6.3 GROWTH, DECAY, AND COOLING

A population of people, a chunk of radioactive material, money in the bank, and a cup of hot soup all share a common trait. In each situation, the rate at which an amount is changing is proportional to the amount:

- the number of births per year is proportional to the number of people in the population
- the number of atoms per hour that release a particle is proportional to the number of atoms present
- the number of dollars of interest per year is proportional to the amount of money in the bank account
- the number of degrees the soup cools per minute is proportional to the temperature difference between the soup and the air.

All of these situations can be modeled with separable differential equations we solved in Section 6.2. In fact, the first three can be modeled with the same differential equation: $y' = ky$. The cooling soup uses $y' = k(y-a)$. In this section our focus is on using the equations and their solutions to answer questions about applied problems. The applications here all involve the rate of change of some quantity with respect to time and the notation is usually changed so the independent variable is time t (instead of x) and the dependent quantity is $f(t)$ (instead of y). The statement $y' = ky$ then becomes $f'(t) = k \cdot f(t)$, and the solution $y = y_0 \cdot e^{kx}$ becomes $f(t) = f(0) \cdot e^{kt}$.

> **Theorem:** If the rate of change of f is proportional to the value of f, $f'(t) = k \cdot f(t)$,
>
> then $f(t) = f(0) \cdot e^{kt}$.

When k is positive, $f(t) = f(0) \cdot e^{kt}$ represents **exponential growth**, and k is called the **growth constant**. When k is negative, $f(t) = f(0) \cdot e^{kt}$ represents **exponential decay**, and k is called the **decay constant**. Fig. 1 shows the graphs of $f(t) = e^{kt}$ for several values of k.

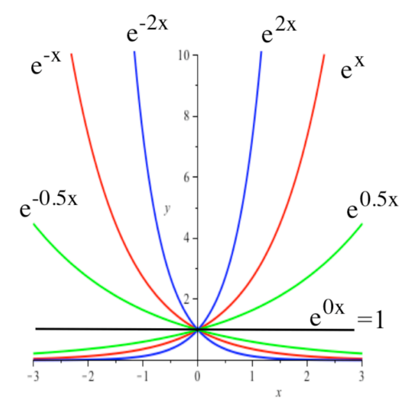

Fig. 1: $y = e^{kx}$

Exponential Growth

When the initial population $f(0)$ and the growth constant k are known, we can write an equation for $f(t)$, the population at any time t, and use it to answer questions about the population.

Example 1: The number of bacteria on a petri plate t hours after the experiment starts is $2000 \cdot e^{0.0488t}$.

(a) How many bacteria are on the plate after 1 hour? 2 hours?

(b) What is the percentage growth of the population from t=0 to t=1? From t=1 to t=2?

(c) How many hours does it take for the population to reach 3000? To double?

Solution: The population after t hours is $f(t) = 2000 \cdot e^{0.0488t}$ (Fig. 2)

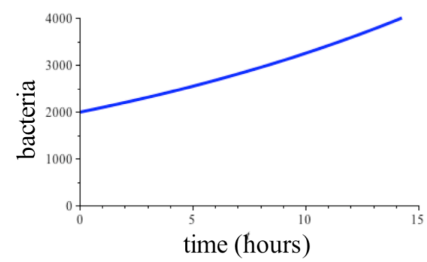

Fig. 2: Growing population of bacteria

(a) $f(1) = 2000 \cdot e^{0.0488(1)} = 2000 \cdot e^{0.0488} \approx 2000(1.0500) = 2100.$
$f(2) = 2000 \cdot e^{0.0488(2)} = 2000 \cdot e^{0.0976} \approx 2000(1.1025) = 2205.$

(b) Percentage growth from t=0 to t=1 is

$$\frac{f(1) - f(0)}{f(0)} \cdot 100 = \frac{2100 - 2000}{2000} \cdot 100 = (0.05)(100) = 5\ \%.$$

Percentage growth from t=1 to t=2 is $\frac{f(2) - f(1)}{f(1)} \cdot 100 = \frac{2205 - 2100}{2100} \cdot 100 = (0.05)(100) = 5\ \%.$

During the first hour, the population grows by 100 and during the second hour it grows by 105, but the percentage growth during each hour is a constant 5 %.

(c) We can find the value of t so $3000 = f(t) = 2000 \cdot e^{0.0488t}$ by dividing each side by 2000:
$1.5 = e^{0.0488t}$

taking logarithms to get t out of the exponent: $\ln(1.5) = \ln(e^{0.0488t}) = 0.0488t \ln(e) = .0488\ t$

and dividing by 0.0488 to solve for t: $t = \frac{1}{0.0488} \ln(1.5) \approx \frac{1}{0.0488}(0.4055) \approx 8.31$ hours.

Since the original population is 2000, the doubled population is 4000. We can find the value of t so that $4000 = f(t) = 2000 \cdot e^{0.0488t}$ by dividing each side by 2000 and taking logarithms: $\ln(2) = \ln(e^{0.0488t}) = 0.0488t \ln(e) = 0.0488\ t$. Then $t = \frac{\ln(2)}{0.0488} \approx \frac{0.693}{0.0488} \approx 14.2$ hours.

