7.2 INVERSE TRIGONOMETRIC FUNCTIONS

Section 7.2 is an introduction to the inverse trigonometric functions, their properties, and their graphs. The discussion focuses on the properties and techniques needed for derivatives and integrals. We emphasize the inverse sine and inverse tangent functions, the two inverse trigonometric functions most used in applications. This section provides the background and foundation for the calculus and applications of the inverse trigonometric functions in the next section.

Inverse Sine: Solving $k = \sin(x)$ for x

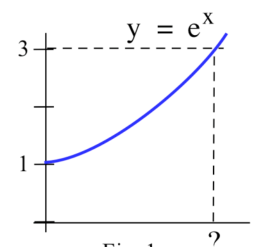


Fig. 1

It is straightforward to solve the equation $3 = e^x$ (Fig. 1): simply take the logarithm, the inverse of the exponential function e^x, of each side of the equation to get $\ln(3) = \ln(e^x) = x$. Since the function $f(x) = e^x$ is one–to–one, the equation $3 = e^x$ has only the one solution $x = \ln(3) \approx 1.1$.

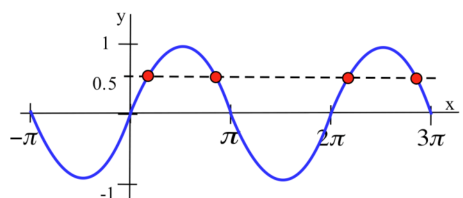


Fig. 2: $y = \sin(x)$

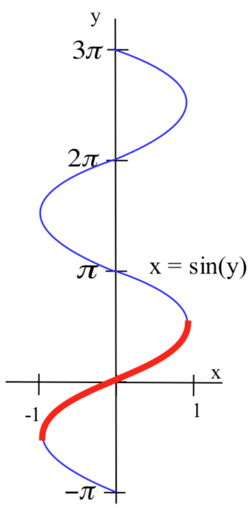


Fig. 3: $x = \sin(y)$

The solution of the equation $0.5 = \sin(x)$ (Fig. 2) presents more difficulties. As Fig. 2 illustrates, the function $f(x) = \sin(x)$ is not one–to–one, and it's reflected graph in Fig. 3 is not the graph of a function. However, sometimes it is important to "undo" the sine function, and we can do so by restricting its domain to the interval $[-\pi/2, \pi/2]$. For $-\pi/2 \le x \le \pi/2$, the function $f(x) = \sin(x)$ is one–to–one and has an inverse function, and the graph of the inverse function (Fig. 4) is the reflection about the line $y = x$ of the (restricted) graph of $y = \sin(x)$.

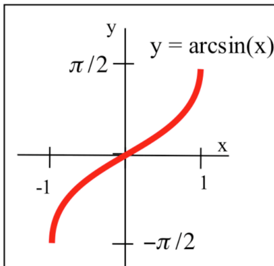


Fig. 4: Inverse sine graph

This inverse of the (restricted) sine function is written as $\mathbf{\sin^{-1}(x)}$ or **arcsin(x)**. Most calculators use keys labeled $\sin^{-1}$ or INV SIN for this function.

Definition of Inverse Sine:

For $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$,

$$y = \arcsin(x) \text{ if } x = \sin(y).$$

The domain of $f(x) = \arcsin(x)$ is $[-1, 1]$. The range of $f(x) = \arcsin(x)$ is $[-\pi/2, \pi/2]$.

The (restricted) sine function and the arcsin function are inverses of each other:

> For $-1 \le x \le 1$, $\sin(\arcsin(x)) = x$.
>
> For $-\pi/2 \le y \le \pi/2$, $\arcsin(\sin(y)) = y$.

Note: The notations $\mathbf{sin^{-1}(x)}$ and $\mathbf{arcsin(x)}$ are both commonly used in mathematical writings, and we use both. The symbol $\sin^{-1}(x)$ should **never** be used to represent $1/\sin(x)$. Represent $1/\sin(x)$ as $\csc(x)$ or, if you really want to use a negative exponent, as $(\sin(x))^{-1}$.

Note: The name "arcsine" comes from the unit circle definition of the sine function. On the unit circle (Fig. 5), if θ is the length of the **arc** whose sine is x, then $\sin(\theta) = x$ and $\theta =$ **arc**$\sin(x)$.

Using the right triangle definition of sine (Fig. 6), θ represents an angle whose sine is x.

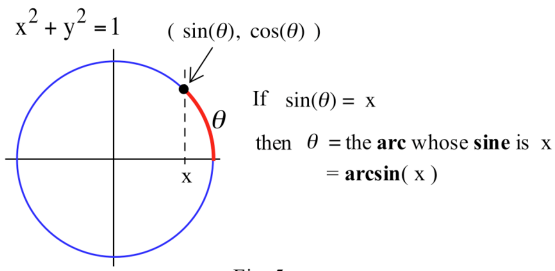


Fig. 5

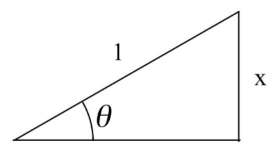


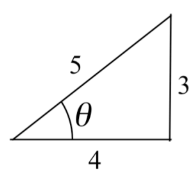
Fig. 6: $\sin(\theta) = x$, $\theta = \arcsin(x)$

