CHAPTER 7: TRANSCENDENTAL FUNCTIONS

Introduction

In the previous chapters we saw how to calculate and use the derivatives and integrals of many of the most important and common functions in mathematics and applications. These common functions include polynomials, exponential functions, logarithms, the six trigonometric functions, and various combinations of them. There are, however, other important and useful functions, and we examine several of them in this chapter. The central purpose of this chapter is to extend the ideas and applications we have already seen to additional functions, primarily the inverse trigonometric functions.

The chapter begins with a short introduction to one–to–one functions. Section 7.1 is an examination of inverse functions and some of their properties. Section 7.2 introduces the inverses of the trigonometric functions, and Section 7.3 shows how to calculate and use the derivatives of the inverse trigonometric functions.

We have already been using a most important pair of one–to–one functions, e^x and $\ln(x)$. We have also used the fact that each of them "undoes" the effect of the other in order to solve equations such as $3 = 5e^{2x}$ and $8 = 2 \cdot \ln(x)$. **The functions e^x and $\ln(x)$ are inverses of each other, and you should keep these functions and their graphs in mind as we discuss general one–to–one and inverse functions.** Section 7.4 (optional) is a calculus–based presentation of the exponential and logarithm functions, and it includes verifications of a number of properties of the exponential and logarithm functions that we have already used, such as the multiplication law for logarithms: $\ln(a \cdot b) = \ln(a) + \ln(b)$.

One–to–One Functions

In earlier courses you saw that some equations have only one solution (for example, $5 - 2x = 3$ and $x^3 = 8$), some have two solutions ($x^2 + 3 = 7$), and some even have an infinite number of solutions ($\sin(x) = 0.8$). The graphs of the functions $y = 5 - 2x$, $y = x^3$, $y = x^2 + 3$, and $y = \sin(x)$ and the solutions of the equations are shown in Fig. 1. The functions f whose equations $f(x) = k$ have only one solution **for each value of k** (each outcome k comes from only one input x) are particularly common in applications, and they have a number of useful mathematical properties. The remainder of this section focuses on those functions and looks at some of their properties.

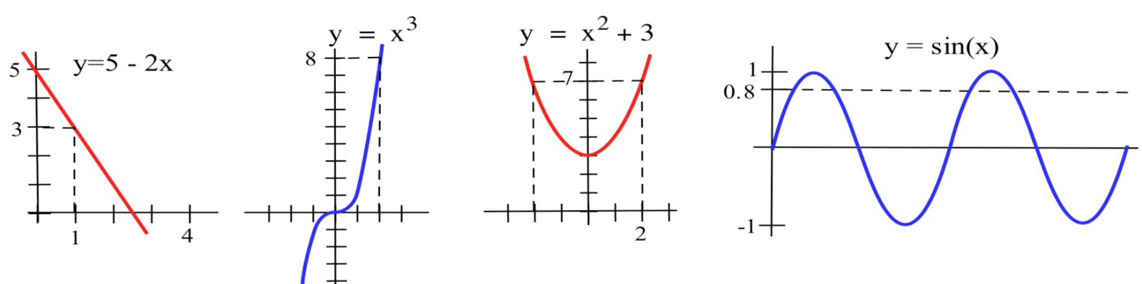

Fig. 1

Example 1: How many solutions does each of the following equations have?

(a) $f(x) = 0$ for $f(x) = x(x-4)$ (b) $g(x) = 3$ for g given by Table 1

(c) $h(x) = 4$ for h given by the graph in Fig. 2. (d) $f(x) = k$ for $f(x) = e^x$.

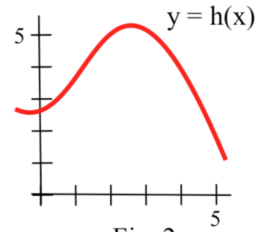

Fig. 2

Solution: (a) Two. $x(4-x) = 0$ if $x = 0$ or $x = 4$. (b) One. $g(x) = 3$ if $x = 2$.

(c) Two. $h(x) = 4$ if $x = 1$ or $x = 5$.

(d) One, $x = \ln(k)$, if $k > 0$. None if $k \le 0$.

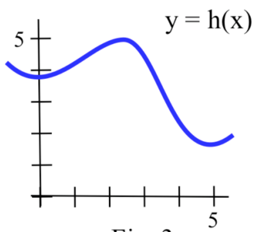

Fig. 3

Practice 1: How many solutions does each of the following equations have?

(a) $f(x) = 4$ for $f(x) = x(4-x)$ (b) $g(x) = 7$ for g given by Table 1

(c) $h(x) = 3$ for h given by the graph in Fig. 3.