The bacteria population will double every 14.2 hours, the **doubling time** for this population.

Practice 1: Use $f(t) = 2000 \cdot e^{0.0488t}$ from Example 1. (a) What is the population when t=5?
(b) How long until the population is 5000? (c) How long until the population triples?

If the value of the growth constant k is not given, usually our first step is to use the given information to find it. Once we know the population at two different times, we can find k.

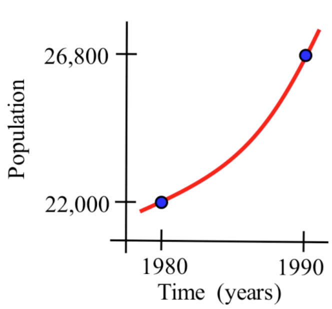

Fig. 3: Growing community

Example 2: The population of a community was 22,000 in 1980 and 26,800 in 1990. Assuming that the community maintains the same rate of exponential growth, (Fig. 3)

(a) what is a formula for the population t years after 1980?

(b) what is the annual percentage rate of growth of the community?

Solution: Let t represent the number of years since 1980, so 1980 corresponds to t=0 and 1990 corresponds to t=10. Then $f(0) = 22{,}000$, $f(10) = 26{,}800$, and $f(t) = f(0) \cdot e^{kt} = 22{,}000 \cdot e^{kt}$.

(a) To find a formula for f(t), we can use the 1990 (t=10) population to find the value for k:

$$26{,}800 = f(10) = 22{,}000 \cdot e^{k(10)} \text{ so } 1.218 = e^{k(10)} \text{ and}$$

$$k = \frac{1}{10} \ln(1.218) \approx \frac{1}{10}(0.197) = 0.0197 . \text{ Then } \mathbf{f(t) = 22{,}000 \cdot e^{(0.0197)t}} .$$

(b) $f(0) = 22{,}000$ and $f(1) = 22{,}000 \cdot e^{(0.0197)1} \approx 22{,}000\,(1.01989) = 22{,}437.58$ so the annual percentage increase was $\frac{f(1) - f(0)}{f(0)} \cdot 100 = \frac{437.58}{22000} \cdot 100 \approx \mathbf{1.989\ \%}$.

Practice 2: An experiment was begun by releasing 12,000 free neutrons into a material, and 2 seconds later, the material contained 18,000 free neutrons. Assuming the number of free neutrons grows exponentially, (a) determine a formula for the number present t seconds after the beginning of the experiment, and (b) determine how long it takes for the number of free neutrons to double.

Compound interest is another example of exponential growth.

Example 3: How long does it take \$1000 to double at an effective annual rate of return of 5%? 10% ? (This assumes that the yield is computed and compounded continuously.)

Solution: Let f(t) be the amount of money after t years. Then $f(0) = 1000$ and $f(t) = 1000 \cdot e^{kt}$.

5%: After 1 year, the investment will be \$1000 + (.05)(\$1000) = \$1050 so $f(1) = 1050 = 1000 \cdot e^{k \cdot 1}$. Solving for k, $1.05 = e^k$ so $k = \ln(1.05)$ and $f(t) = 1000 \cdot e^{\ln(1.05)t}$. Solving

$$2000 = 1000 \cdot e^{\ln(1.05)t} \text{ for t gives } t = \frac{\ln(2)}{\ln(1.05)} \approx \mathbf{14.2 \text{ years}}.$$

10%: After 1 year the investment will be \$1100. Then $k = \ln(1.10)$ so $f(t) = 1000e^{\ln(1.10)t}$

and the doubling time is $t = \frac{\ln(2)}{\ln(1.10)} \approx \mathbf{7.27 \text{ years}}$.

Practice 3: How long does it take an investment to double if the rate of return is 12%?

When we know the growth constant k, the doubling time is simple to find. If $f(t) = f(0) \cdot e^{kt}$ then the doubling time is the time t so that $2f(0) = f(t) = f(0) \cdot e^{kt}$. Then

$$2 = e^{kt} \text{ and } \ln(2) = kt \text{ so } t = \frac{\ln(2)}{k} .$$

Doubling Time: If $f(t) = f(0) \cdot e^{kt}$, then the doubling time is $t = \frac{\ln(2)}{k}$. (Fig. 4)

An important aspect of exponential growth is that the **doubling time depends only on the growth constant k** and not on the population or the starting time. The previous Example and Practice problem illustrate the basis for a "rule" used in business:

Rule of 72: An investment with an annual rate of return of R% takes about $\frac{72}{R}$ years to double in value.

Table 1 shows the exact values for doubling times obtained using exponential growth with those obtained using the Rule of 72. The Rule of 72 gives good approximations and is easy to use. Problem 12 asks you to show why this "rule" works, and problem 13 asks you to find a "rule" for an investment to triple in value.