Right Triangles and Arcsine

For the right triangle in Fig. 7, $\sin(\theta) = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{3}{5}$ so $\theta = \arcsin(3/5)$. It is possible to evaluate other trigonometric functions, such as cosine and tangent, of an arcsine without explicitly solving for the value of the arcsine. For example,

$$\cos(\arcsin(3/5)) = \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5} \quad \text{and}$$

$$\tan(\arcsin(3/5)) = \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}.$$



$\sin(\theta) = \dfrac{3}{5}$

$\theta = \arcsin\left(\dfrac{3}{5}\right)$

$(\theta \approx 0.6435)$

Fig. 7

Once we know the sides of the right triangle, the values of the other trigonometric functions can be evaluated using their standard right triangle definitions:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\text{adjacent}}{\text{opposite}}$$

If the angle θ is given as the arcsine of a number, but the sides of a right triangle are not given, we can construct our own triangle with the given angle: select values for the opposite side and hypotenuse so the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ is the value whose arcsin we want: arcsin(opposite/hypotenuse). The length of the other side can be calculated using the Pythagorean Theorem.

Example 1: Determine the lengths of the sides of a right triangle so one angle is $\theta = \arcsin(\frac{5}{13})$. Use the triangle to determine the values of $\tan(\arcsin(\frac{5}{13}))$ and $\csc(\arcsin(\frac{5}{13}))$.

Solution: We want the sine, the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$, of θ to be 5/13 so we can choose the opposite side to be 5 and the hypotenuse to be 13 (Fig. 8). Then $\sin(\theta) = 5/13$, the value we want. Using the Pythagorean Theorem, the length of the adjacent side is $\sqrt{13^2 - 5^2} = 12$.

Finally, $\tan(\theta) = \tan(\arcsin(\frac{5}{13})) = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{12}$ and

$$\csc(\theta) = \csc(\arcsin(\frac{5}{13})) = \frac{1}{\sin(\arcsin(\frac{5}{13}))} = \frac{13}{5}.$$

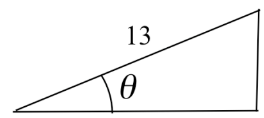


Fig. 8: $\sin(\theta) = \frac{5}{13}$

Any choice of values for the opposite side and the hypotenuse is fine as long as the ratio of the opposite to the hypotenuse is 5/13 .

Practice 1: Determine the lengths of the sides of a right triangle so one angle is $\theta = \arcsin(\frac{6}{11})$. Use the triangle to determine the values of $\tan(\arcsin(\frac{6}{11}))$, $\csc(\arcsin(\frac{6}{11}))$, and $\cos(\arcsin(\frac{6}{11}))$.

Example 2: Determine the lengths of the sides of a right triangle so one angle is $\theta = \arcsin(x)$. Use the triangle to determine the values of $\tan(\arcsin(x))$ and $\cos(\arcsin(x))$.

Solution: We want the sine, the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$, of θ to be x so we can choose the opposite side to be x and the hypotenuse to be 1 (Fig. 9). Then $\sin(\theta) = x/1 = x$, and, using the Pythagorean Theorem, the length of the adjacent side is $\sqrt{1 - x^2}$.

$$\tan(\arcsin(x)) = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{\sqrt{1 - x^2}} \quad \text{and}$$

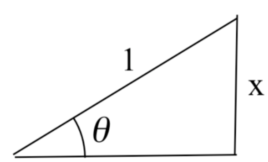


Fig. 9: $\theta = \arcsin(x)$

$$\cos(\arcsin(x)) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{1 - x^2}}{1} = \sqrt{1 - x^2}.$$

Practice 2: Use the triangle in Example 2 to help evaluate $\sec(\arcsin(x))$ and $\csc(\arcsin(x))$.

Inverse Tangent: Solving $k = \tan(x)$ for x

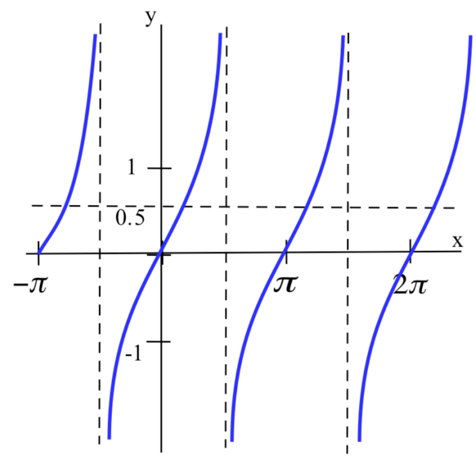


Fig. 10: $y = \tan(x)$

The equation $0.5 = \tan(x)$ (Fig. 10) has many solutions; the function $f(x) = \tan(x)$ is not one–to–one; and it's reflected graph in Fig. 11 is not the graph of a function. However, if the domain of the tangent function is restricted to the interval $(-\pi/2, \pi/2)$, then $f(x) = \tan(x)$ is one–to–one and has an inverse function. The graph of the inverse tangent function (Fig. 12) is the reflection about the line $y = x$ of the (restricted) graph of $y = \tan(x)$.