(d) $f(x) = 5$ for $f(x) = \ln(x)$

x	g(x)
0	5
1	7
2	3
3	5
4	0
5	7

Table 1

Horizontal Line Test for One–to–one (Definition of One–to–one)

A function is **one–to–one** if each **horizontal** line intersects the graph of the function at most once.

Equivalently, a function $y = f(x)$ is one–to–one if two **distinct** x–values always result in two **distinct** y–values: $a \ne b$ implies $f(a) \ne f(b)$. This immediately tells us that every strictly increasing function is one–to–one, and every strictly decreasing function is one–to–one. (Why?)

For any function, if we know the input x–value, we can calculate the output y–value, but an output may have come from any of several different inputs. With a one–to–one function, each output y–value comes from only one input x–value.

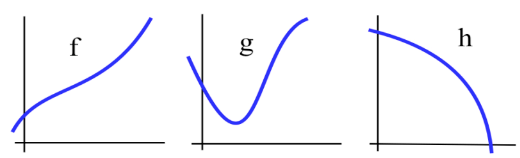

Fig. 4

Example 2: (a) Which functions in Fig. 4 are one–to–one?

(b) Which functions in Table 2 are one–to–one?

Solution: (a) In Fig. 4, functions f and h are one–to–one.

Function g is not one–to–one.

(b) In Table 2, function h is one–to–one. Functions f and g are not one–to–one.

x	f(x)	g(x)	h(x)
0	5	7	2
1	2	3	–1
2	3	0	5
3	5	1	4
4	0	6	3
5	1	3	0

Table 2

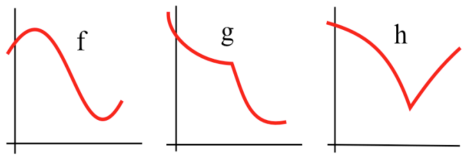

Fig. 5

Practice 2:

(a) Which functions in Fig. 5 are one–to–one?

(b) Which functions in Table 3 are one–to–one?

x	f(x)	g(x)	h(x)
0	4	2	2
1	2	3	–5
2	–2	0	1
3	5	4	14
4	3	6	3
5	1	7	1

Table 3

Example 3: Let $f(x) = 2x + 1$ (Fig. 6). Find the values of x so that

(a) $f(x) = 9$ and (b) $f(x) = a$.

(c) Solve $f(y) = x$ for y. ($f(y) = 2y + 1$)

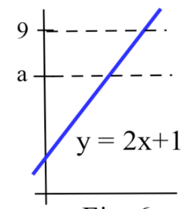

Fig. 6

Solution: (a) $9 = f(x) = 2x + 1$ so $8 = 2x$ and $x = 8/2 = 4$. (b) $a = 2x + 1$ so $2x = a - 1$ and $x = (a - 1)/2$.

(c) $x = f(y) = 2y + 1$ so $2y = x - 1$ and $y = (x - 1)/2$.

Practice 3: Let $f(x) = 3x - 5$. Find values of x so (a) $f(x) = 7$ and (b) $f(x) = a$. (c) Solve $f(y) = x$ for y.

Practice 4: Show that exponential growth, $f(x) = e^{3x}$, and exponential decay, $g(x) = e^{-2x}$, are one–to–one.

PROBLEMS

In problems 1 – 4, state whether the given functions are one–to–one.

1. $f(x) = 3x - 5$, $y = 3 - x$, $g(x)$ given by Table 4, and $h(x)$ given by the graph in Fig. 7.

x	g(x)
0	3
1	4
2	5
3	2
4	4

Table 4

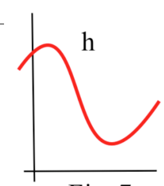

Fig. 7

2. $f(x) = x/4$, $y = x^2 + 3$, $g(x)$ given by Table 5, and $h(x)$ given by the graph in Fig. 8.

x	g(x)
0	3
1	2
2	0
3	–2
4	1

Table 5

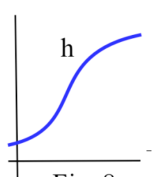

Fig. 8

3. $f(x) = \sin(x)$, $y = e^x - 2$, $g(x)$ given by Table 6, and $h(x)$ given by the graph in Fig. 9.

x	g(x)
0	–1
1	5
2	3
3	1
4	0

Table 6

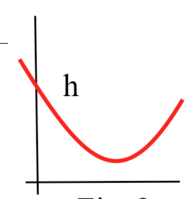

Fig. 9

4. $f(x) = 17$, $y = x^3 - 1$, $g(x)$ given by Table 7, and $h(x)$ given by the graph in Fig. 10.