	Doubling Time (years)	
Rate of return (%)	Exact	Rule of 72
4	17.7	18.0
5	14.2	14.4
6	11.9	12.0
7	10.2	10.3
9	8.0	8.0
10	7.3	7.2
12	6.1	6.0
20	3.8	3.6

Table 1: Time for an investment to double in value

Exponential Decay

Exponential decay occurs when the rate of loss of something is proportional to the amount present. One example of exponential decay is radioactive decay: the number of atoms of a radioactive substance that "decay" (split into nonradioactive atoms and release p proportional to the number of radioactive atoms present. Exponentia quickly some medicines are absorbed from the bloodstream and even Exponential decay calculations are similar to those for growth, but th about "**half–life**", the time for half of the material to decay or be abso Table 2 shows the half–lives of some isotopes.

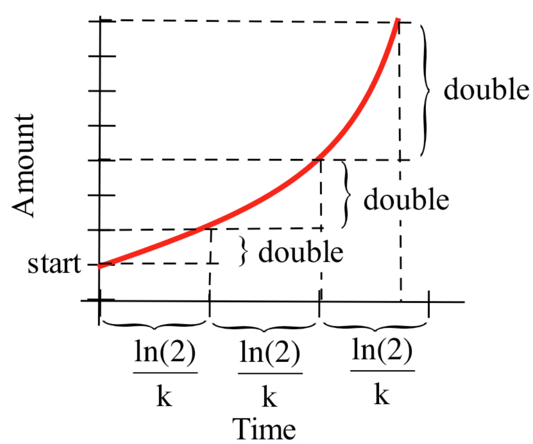

Fig. 4: Constant doubling times

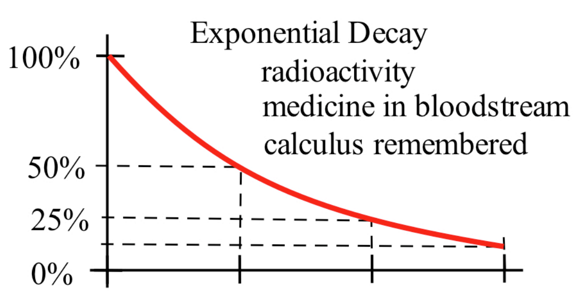

Fig. 5: Exponential decay

strontium–90	29 years
argon–39	265 years
carbon–14	5700 years
plutonium–239	24,400 years
uranium–238	4.51×10^5 years
uranium–234	2.47×10^9 years

Table 2: Half–lives of some isotopes

Example 4: You started with 10 g of radioactive Q, but after 6 days of decay there were only 3 g left (Fig. 6).

(a) Find a formula for the amount of Q present after t days.

(b) What is the half–life of Q?

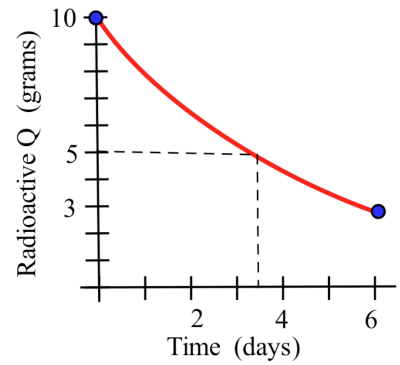

Fig. 6: Radioactive decay

Solution: Let f(t) be the amount of Q present after t days.

Then $f(t) = f(0) \cdot e^{kt} = 10 \cdot e^{kt}$.

(a) $3 = f(6) = 10 \cdot e^{k6}$ so $0.3 = e^{6k}$ and $\ln(0.3) = 6k$. Then

$k = \frac{1}{6} \ln(0.3) \approx -0.2007$ and $\mathbf{f(t) = 10 \cdot e^{(-0.2007)t}}$.

(b) The half–life is the time required for half of the material to decay, so we need to solve

$5 = 10 \cdot e^{(-0.2007)t}$ for t. Dividing by 10 and then taking logarithms,

$1/2 = e^{(-0.2007)t}$ and $\ln(1/2) = (-0.2007)t$ so $t = \frac{\ln(0.5)}{-0.2007} \approx 3.45$ days.

Carbon–14 Dating: If the half life of a substance is known and we know how much of the substance is present in a sample now, we can determine how much was present at some past time or determine how long ago the sample contained a particular amount of the substance. Radioactive carbon–14 with a half–life of about 5700 years is used in this way to estimate how long ago plants and animals lived. When a plant is alive it continually exchanges carbon–14 and ordinary carbon with the atmosphere so the ratio of carbon–14 to nonradioactive carbon stays relatively constant. But once the plant dies, this exchange stops. The ordinary carbon remains in the material, but the carbon–14 decays so the ratio of carbon–14 to ordinary carbon decreases at a known rate. By measuring the ratio of carbon–14 to ordinary carbon in a sample of plant tissue, scientists can determine how long ago the plant died and obtain an estimate for the age of the sample.