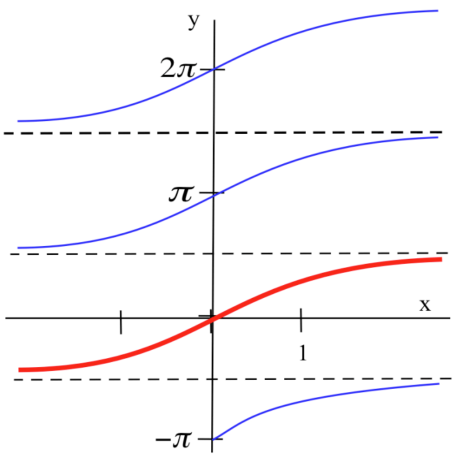


Fig. 11: $x = \tan(y)$

This inverse tangent function is written as $\mathbf{tan^{-1}(x)}$ or **arctan(x)**.

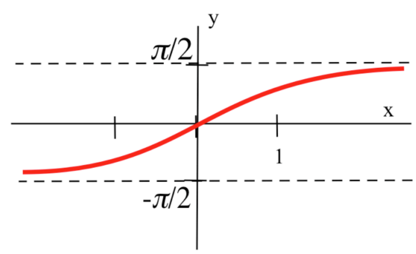


Fig. 12: $y = \arctan(x)$

Note: The symbol $\tan^{-1}(x)$ should **never** be used to represent $\frac{1}{\tan(x)}$.

Definition of Inverse Tangent:

For all x and for $-\pi/2 < y \le \pi/2$,

$$y = \arctan(x) \text{ if } x = \tan(y).$$

The domain of $f(x) = \arctan(x)$ is all real numbers. The range of $f(x) = \arctan(x)$ is $(-\pi/2, \pi/2)$.

The (restricted) tangent function and the arctan function are inverses of each other.

For all values of x, $\tan(\arctan(x)) = x$.

For $-\pi/2 < y < \pi/2$, $\arctan(\tan(y)) = y$.

Arctan(x) is the length of the arc on the unit circle whose tangent is x. Arctan(x) is the angle whose tangent is x: $\tan(\arctan(x)) = x$.

Right Triangles and Arctan

For the right triangle in Fig. 13, $\tan(\theta) = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{3}{2}$ so $\theta = \arctan(3/2)$.

Then $\sin(\arctan(3/2)) = \sin(\theta) = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{3}{\sqrt{13}} \approx 0.83$ and

$$\cot(\arctan(3/2)) = \frac{1}{\tan(\arctan(3/2))} = \frac{1}{3/2} = \frac{2}{3} \approx 0.67 .$$

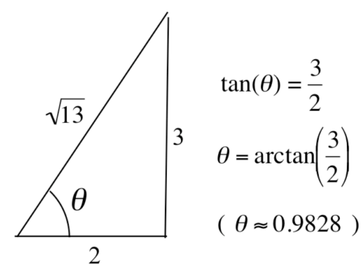


Fig. 13

Practice 3: Determine the lengths of the sides of a right triangle so one angle is $\theta = \arcsin(\frac{3}{5})$. Use the triangle to determine the values of $\tan(\arcsin(\frac{3}{5}))$, $\csc(\arcsin(\frac{3}{5}))$, and $\cos(\arcsin(\frac{3}{5}))$.

Example 3: On a wall 8 feet in front of you, the lower edge of a 5 foot tall painting is 2 feet above your eye level (Fig. 14). Represent your viewing angle θ using arctangents.

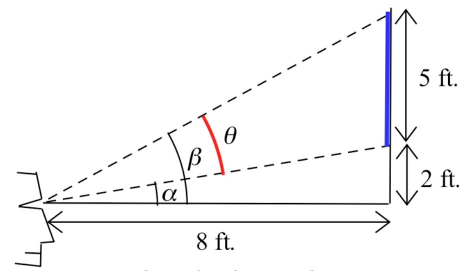


Fig. 14: Viewing a picture

Solution: The viewing angle α to the bottom of the painting satisfies $\tan(\alpha) = \text{opposite/adjacent} = 2/8$, so $\alpha = \arctan(1/4)$. Similarly, the angle β to the top of the painting satisfies $\tan(\beta) = 7/8$ so $\beta = \arctan(7/8)$. Finally, the viewing angle θ for the painting is $\theta = \beta - \alpha = \arctan(7/8) - \arctan(1/4) \approx 0.719 - 0.245 = 0.474$, or about $27°$.

Practice 4: Determine the scoring angle for the soccer player in Fig. 15.

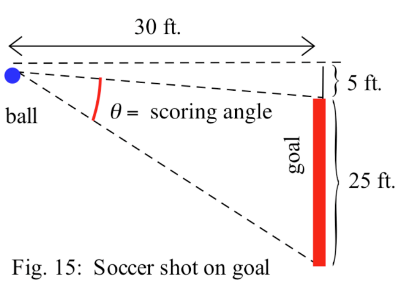


Fig. 15: Soccer shot on goal

Example 4: Determine the lengths of the sides of a right triangle so one angle is $\theta = \arctan(x)$. Use the triangle to determine the values of $\sin(\arctan(x))$ and $\cos(\arctan(x))$.