5. Are Social Security numbers one–to–one? Telephone numbers?

6. What would it mean if the scores on a calculus test were one–to–one?

7. What is the legal/social term for one–to–one in marriage?

x	g(x)
0	2
1	5
2	4
3	1
4	2

Table 7

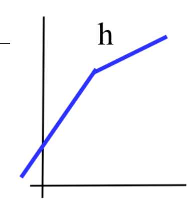

Fig. 10

8. The function given below represents "y is married to x."

(a) Is this f a function? (b) Is f one–to–one? (c) Is P breaking the law?

(d) Is A breaking the law?

x	A	B	C	D
y	P	Q	P	R

9. How many places can a one–to–one function touch the x–axis?

10. Can a **continuous** one–to–one function have the values given below Explain why it is possible or why it is not possible?

x	1	3	5
f(x)	2	7	3

11. The graph of $f(x) = x - 2 \cdot \text{INT}(x)$ for $-2 \le x \le 3$ is given in Fig. 11.

(a) Is f a one–to–one function?

(b) Is f an increasing function? a decreasing function?

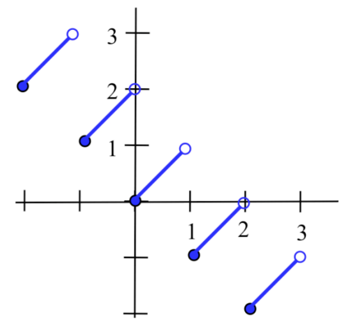

Fig. 11: $y = x - 2 \cdot \text{INT}(x)$

12. Is **every** linear function $f(x) = ax + b$ one–to–one?

13. Show that the function $f(x) = \ln(x)$ is one–to–one for $x > 0$.

14. Show that the function $f(x) = e^x$ is one–to–one.

In problems 15 – 18, rules are given for encoding a 6 letter alphabet. For each problem:

(a) Is the encoding rule a function? (b) Is the encoding rule one–to–one? (c) Encode the word "bad."

(d) Write a table for decoding the encoded letters and use it to decode your answer to part (c).

(e) Graph the encoding rule and the decoding rule. (Fig. 12 shows the graphs for the code in problem 15). How are the encoding and decoding graphs related?

15.

original letter	a	b	c	d	e	f
encoded letter	d	c	f	e	b	a

16.

original letter	a	b	c	d	e	f
encoded letter	b	d	f	b	a	c

17.

original letter	a	b	c	d	e	f
encoded letter	d	f	e	a	c	b

How does your decoding rule compare with the encoding rule?

What happens if you encode a word and then encode the encoded word: for example, Encode(Encode("bad")) = ?

18.

original letter	a	b	c	d	e	f
ncoded letter	e	a	f	c	b	d

What happens if you apply this coding rule three times:

Encode(Encode(Encode("bad"))) = ?

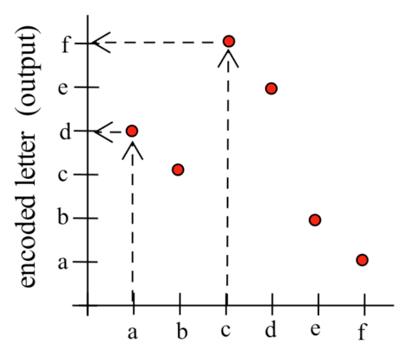

Fig. 12: Graph of a code

Section 7.0 **PRACTICE Answers**

Practice 1: (a) One. Solve $x(4 - x) = 4$ to get $x = 2$.

(b) At least two, $x = 1$ and $x = 5$.

(c) At least one, $x \approx 4$.

(d) Exactly one. Solve $5 = \ln(x)$ to get $x = e^5 \approx 148.4$.

Practice 2: (a) Only g is 1–1.

(b) From the values in the table, f anf g are one–to–one.

(However, if f and g are continuous on [0,5], then neither of them is one–to–one. Why?)

Practice 3: (a) $3x - 5 = 7$ so $x = 4$.

(b) $3x - 5 = a$ so $x = \frac{a+5}{3}$.

(c) $f(x) = 3x - 5$ so $f(y) = 3y - 5$. $f(y) = x$ means that $3y - 5 = x$ and $y = \frac{x+5}{3}$.

Practice 4: $f(x) = e^{3x}$. $f'(x) = 3 \cdot e^{3x} > 0$ so f is increasing, one–to–one, and has an inverse.
$g(x) = e^{-2x}$. $g'(x) = -2 \cdot e^{-2x} < 0$ so g is decreasing, one–to–one, and has an inverse.