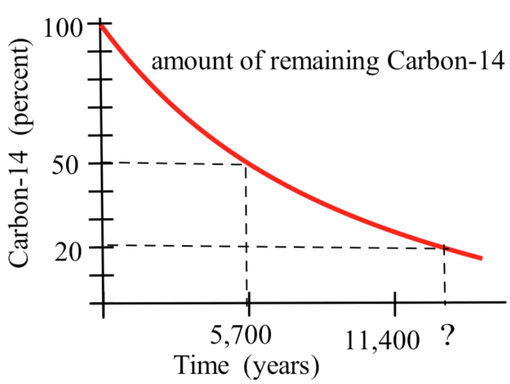

Fig. 7: Determining the age of a basket

Example 5: The amount of carbon–14 in plant fiber of a woven basket is 20% of the amount present in a living plant. Estimate the age of the basket. (Fig. 7)

Solution: Let f(t) represent the amount of carbon–14 in a sample that is t years old. Since we know the half–life is 5700 years, then

$f(5700) = f(0) \cdot e^{k \cdot 5700} = \frac{1}{2} f(0)$ so $e^{k \cdot 5700} = \frac{1}{2}$.

Solving for k,

$5700\,k = \ln(1/2)$ and $k = \frac{\ln(.5)}{5700} \approx -0.0001216$ so $f(t) = f(0) \cdot e^{(-0.0001216)t}$.

Since 20% of the carbon–14 remains in our sample, we want the value of t so that

$$0.20 \cdot f(0) = f(t) = f(0) \cdot e^{(-0.0001216)t} .$$

Dividing by f(0), taking logarithms, and solving for t, we get $t = \frac{\ln(0.2)}{-0.0001216} \approx 13{,}235$ years.

The basket was made from a plant that died about 13,200 years ago. (Does that mean the basket was made about 13,200 years ago?) This dating method is very sensitive to small changes in the measured amount of carbon–14.

When the decay constant k is known, the half–life is simple to find. If $f(t) = f(0) \cdot e^{kt}$ then the half–life is the time t so that $\frac{1}{2} f(0) = f(t) = f(0) \cdot e^{kt}$. Solving for t, we have $t = \frac{\ln(1/2)}{k}$.

Half–life: If $f(t) = f(0) \cdot e^{kt}$, then the half–life is $t = \frac{\ln(1/2)}{k}$. (Fig. 8)

The half–life depends only on the decay constant k and not on the amount of material we have. If the half–life is known, then $k = \frac{\ln(1/2)}{\text{half–life}}$.

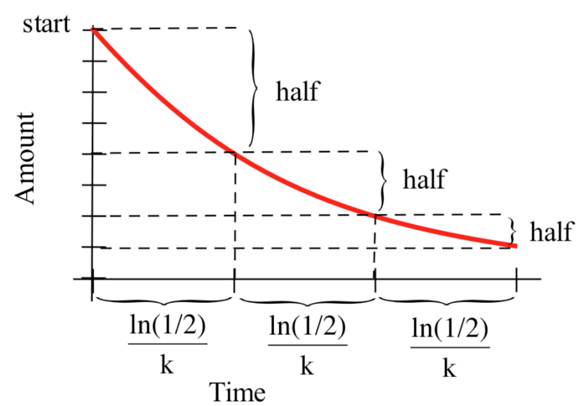

Fig. 8: Constant halfing times

Practice 4: The half–life of an isotope is 8 days. Write a formula for the amount of the isotope present t days after you began with 10 mg.

The rate at which many medicines are absorbed from the blood is proportional to the concentration of the medicine in the blood: the higher the concentration in the blood, the faster it is absorbed from the blood.

Example 6: Suppose medicine M has an absorption (decay) constant of –0.17 (determined experimentally), and that the lowest concentration of M that is "effective" is 0.3 mg/l (milligrams of M per liter of blood). If a patient who has 8 liters of blood is injected with 20 mg of M, how long will the M be effective?

Solution: Since the patient is starting with 20 mg of M in 8 liters of blood, the initial concentration is 20 mg/8 l = 2.5 mg/l . Then the amount of M at time t hours is $f(t) = 2.5e^{-0.17t}$, and we want to find t so that f(t) = 0.3 mg/l : $0.3 = 2.5e^{-0.17t}$ so $t = \frac{1}{-0.17} \ln(0.3/2.5) \approx 12.5$ hours.

Many medicines have a "safe and effective" range of concentrations (Fig. 9), and the goal of a schedule for taking the medicine is to keep the concentration near the middle of that range. Taking doses too close together in time can result in an overdose (Fig. 10), and taking them too far apart is eventually ineffective.