Solution: We want the tangent (the ratio opposite/adjacent) of θ to be x so we can choose the opposite side to be x and the adjacent side to be 1 (Fig. 16). Then $\tan(\theta) = x/1 = x$, and, using the Pythagorean Theorem, the length of the hypotenuse is $\sqrt{1+x^2}$. Finally,

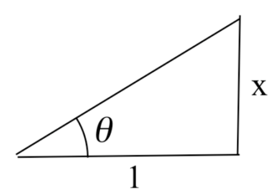


Fig. 16: $\theta = \arctan(x)$

$$\sin(\arctan(x)) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{\sqrt{1+x^2}} \quad \text{and} \quad \cos(\arctan(x)) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{1+x^2}} .$$

Practice 5: Use the triangle in Fig. 16 to help evaluate $\sec(\arctan(x))$ and $\cot(\arctan(x))$.

Inverse Secant: Solving k = sec(x) for x

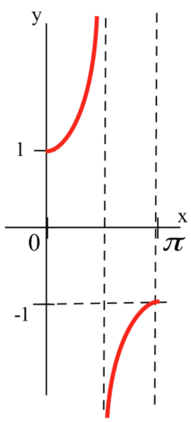


Fig. 18: y = sec(x)

The equation 2 = sec(x) (Fig. 17) has many solutions, but we can create an inverse **function** for secant in the same way we did for sine and tangent, by suitably restricting the domain of the secant function so it is one–to–one. Fig. 18 shows the restriction $0 \le x \le \pi$ and $x \ne \pi/2$ which results in a one–to–one function which has an inverse. The graph of the inverse function (Fig. 19) is the reflection about the line y = x of the (restricted) graph of y = sec(x).

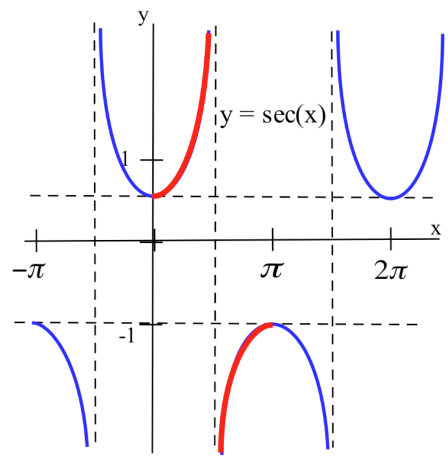


Fig. 17: y = sec(x)

This inverse secant function is written as $\mathbf{sec^{-1}(x)}$ or **arcsec(x)**.

Definition of Inverse Secant:

For $| x | \ge 1$ and for $0 \le y \le \pi$ with $y \ne \pi/2$,

y = arcsec(x) if x = sec(y).

The domain of f(x) = arcsec(x) is all x with $| x | \ge 1$.

The range of f(x) = arcsec(x) is $[0, \pi/2)$ and $(\pi/2, \pi]$.

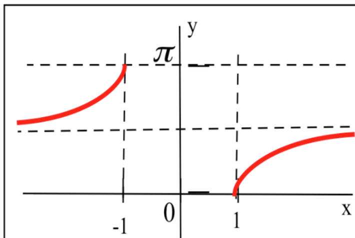


Fig. 19: x = arcsec(y)

The (restricted) secant function and the arcsec function are inverses of each other.

For all values of x with $| x | \ge 1$, sec(arcsec(x)) = x.

For all values of y with $0 \le y \le \pi$ and $y \ne \pi/2$, arcsec(sec(y)) = y.

Note: There are alternate ways to restrict secant to get a one–to–one function, and they lead to slightly different definitions of the inverse secant. We chose to use this restriction because it seems more "natural" than the alternatives, it is easier to evaluate on a calculator, and it is the most commonly used.

Example 5: Evaluate tan(arcsec(x)).

Solution: We want the secant (the ratio hypotenuse/adjacent) of θ to be x so we can choose the hypotenuse to be x and the adjacent side to be 1 (Fig. 20) . Then sec(θ) = x/1 = x. Using the Pythagorean Theorem,

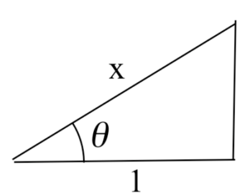


Fig. 20: θ = arcsec(x)

the length of the opposite side is $\sqrt{x^2 - 1}$, so

$$\tan(\operatorname{arcsec}(x)) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{x^2 - 1}}{1} = \sqrt{x^2 - 1} \ .$$

Practice 6: Evaluate $\sin(\operatorname{arcsec}(x))$ and $\cot(\operatorname{arcsec}(x))$.

The Other Inverse Trigonometric Functions

The inverse tangent and inverse sine functions are by far the most commonly used of the six inverse trigonometric functions. In particular, you will often encounter the arctangent function when you integrate rational functions such as $1/(1 + x^2)$ and $(4x + 7)/(x^2 + 6x + 10)$. The inverse secant function is used less often.

