are solved here and some of their applications are explored in section 6.3.

The Differential Equation $y' = ky$:

The differential equation $y' = ky$ describes a function y whose rate of change is proportional to the value of y. Fig. 3 shows direction fields for $y' = 1y$ (growth) and $y' = -2y$ (decay). The differential equation $y' = ky$ models the behavior of populations (the number of babies born is proportional to the number of people in the population) , radioactive decay (the number of atoms which decay is proportional to the number of atoms present), the absorption of some medicines by our bodies, and many other situations. The solutions of $y' = ky$ will help us determine how long it takes a population to double in size, how old some prehistoric artifacts are, and even how often some medicines should be taken in order to maintain a safe and effective concentration of medicine in our bodies.



$y' = 1y$, $y(0)=1$ $y' = -2y$, $y(0)=1$ $y' = 0y$, $y(0)=1$

Fig. 3: Direction fields for $y' = ky$

> If $y' = ky$ $(y > 0)$, then $y = y(0) \cdot e^{kx}$.

Proof: $y' = ky$ can be rewritten as $\frac{1}{y}\, y' = k$ so it is a separable differential equation can be written using differentials as

$$\frac{1}{y}\, dy = k\, dx . \text{ Then } \int \frac{1}{y}\, dy = \int k\, dx \text{ so } \ln(y) = kx + C.$$

When $x = 0, y = y(0)$ so $\ln(y(0)) = k \cdot 0 + C$ and $C = \ln(y(0))$. Then

$$\ln(y) = kx + \ln(y(0)), \text{ so } e^{\ln(y)} = e^{kx + \ln(y(0))} = e^{kx} \cdot e^{\ln(y(0))} \text{ and } y = y(0) \cdot e^{kx} .$$

Practice 3: The population of a town is 7,000 people and it is growing at a rate so $P' = 0.08 \cdot P$ people/year. Write an equation for the population of the town t years from now and use the equation to estimate the towns population in 10 years.

The Differential Equation $y' = k(y - a)$:

The differential equation $y' = k(y - a)$ describes a function y whose rate of change is proportional to the difference of y and the number a. Figure 4 shows the direction fields for $y' = 1(y - a) = y - a$ and $y' = -1(y - a) = a - y$. In the first case, the solution curves are "repelled" by the horizontal line $y = a$, and in the second case they are "attracted" by the line. The differential equation $y' = k(y - a)$ models the changing temperature of a cup of tea (the rate of cooling is proportional to the difference in temperature of the tea and the surrounding air) and the changing pressure within a balloon (the rate of pressure change is proportional to the difference in pressure between the inside and outside of the balloon). The solutions of the differential equation will help us determine how long it takes the hot tea to cool (or cold tea to warm up) to any given temperature and how long it takes a slowly leaking balloon (or tire) to lose half of its air.

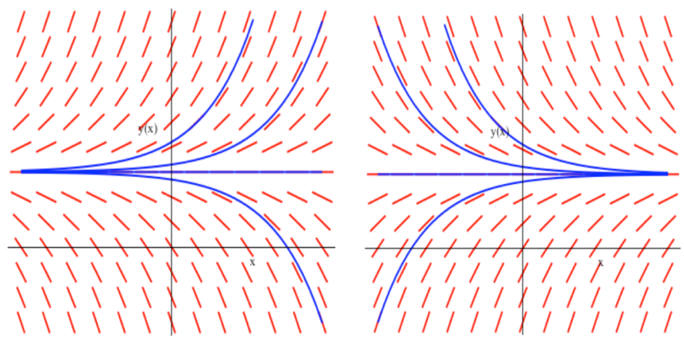


Fig. 4: Direction fields for $y' = k(y - a)$

If $y' = k(y - a)$, then $y - a = (y_0 - a) \cdot e^{kx}$.

Proof: Since the differentiable equation $y' = k(y - a)$ is separable, we can separate the variables, integrate, and solve for y. The equation can be written as

$$\frac{1}{y-a}\, dy = k\, dx. \text{ Then } \int \frac{1}{y-a}\, dy = \int k\, dx \text{ so } \ln(y - a) = k \cdot x + C .$$

When $x = 0, y = y_0$ so $\ln(y_0 - a) = k \cdot 0 + C$ and $C = \ln(y_0 - a)$. Then

$$\ln(y - a) = k \cdot x + \ln(y_0 - a) \text{ so } e^{\ln(y-a)} = e^{k \cdot x + \ln(y_0 - a)} = e^{k \cdot x}\, e^{\ln(y_0 - a)} = (y_0 - a) \cdot e^{k \cdot x}$$

and $y - a = (y_0 - a) \cdot e^{kx}$.