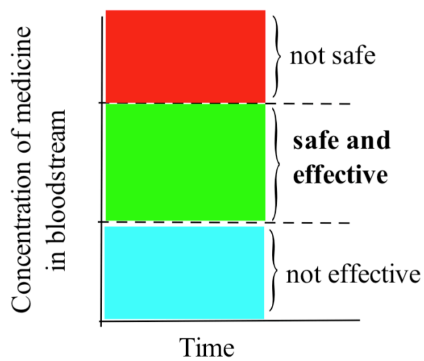

Fig. 9 : Safe and effective region

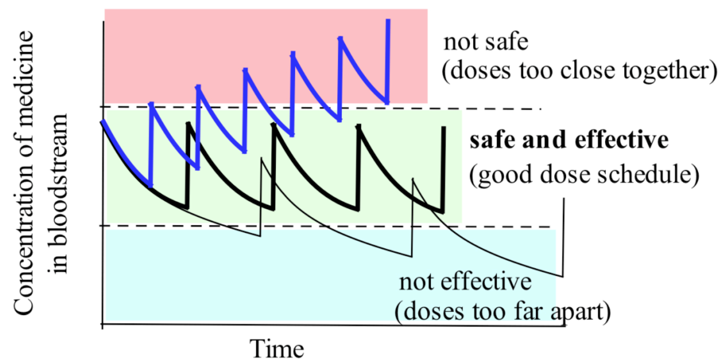

Fig. 10 : Safe and effective dosing schedule

Newton's Law of Cooling/Warming

Some rates of change depend on **how far** a value is from a fixed value. The rate at which a hot cup of soup cools (or a cool cup of milk warms up) is proportional to the difference in temperature between the soup and the surrounding air. This principle is called Newton's Law of Cooling/Warming.

Newton's Law of Cooling/Warming

If $f(t)$ is the temperature at time t of an object in an atmosphere with temperature a,

then the rate of change of f is proportional to the difference between f and a, $f'(t) = k\{ f(t)-a \}$,

and $f(t) = a + \{f(0) - a\} \cdot e^{kt}$.

The statement that the rate of change is proportional to the difference, $f'(t) = k\{ f(t) - a \}$, is a result from physics. The differential equation is separable, and was solved in the last section. Figure 11 shows some functions that have different initial values and that satisfy the differential equation $f'(t) = f(t) - 5$.

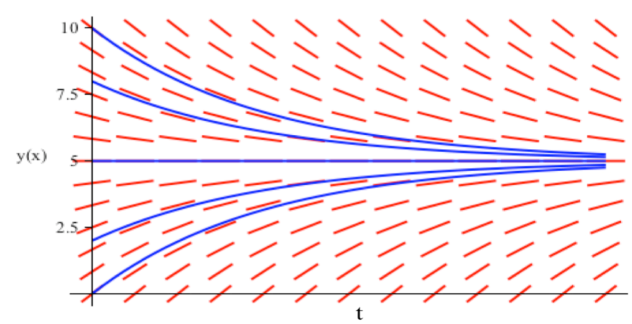

Fig. 11: Solutions of $f'(t)= (-1)*(f(t) - 5)$

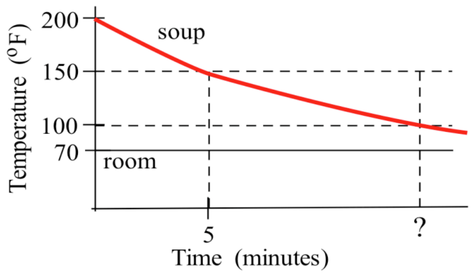

Fig. 12: Cooling cup of soup

Example 7: A cup of hot soup is in a room with a temperature of 70° F. When first poured, the soup was 200° F, and 5 minutes later it was 150° F (Fig. 12).

(a) Find an equation for the temperature f of the soup at any time t.

(b) How long does it take the 200° F soup to cool to 100° F?

(c) What will the temperature of the soup be after a "long" time?

Solution: (a) In this example, a = 70° F and f(0) = 200° F so $f'(t) = k\{ f(t) - 70 \}$ and $f(t) = 70 + \{200 - 70\} \cdot e^{kt} = 70 + 130 \cdot e^{kt}$. We can use the information that f(5) = 150° F to find the value of k and an equation for f(t) :

$$150 = f(5) = 70 + 130 \cdot e^{k5} \quad \text{so} \quad k = \frac{1}{5} \cdot \ln(80/130) \approx -0.0971 \quad \text{and} \quad \mathbf{f(t) = 70 + 130 \cdot e^{(-0.0971)t}}.$$

(b) We want to find the t so that f(t) = 100. Using the result from part (a),

$$100 = f(t) = 70 + 130 \cdot e^{(-0.0971)t} \quad \text{so} \quad 30 = 130 \cdot e^{(-0.0971)t} \quad \text{and} \quad t = \frac{\ln(30/130)}{-0.0971} \approx 15.1 \text{ minutes}.$$

(c) "After a long time" means for very large values of t.

$$\lim_{t \to \infty} 70 + 130 \cdot e^{(-0.0971)t} = \lim_{t \to \infty} 70 + \frac{130}{e^{(0.0971)t}} \longrightarrow 70 + 0 = 70° \text{ F}.$$

Eventually, the soup will cool down to (almost) the temperature of the room.

PROBLEMS

1. Fig. 13 shows the growth of a city over several decades. How long did it take the city to double in population from 10,000 to 20,000? How long did it take to double from 15,000 to 30,000? What is the approximate doubling time of this population?