The other three inverse trigonometric functions ($\cos^{-1}$, $\cot^{-1}$, and $\csc^{-1}$) can be defined as the inverses of a suitably restricted parts of cosine, cotangent, and cosecant, respectively. The graph of each inverse function is the reflection about the line $y = x$ of the graph of the restricted trigonometric function.

Inverse Trigonometric Functions, Calculators, and Computers

Most scientific calculators only have keys for $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ (or INVerse SIN, COS, TAN), but those are enough to enable us to calculate the values of the other three inverse trigonometric functions by using the following identities.

If $x \neq 0$ and x is in the domains of the functions, then

$\operatorname{arccot}(x) = \arctan(1/x)$, $\operatorname{arcsec}(x) = \arccos(1/x)$, $\operatorname{arccsc}(x) = \arcsin(1/x)$

And some programming languages only have a single inverse trigonometric function, the arctangent function ATN(X), but even that is enough to enable us to evaluate the other five inverse trigonometric functions. Formulas for evaluating each inverse trigonometric function just in terms of ATN are given in an Appendix after the Practice Answers as are several additional identities involving the inverse trigonometric functions.

PROBLEMS

1. (a) List the three smallest positive angles θ that are solutions of the equation $\sin(\theta) = 1$.
   (b) Evaluate $\arcsin(1)$ and $\operatorname{arccsc}(1)$.

2. (a) List the three smallest positive angles θ that are solutions of the equation $\tan(\theta) = 1$.
   (b) Evaluate $\arctan(1)$ and $\operatorname{arccot}(1)$.

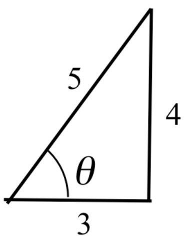


Fig. 21

3. Find the values of x **between 1 and 7** so (a) $\sin(x) = 0.3$, (b) $\sin(x) = -0.4$, and (c) $\sin(x) = 0.5$.

4. Find the values of x **between 5 and 12** so (a) $\sin(x) = 0.8$, and (b) $\sin(x) = -0.9$.

5. Find the values of x **between 2 and 7** so (a) $\tan(x) = 3.2$, and (b) $\tan(x) = -0.2$.

6. Find the values of x **between 1 and 5** so (a) $\tan(x) = 8$, and (b) $\tan(x) = -3$.

7. In Fig. 21, angle θ is (a) the arcsine of what number? (b) the arctangent of what number? (c) the arcsecant of what number? (d) the arccosine of what number?

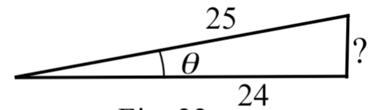


Fig. 22

8. In Fig. 22, angle θ is (a) the arcsine of what number? (b) the arctangent of what number? (c) the arcsecant of what number? (d) the arccosine of what number?

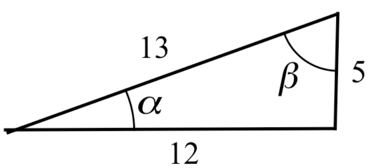


Fig. 23

9. For the angle α in Fig. 23 , evaluate (a) $\sin(\alpha)$, (b) $\tan(\alpha)$, (c) $\sec(\alpha)$, and (d) $\cos(\alpha)$.

10. For the angle β in Fig. 23, evaluate (a) $\sin(\beta)$, (b) $\tan(\beta)$, (c) $\sec(\beta)$, and (d) $\cos(\beta)$.

11. For $\theta = \sin^{-1}(2/7)$, find the exact values of (a) $\tan(\theta)$, (b) $\cos(\theta)$, (c) $\csc(\theta)$, and (d) $\cot(\theta)$.

12. For $\theta = \tan^{-1}(9/2)$, find the exact values of (a) $\sin(\theta)$, (b) $\cos(\theta)$, (c) $\csc(\theta)$, and (d) $\cot(\theta)$.

13. For $\theta = \cos^{-1}(1/5)$, find the exact values of (a) $\tan(\theta)$, (b) $\sin(\theta)$, (c) $\csc(\theta)$, and (d) $\cot(\theta)$.

14. For $\theta = \sin^{-1}(a/b)$ with $0 < a < b$, find the exact values of (a) $\tan(\theta)$, (b) $\cos(\theta)$, (c) $\csc(\theta)$, and (d) $\cot(\theta)$.

15. For $\theta = \tan^{-1}(a/b)$ with $0 < a < b$, find the exact values of (a) $\tan(\theta)$, (b) $\sin(\theta)$, (c) $\cos(\theta)$, and (d) $\cot(\theta)$.

16. For $\theta = \tan^{-1}(x)$, find the exact values of (a) $\sin(\theta)$, (b) $\cos(\theta)$, (c) $\sec(\theta)$, and (d) $\cot(\theta)$.

17. Find the exact values of (a) $\sin(\cos^{-1}(x))$, (b) $\cos(\sin^{-1}(x))$, and (c) $\sec(\cos^{-1}(x))$.

18. Find the exact values of (a) $\tan(\cos^{-1}(x))$, (b) $\cos(\tan^{-1}(x))$, and (c) $\sec(\sin^{-1}(x))$.

19. (a) Does $\arcsin(1) + \arcsin(1) = \arcsin(2)$?
(b) Does $\arccos(1) + \arccos(1) = \arccos(2)$?