Practice 4: When a pan of $90^{o}C$ water ($T_{o} = 90$) is placed in a $70^{o}C$ room ($a = 70$), the rate at which the water cools is $T' = -0.15(T - 70)$ degrees per minute (Fig. 5). Write a formula for the temperature T of the water t minutes after it is placed in the room and use the equation to estimate the temperature of the water after 5 minutes, 10 minutes, and 15 minutes.

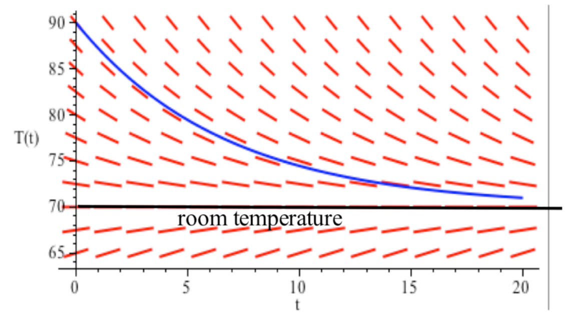


Fig. 5: Temperature of warm water in a cool room

The solutions of these two differential equations are used in applied problems in Section 6.3.

PROBLEMS

1. Fig. 6 shows the direction field of the separable differential equation $y' = 2xy$. Sketch the solutions of the differential equation which satisfy the initial conditions $y(0) = 3$, $y(0) = 5$, and $y(1) = 2$.

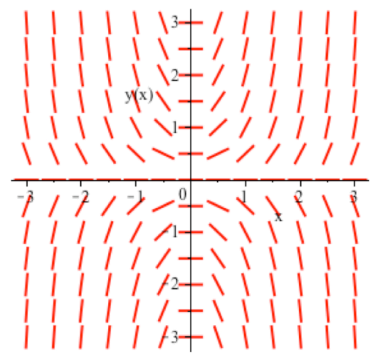


Fig. 6: Direction field for $y' = 2xy$

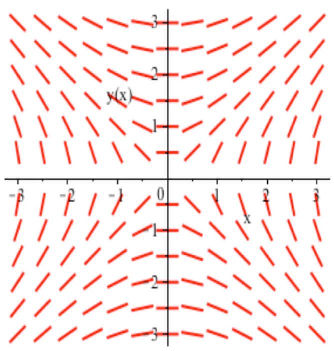


Fig. 7: Direction field for $y' = x/y$

2. Fig. 7 shows the direction field of the separable differential equation $y' = x/y$. Sketch the solutions of the differential equation which satisfy the initial conditions $y(0) = 3$, $y(0) = 5$, and $y(1) = 2$.

In problems 3 – 10, (a) separate the variables and rewrite the differential equation in the form $g(y) \cdot y' = f(x)$, and (b) solve the resulting differential equation. (Assume that x and y are restricted so that division by zero does not occur.)

3. $y' = 2xy$
4. $y' = x/y$
5. $(1 + x^2) \cdot y' = 3$
6. $xy' = y + 3$
7. $y' \cos(x) = e^y$
8. $y' = x^2y + 3y$
9. $y' = 4y$
10. $y' = 5(2 - y)$

In problems 11 – 18, solve the initial value separable differential equations.

11. $y' = 2xy$ for $y(0) = 3$, $y(0) = 5$, and $y(1) = 2$.

12. $y' = x/y$ for $y(0) = 3$, $y(0) = 5$, and $y(1) = 2$.

13. $y' = 3y$ for $y(0) = 4$, $y(0) = 7$, and $y(1) = 3$.

14. $y' = -2y$ for $y(0) = 4$, $y(0) = 7$, and $y(1) = 3$.

15. $y' = 5(2 - y)$ for $y(0) = 5$ and $y(0) = -3$.

16. $y' = 7(1 - y)$ for $y(0) = 4$ and $y(0) = -2$.

17. $(1 + x^2) \cdot y' = 3$ for $y(1) = 4$ and $y(0) = 2$.

18. $xy' = y + 3$ for $y(1) = 20$. Can $y(0) = 2$?

19. The rate of growth of a population $P(t)$ which starts with 3,000 people and increases by 4% per year (Fig. 8) is $P'(t) = 0.0392 \cdot P(t)$. Solve the differential equation and use the solution to estimate the population in 20 years.

20. The rate of growth of a population $P(t)$ which starts with 5,000 people and increases by 3% per year is $P'(t) = 0.0296 \cdot P(t)$. Solve the differential equation and use the solution to estimate the population in 20 years.