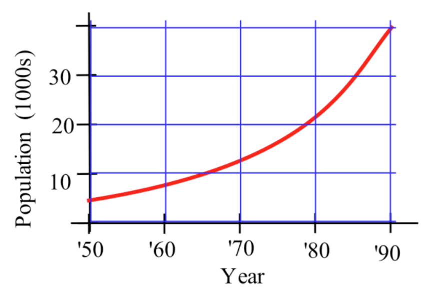

Fig. 13: Population growth of a city

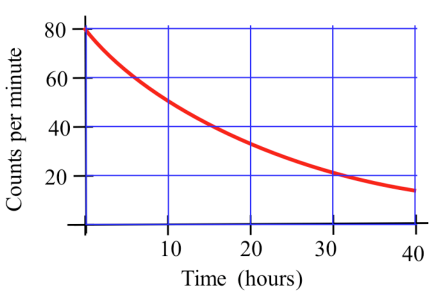

Fig. 14: Radioactive decay

2. Fig. 14 shows the counts per minute from a piece of radioactive material. How long did it take the counts to decrease from 80 per minute to 40? From 60 to 30? From 40 to 20? What is the half–life of this material?

3. The population of a community in 1970 was 48,000 people and in 1990 it was 64,000 people. (a) Write a formula for the population of the community t years after 1970. (b) Estimate the population in the year 2000. (c) Approximately when will the population be 100,000? (d) What is the doubling time of the population of this community?

4. The population of a community in 1970 was 40,000 people and in 1990 it was 60,000 people. (a) Write a formula for the population of the community t years after 1970. (b) Estimate the population in the year 2000. (c) Approximately when will the population be 100,000? (d) What is the doubling time of the population of this community?

5. You have found a terrific investment which pays at an effective annual rate of 15%. (a) Use the Rule of 72 and the exponential growth model to calculate how long it will take a $5,000 investment to double. (b) How long will it take for the investment to triple?

6. You have $3,000 invested for 10 years at an effective annual rate of 7.5% and a friend has the same amount invested at an effective annual interest rate of 7.75% . Your friend will get back how much more money than you at the end of (a) 10 years? (b) 20 years?

7. Find a formula for the population of the city in Fig. 13.

8. (Without using calculus.) Each bacterium of a certain species splits into two bacteria at the end of each minute. If we start with a few bacteria in a bowl at 3 pm and the bowl is full of bacteria at 4:30 pm, when was the bowl half full?

9. The newscaster said that the population of the world is now doubling every 20 years. What annual rate of growth results in a 20 year doubling time?

10. Group A has a population of 150,000 and a growth rate of 4%. Group B has a population of 100,000 and a growth rate of 7%. In how many years will the two groups be the same size? (Fig. 15)

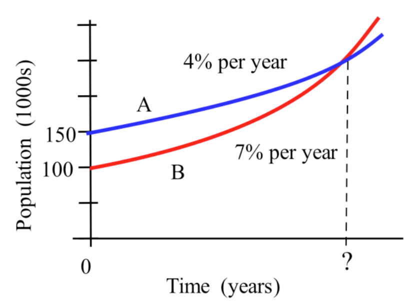

Fig. 15: Two growing populations

11. Group A has a population of 600,000 and a growth rate of 3%. Group B has a population of 400,000 and a growth rate of 6%. In how many years will the two groups be the same size?

12. Derive the "Rule of 72." For an investment with an annual rate of return of $R\%$, show that the value of the growth constant is $k = \ln(1 + R/100)$ so the doubling time is $\ln(2)/\ln(1 + R/100)$. Calculate the values of k for R between 5 and 15, and observe that for these values of R the exact doubling time $\dfrac{\ln(2)}{\ln(1 + R/100)}$ is approximately equal to $\dfrac{72}{R}$.

13. Develop a "Rule of M" for the **tripling** time of a investment. Find a value for M so M/R is a good approximation of the time its takes an investment with a rate of return of $R\ \%$ to triple in value. Assume that R is between 5 and 15.

14. The unregulated population of fish in a certain lake grows by 30% per year under optimum conditions, and the result of a fish census is that there are approximately 20,000 fish in the lake. How many fish can be harvested (Fig. 16) at the end of the year in order to maintain a stable population from year to year? (This is an example of calculating the yield for a "renewable resource." In practice, the calculations are more sophisticated and also take into account the distribution of species, ages and genders.)

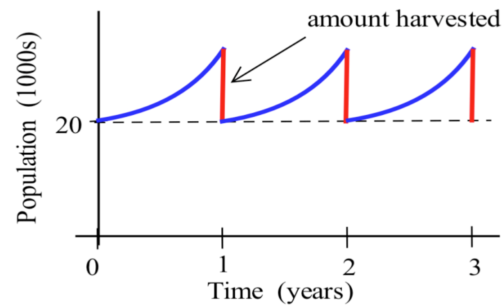

Fig. 16: Harvesting a fish growing population

15. The annual growth constant for the population of snails is $k = 0.14$, and currently we have 8,000 snails.
    (a) Graph the snail population for 20 months if we harvest 2,000 snails at the end of every 2 months.
    (b) Graph the snail population for 20 months if we harvest 3,000 snails at the end of every 2 months.
    (c) How many snails can we harvest every 2 months in order to maintain a stable population?