20. (a) What is the viewing angle for the tunnel sign in Fig. 24? (b) Use arctangents to describe the viewing angle when the observer is x feet from the entrance of the tunnel.

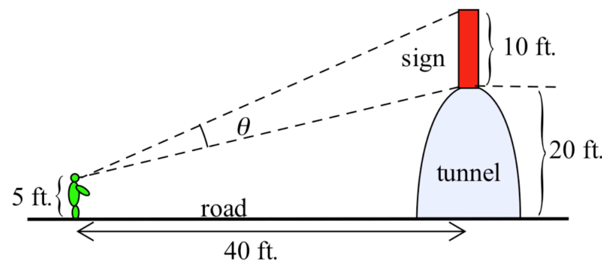


Fig. 24: On the road

21. (a) What is the viewing angle for the chalk board in Fig. 25 ? (b) Use arctangents to describe the viewing angle when the student is x feet from the front wall.

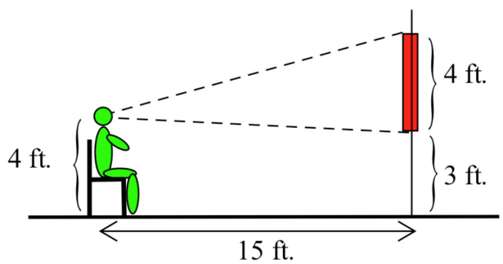


Fig. 25: In the classroom

22. Graph $y = \arcsin(2x)$ and $y = \arctan(2x)$.

23. Graph $y = \arcsin(x/2)$ and $y = \arctan(x/2)$.

24. Which curve is longer, $y = \sin(x)$ from $x = 0$ to π or $y = \arcsin(x)$ from $x = -1$ to 1 ?

For problems 25 – 28, $\frac{d\theta}{dt} = 12$ when $\theta = 1.3$, and θ and h are related by the given formula. Find $\frac{dh}{dt}$.

25. $\sin(\theta) = h/20$. 26. $\tan(\theta) = h/50$. 27. $\cos(\theta) = 3h + 20$. 28. $3 + \tan(\theta) = 7h$.

For problems 29 – 32 , $\frac{dh}{dt} = 4$ when $\theta = 1.3$, and θ and h are related by the given formula. Find $\frac{d\theta}{dt}$.

29. $\sin(\theta) = h/38$. 30. $\tan(\theta) = h/40$.

31. $\cos(\theta) = 7h - 23$. 32. $\tan(\theta) = h^2$.

33. You are observing a rocket launch from a point 4000 feet from the launch pad (Fig. 26). When the observation angle is $\pi/3$, the angle is increasing at $\pi/12$ feet per second. How fast is the rocket traveling? (Hint: θ and h are functions of t. $\tan(\theta) = \frac{h}{4000}$ so $\frac{d}{dt}\tan(\theta) = \frac{d}{dt}\left(\frac{h}{4000}\right)$. Then solve for $\frac{dh}{dt}$.)

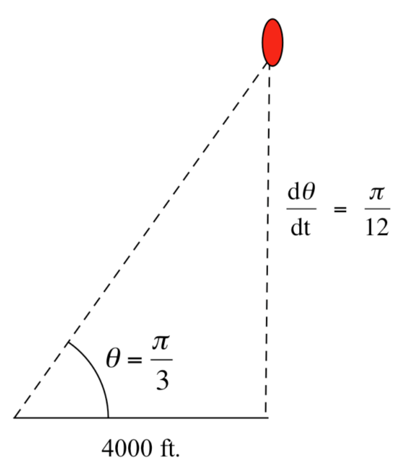


Fig. 26: Rocket launch

34. You are observing a rocket launch from a point 3000 feet from the launch pad. You heard on the radio the when the rocket is 5000 feet high its velocity is 100 feet per second. (a) What is the angle of elevation of the rocket when it is 5000 feet up? (b) How fast is the angle of elevation increasing when the rocket is 5000 feet up? (See the previous hint, but now solve for $d\theta/dt$.)

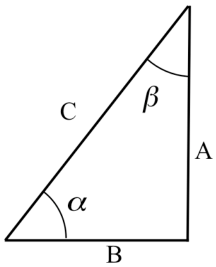


Fig. 27

35. In Fig. 27, (a) angle α is arcsine of what number?
    (b) angle β is the arccosine of what number?
    (c) For positive numbers A and B, evaluate arcsin(A/C) + arccos(A/C).

36. In Fig. 27, (a) angle α is arctangent of what number?
    (b) angle β is the arccotangent of what number?
    (c) For positive numbers A and B, evaluate arctan(A/B) + arccot(A/B).

37. In Fig. 27, (a) angle α is arcsecant of what number?
    (b) angle β is the arccosecant of what number?
    (c) For positive numbers A and B, evaluate arcsec(C/B) + arccsc(C/B).