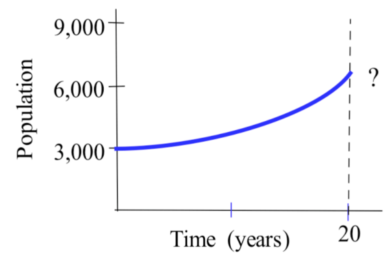


Fig. 8

21. The rate of decay of a piece of carbon–14 in a piece of material containing 3 grams of carbon–14 is $C'(t) = (-0.00012) \cdot C(t)$

where $C(t)$ is the number of grams present after t years (Fig. 9). Solve the differential equation and use the solution to estimate the amount of carbon–14 present after 10,000 years.

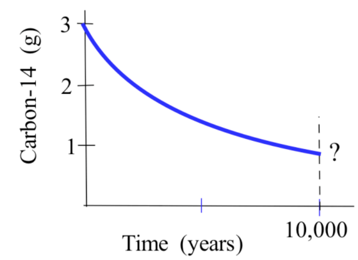


Fig. 9

22. The rate of decay of iodine–131 is $I'(t) = -0.086 \cdot I(t)$ where $I(t)$ is the number of grams present after t days. Solve the differential equation. If we start with 5 grams of iodine–131, how much will be present after 2 hours and after 10 hours? (First find a formula for $I(t)$.)

23. The rate of temperature change of a bowl of soup in a 25°C room is $T' = -0.12(T - 25)$ where T is the temperature of the soup after t minutes. If the soup originally is 80°C, find a formula for T and use it to estimate the temperature of the soup after 5 minutes (Fig. 10).

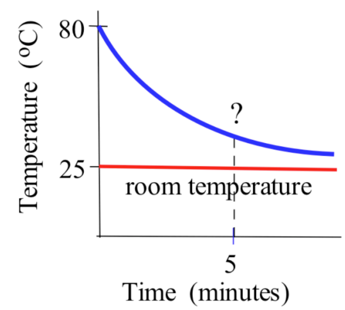


Fig. 10

24. When the switch is closed in an electrical circuit with a constant voltage source of 9 volts, a resistance of 2 ohms and an inductance of 3 henries, the rate of change of the current i (in amperes) is described by the differential equation $3\frac{di}{dt} + 2i = 9$ where t is the time in seconds. Solve the separable differential equation for i.

Section 6.2 **PRACTICE Answers**

Practice 1: (a) $y' = x^3(y-5)$ so $\frac{1}{y-5}\, y' = x^3$. In the pattern, $g(y) = \frac{1}{y-5}$ and $f(x) = x^3$.

(b) $y' = \dfrac{3}{2x + x\cdot\sin(y+2)}$ so $(2+\sin(y+2))\cdot y' = \frac{3}{x}$. $g(y) = 2 + \sin(y+2)$ and $f(x) = \frac{3}{x}$.

Practice 2: $y' = \dfrac{1-\sin(x)}{3y^2}$ so $3y^2\cdot y' = 1-\sin(x)$ and $3y^2\, dy = (1-\sin(x))\, dx$. Then

General solution: $\int 3y^2\, dy = \int (1-\sin(x))\, dx$ so $\mathbf{y^3 = x + \cos(x) + C}$.

Particular solution (0,2): $(2)^3 = 0 + \cos(0) + C$ so $8 = 1 + C$ and $C = 7$.

$\mathbf{y^3 = x + \cos(x) + 7}$.

Practice 3: ($y' = ky$ so $y = y(0)\cdot e^{kx}$.) $P' = 0.08\cdot P$ so $P(t) = P(0)\cdot e^{0.08t}$ with $P(0) = 7{,}000$.

$\mathbf{P(t) = 7{,}000\cdot e^{0.08t}}$. $P(10) = 7{,}000\cdot e^{0.08(10)} = 7{,}000\cdot e^{0.8} \approx 7{,}000(2.22554) = \mathbf{15{,}579}$**.**

Practice 4: ($y' = k(y-a)$ so $y - a = \{y_0 - a\}\cdot e^{kx}$)

$T' = -0.15(T-70)$. $k = -0.15$, $a = 70$ and $T_0 = 90$. Then

$T - 70 = (T_0 - 70)e^{-0.15t} = (90-70)e^{-0.15t} = 20e^{-0.15t}$ and $\mathbf{T = 70 + 20e^{-0.15t}}$.

When $t = 5$, $T = 70 + 20e^{-0.15t} = 70 + 20e^{-0.15(5)} = 70 + 20e^{-0.75} \approx 70 + 20(0.472) \approx 79.4°$.

When $t = 10$, $T = 70 + 20e^{-0.15t} = 70 + 20e^{-0.15(10)} = 70 + 20e^{-1.5} \approx 70 + 20(0.223) \approx 74.5°$.

When $t = 15$, $T = 70 + 20e^{-0.15t} = 70 + 20e^{-0.15(15)} = 70 + 20e^{-2.25} \approx 70 + 20(0.105) = 72.1°$.

After a "long" time, the temperature will be very close to (and slightly above) 70°.