16. An exponential growth function $f(t) = A\,e^{kt}$ has a constant doubling time, but there are functions with constant doubling times which are not exponential. (a) Show that the exponential function $f(t) = 2^t = e^{\ln(2)t}$ has a constant doubling time of 1. (Show that $f(t+1) = 2f(t)$.) (b) Graph the function $g(t) = 2^t(1 + A\bullet\sin(2\pi t))$ for $A = 0.5$ and 1.5 . Show that g has a constant doubling time 1 for every choice of A.

17. We started an experiment with 10 grams of a radioactive material and 14 days later there were 2 grams left. (a) Find an equation for the amount of material remaining t days after the beginning of the experiment. (b) Find the half–life of the material. (c) How long after the beginning of the experiment will there be 0.7 grams of the material left?

18. We start with 8 mg of a radioactive substance and 10 days later determine that there is 6.3 mg of the substance left. (a) Find an equation for the amount left t days after the start. (b) Find the half–life of the material. (c) How long after the start will there be one milligram of the substance left?

19. The Geiger counter initially recorded 187 counts per minute from a radioactive material, but 2 days later the count was down to 143 counts per minute. (a) What is the half–life of the material?
    (b) When will the count be down to 20 counts per minute? (The count per minute is proportional to the amount of radioactive material present.)

20. The initial measurement from a radioactive material was 540 counts per minute, and a week later it was 500 counts per minute.
    (a) What is the half–life of the material?
    (b) When will the count be down to 100 counts per minute?

21. Determine an equation for the counts per minute for the radioactive material A in Fig. 17.

22. Determine an equation for the counts per minute for the radioactive material B in Fig. 17.

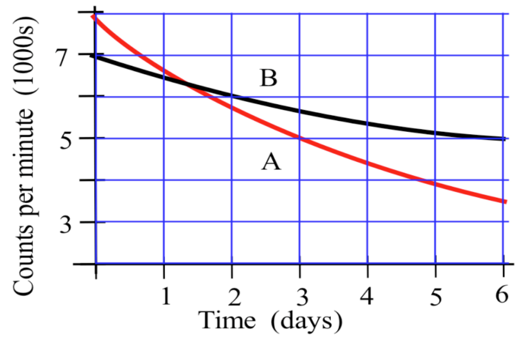

Fig. 17: Radioactive decay of two materials

23. A friend is considering purchasing a letter reputedly written by Isaac Newton (1642–1727), but an analysis of the paper shows that it contains 97.5% of the proportion of carbon–14 present in new paper. Can we be certain the letter is a forgery? If the paper is the right age, can we be certain the letter is genuine?

24. For several centuries the Shroud of Turin was widely believed to be the shroud of Jesus. Three independent laboratories in England, Switzerland, and the United States used carbon–14 dating on a few square centimeters of the cloth, and in 1988 they reported that the Shroud of Turin was probably made in the early 1300s and certainly after 1200 A.D. (a) If the Shroud was made in 1300 A.D., what percentage of the original carbon–14 was still present in 1988? (b) If the Shroud was made in 30 A.D., what percentage of the original carbon–14 was still present in 1988? (Science 21, October 1988, Vol. 242, p. 378)

25. Half of a particular medicine is used up by the body every 6 hours, and the medicine is not effective if the concentration in the blood is less than 10 mg/l. If an ill person is given an initial dose of medicine to raise the concentration to 30 mg/l, how long will the medicine be effective?

26. A particular illegal substance has a half–life of 12 hours, and it can be detected in concentrations as low as 0.002 mg/l in the blood. (a) If a person has an initial concentration of the substance of 15 mg/l in the blood, how long can it be detected? (b) If the detection test is improved by a factor of 100 so it can detect a concentration of 0.00002 mg/l, how long can an initial concentration of 15 mg/l be detected?

27. A doctor gave a patient 9 mg of a medicine which has half–life of 15 hours in the body. How much of the medicine does the patient need to take **every 8 hours** in order to maintain between 6 and 9 mg of the medicine in the body all of the time? (Fig. 18)

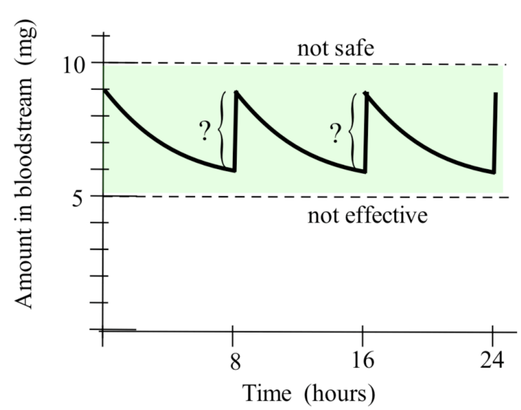

Fig. 18 : Finding a safe and effective dose

28. Each layer of a dark film transmits 40% of the light that strikes it. (a) How many layers are needed for an eye shield to transmit 10% of the light? (b) How many layers are needed to transmit 2% of the light?