38. Describe the pattern of your results for the previous three problems.

39. For the angle θ in Fig. 28, (a) θ is the arctangent of what number? (b) θ is arccotangent of what number?

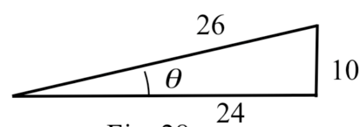


Fig. 28

40. For the angle θ in Fig. 28, (a) θ is the arcsine of what number?
    (b) θ is arccosecant of what number?

41. For the angle θ in Fig. 28, (a) θ is the arccosine of what number?
    (b) θ is arcsecant of what number?

42. Describe the pattern of your results for the previous three problems.

In problems 43 – 51, use your calculator and the appropriate identities to evaluate the functions.

43. $\sec^{-1}(3)$	44. $\sec^{-1}(-2)$	45. $\sec^{-1}(-1)$	46. $\cos^{-1}(0.5)$	47. $\cos^{-1}(-0.5)$
48. $\cos^{-1}(1)$	49. $\cot^{-1}(1)$	50. $\cot^{-1}(0.5)$	51. $\cot^{-1}(-3)$	

52. For the triangle in Fig. 29, (a) θ = arctan(_____) . (b) θ = arccot(_____) .
    (c) arccot (_____) = arctan(_____) .

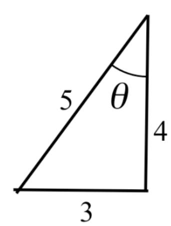


Fig. 29

53. For the triangle in Fig. 30, (a) θ = arcsin(_____) . (b) θ = arccos(_____) .
    (c) arccos(_____) = arcsin(_____) .

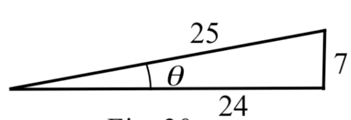


Fig. 30

$\tan(\arcsin(x)) = \dfrac{x}{\sqrt{1-x^2}}$ so $\arcsin(x) = \arctan\left(\dfrac{x}{\sqrt{1-x^2}}\right)$. Imitate this reasoning in problems 54 and 55.

54. Evaluate $\tan(\operatorname{arccot}(x))$ and use the result to find a formula for $\operatorname{arccot}(x)$ in terms of arctangent.

55. Evaluate $\tan(\operatorname{arcsec}(x))$ and use the result to find a formula for $\operatorname{arcsec}(x)$ in terms of arctangent.

56. Let $a = \arctan(x)$ and $b = \arctan(y)$. Use the identity $\tan(a+b) = \dfrac{\tan(a) + \tan(b)}{1 - \tan(a)\cdot\tan(b)}$ to show that

$$\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right).$$

Section 7.2 PRACTICE Answers

Practice 1: See Fig. 31. $\sin(\theta) = \dfrac{\text{OPP}}{\text{HYP}} = \dfrac{6}{11}$.

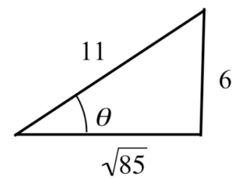


Fig. 31

$\tan(\arcsin(\frac{6}{11})) = \tan(\theta) = \dfrac{\text{OPP}}{\text{ADJ}} = \dfrac{6}{\sqrt{85}}$.

$\csc(\arcsin(\frac{6}{11})) = \dfrac{1}{\sin(\arcsin(\frac{6}{11}))} = \dfrac{1}{6/11} = \dfrac{11}{6}$. $\cos(\arcsin(\frac{6}{11})) = \cos(\theta) = \dfrac{\text{ADJ}}{\text{HYP}} = \dfrac{\sqrt{85}}{11}$.

Practice 2: See Fig. 32: OPP = x, HYP = 1, ADJ = $\sqrt{1-x^2}$.

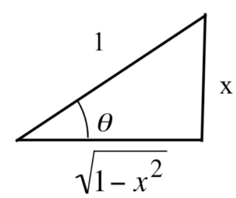


Fig. 32

$\sec(\arcsin(x)) = \sec(\theta) = \dfrac{1}{\cos(\theta)} = \dfrac{1}{\text{ADJ/HYP}} = \dfrac{\text{HYP}}{\text{ADJ}} = \dfrac{1}{\sqrt{1-x^2}}$.

$\csc(\arcsin(x)) = \dfrac{1}{\sin(\arcsin(x))} = \dfrac{1}{x}$.

Practice 3: See Fig. 33. $\sin(\theta) = \dfrac{\text{OPP}}{\text{HYP}} = \dfrac{3}{5}$. $\tan(\arcsin(\frac{3}{5})) = \tan(\theta) = \dfrac{\text{OPP}}{\text{ADJ}} = \dfrac{3}{4}$.

$\csc(\arcsin(\frac{3}{5})) = \dfrac{1}{\sin(\arcsin(\frac{3}{5}))} = \dfrac{1}{3/5} = \dfrac{5}{3}$.

$\cos(\arcsin(\frac{3}{5})) = \cos(\theta) = \dfrac{\text{ADJ}}{\text{HYP}} = \dfrac{4}{5}$.

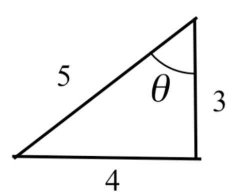


Fig. 33

Practice 4: See Fig. 34. $\tan(\alpha) = \frac{5}{30}$ so $\alpha = \arctan(\frac{5}{30}) \approx 0.165$ (or $\approx 9.46^o$).