29. A region has been contaminated with radioactive iodine–131 to a level 5 times the safe level. How long will it take until the area is safe?

30. A region has been contaminated with radioactive strontium–90 to a level 5 times the safe level. How long will it take until the area is safe?

31. The population of a country is 4 million and is growing at 5% per year. Currently the country has 10 million acres of forests which are being cut down (and not replanted) at a rate of 300,000 acres per year. (a) Find an equation for the number of acres of forest per person.
 (b) How fast is the number of acres of forest per person changing?
 (c) If the population and harvest rates remain constant, in approximately how many years will there be one acre of forest per person?

32. When a pan of hot (200° F) water is removed from the stove in a 70° F kitchen, it takes 4 minutes to cool to a temperature of 150° F.
 (a) Find an equation for the temperature of the water t minutes after it is removed from the stove.
 (b) When will the water be 100° F? (c) When will the water be 80° F?
 (d) When will be water be 60° F?

33. When the pan of 200° F water is taken outside on a cool (40° F) day, it only takes 4 minutes to cool to 150 F.
 (a) Find an equation for the temperature of the water t minutes after it is removed from the stove.
 (b) When will the water be 100° F? (c) When will the water be 80° F?
 (d) When will be water be 60° F?

34. When a pitcher of orange juice is taken out of a 40° F refrigerator in a 70° F kitchen, it takes 5 minutes to warm up to 60° F.
 (a) Find an equation for the temperature of the juice t minutes after it is removed from the refrigerator.
 (b) How long does it take to warm up 50° F?
 (c) How long does it take to warm up to 65° F?

Section 6.3 **PRACTICE Answers**

Practice 1: $f(t) = 2000 \cdot e^{0.0488t}$.

(a) When $t = 5$, $f(5) = 2000 \cdot e^{0.0488(5)} = 2000 \cdot e^{0.244} \approx 2000(1.276) = \mathbf{2{,}552}$.

(b) $f(t) = 5000$: $5000 = 2000 \cdot e^{0.0488t}$ so $\frac{5000}{2000} = \frac{5}{2} = e^{0.0488t}$.

Taking the natural log of each side, $\ln(5/2) = 0.0488t$ and $t = \frac{1}{0.0488} \ln(5/2) \approx \mathbf{18.78}$.

(c) $f(0) = 2{,}000$. Triple $= 6{,}000$. $6000 = 2000 \cdot e^{0.0488t}$ so $\frac{6000}{2000} = 3 = e^{0.0488t}$.

Taking the natural log of each side, $\ln(3) = 0.0488t$ and $t = \frac{1}{0.0488} \ln(3) \approx \mathbf{22.51}$.

Practice 2: $f(0) = 12{,}000$ and $f(2) = 18{,}000$.

(a) $f(t) = 12{,}000 \cdot e^{kt}$. $18{,}000 = 12{,}000 \cdot e^{k(2)}$ so $\frac{18000}{12000} = 1.5 = e^{2k}$.

Taking the natural log of each side, $\ln(1.5) = 2k$ and $k = \frac{1}{2} \ln(1.5) \approx 0.2027$.

$f(t) = 12{,}000 \cdot e^{(0.5 \ln(1.5)) t} \approx 12{,}000 \cdot e^{0.2027t}$.

(b) Double = $2(12{,}000) = 24{,}000$. $24{,}000 = 12{,}000 \cdot e^{0.2027t}$ so $\frac{24000}{12000} = 2 = e^{0.2027t}$.

Then $\ln(2) = 0.2027t$ so $t = \frac{1}{0.2027} \ln(2) \approx$ **3.42** .

Practice 3: After 1 year, each \$1 investment will be \$1 + (.12)(\$1) = \$1.12 so $f(1) = 1.12 = 1 \cdot e^{k \cdot 1}$.
Solving for k, $1.12 = e^k$ so $k = \ln(1.12)$ and $f(t) = e^{\ln(1.12)t}$.
Solving $2 = e^{\ln(1.12)t}$ for t gives $t = \frac{\ln(2)}{\ln(1.12)}$ $\approx$ **6.12 years**.

Practice 4: $f(t) = f(0) \cdot e^{kt} = 10 \cdot e^{kt}$ with $k = \frac{\ln(1/2)}{\text{half life}} = \frac{\ln(1/2)}{8} \approx -0.0866$.
$\mathbf{f(t) \approx 10 \cdot e^{-0.0866t}}$.

Differential equations in "literature":

From the murder mystery, <u>The Calculus of Murder</u> by Erik Rosenthal, St. Martin's Press, 1986:

"Maybe we could do some calculations before you call. From what you said, the rate of absorption would be proportional to the amount present and inversely proportional to the content of the stomach."

"Daniel, speak English."

"The more poison, the faster the rate of absorption: the greater the content of the stomach, the slower the . . . "

"Got it. Sounds right. So?"

"There's probably a constant of absorbency well known for arsenic and any given set of conditions. **It's a simple differential equation**."

"You're kidding."

The detective, a calculus teacher (!), then goes on to solve the differential equation $y'=cy$, to find the absorption constant c, and to figure out "whodunit".