$\tan(\alpha + \theta) = \frac{30}{30} = 1$ so $\alpha + \theta = \arctan(1) \approx 0.785$ (or 45^o).

Finally, $\theta = (\alpha + \theta) - (\alpha) \approx 0.785 - 0.165 = 0.62$

(or $\theta \approx 45^o - 9.46^o = 35.54^o$).

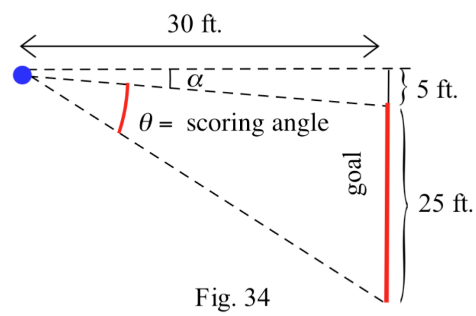


Fig. 34

Practice 5: See Fig. 35. $\tan(\theta) = \frac{x}{1} = x$ so $\theta = \arctan(x)$.

$\sec(\arctan(x)) = \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{\text{ADJ/HYP}}$

$= \frac{\text{HYP}}{\text{ADJ}} = \frac{\sqrt{1+x^2}}{1} = \sqrt{1+x^2}$.

$\cot(\arctan(x)) = \cot(\theta) = \frac{\text{ADJ}}{\text{HYP}} = \frac{1}{x}$.

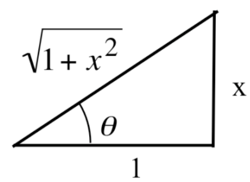


Fig. 35

Practice 6: See Fig. 20 in text. $\sin(\text{arcsec}(x)) = \frac{\sqrt{x^2-1}}{x}$. $\cot(\text{arcsec}(x)) = \frac{1}{\sqrt{x^2-1}}$

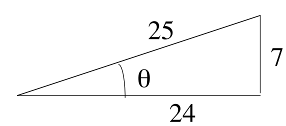


Fig. 30

Appendix: Identities for Inverse Trigonometric Functions

Theorem: arccot(x) = arctan(1/x) , arcsec(x) = arccos(1/x) , arccsc(x) = arcsin(1/x) .

Proof: $\tan(\operatorname{arccot}(x)) = \dfrac{1}{\cot(\operatorname{arccot}(x))} = \dfrac{1}{x}$ so, by taking the arctan of each side of the equation,

$$\arctan\{\tan(\operatorname{arccot}(x))\} = \arctan\{1/x\} \text{ and } \operatorname{arccot}(x)) = \arctan\{1/x\} .$$

Similarly, $\cos(\operatorname{arcsec}(x)) = \dfrac{1}{\sec(\operatorname{arcsec}(x))} = \dfrac{1}{x}$ so arcsec(x) = arccos(1/x), and

$\sin(\operatorname{arccsc}(x)) = \dfrac{1}{\csc(\operatorname{arccsc}(x))} = \dfrac{1}{x}$ so arccsc(x) = arcsin(1/x).

Theorem: For $0 \le x$, (a) arcsin(x) + arccos(x) = $\pi/2$, (b) arctan(x) + arccot(x) = $\pi/2$, and (c) arcsec(x) + arccsc(x) = $\pi/2$.

Partial proof: If A and B are the complementary angles in a right triangle ($A + B = \pi/2$), then sin(A) = cos(B). Let x = sin(A) = cos(B). Then A = arcsin(x) and B = arccos(x) so the fact that $A + B = \pi/2$ becomes arcsin(x) + arccos(x) = $\pi/2$.

(This "partial proof" assumed that A and B were nonnegative acute angles, so the proof is only valid if x is the sine or cosine of such angles, that is, that $0 \le x \le 1$. In fact, (a) is true for $-1 \le x \le 1$.)

If $A + B = \pi/2$, then tan(A) = cot(B). Let x = tan(A) = cot(B). Then A = arctan(x) and B = arccot(x) so $A + B = \pi/2$ implies that arctan(x) + arccot(x) = $\pi/2$. (This "partial proof" is valid for $0 \le x$.)

Theorem: arcsin(–x) = – arcsin(x) , arctan(–x) = – arctan(x) , arcsec(– x) = π – arcsec(x) (if $x \ge 1$).

Most versions of the programming language BASIC only have a single inverse trigonometric function, the arctangent function ATN(X), but that is enough to enable us to evaluate the other five inverse trigonometric functions.

$\arcsin(x) = \mathrm{ATN}\left(\dfrac{X}{\sqrt{1-X^2}}\right)$ $\arccos(x) = \pi/2 - \arcsin(x) = \pi/2 - \mathrm{ATN}\left(\dfrac{X}{\sqrt{1-X^2}}\right)$

$\operatorname{arccot}(x) = \mathrm{ATN}(1/X)$

$\operatorname{arcsec}(x) = \mathrm{ATN}\left(\sqrt{X^2-1}\right)$ $\operatorname{arccsc}(x) = \pi/2 - \operatorname{arcsec}(x) = \pi/2 - \mathrm{ATN}\left(\sqrt{X^2-1}\right)$