7.3 CALCULUS WITH THE INVERSE TRIGONOMETRIC FUNCTIONS

The three previous sections introduced the ideas of one–to–one functions and inverse functions and used those ideas to define arcsine, arctangent, and the other inverse trigonometric functions. Section 7.3 presents the **calculus** of inverse trigonometric functions. In this section we obtain derivative formulas for the inverse trigonometric functions and the associated antiderivatives. The applications we consider are both classical and sporting.

Derivative Formulas for the Inverse Trigonometric Functions

Derivative Formulas

(1) $\mathbf{D}(\arcsin(x)) = \dfrac{1}{\sqrt{1-x^2}}$ (for $|x| < 1$)

(4) $\mathbf{D}(\arccos(x)) = -\dfrac{1}{\sqrt{1-x^2}}$ (for $|x| < 1$)

(2) $\mathbf{D}(\arctan(x)) = \dfrac{1}{1+x^2}$ (for all x)

(5) $\mathbf{D}(\text{arccot}(x)) = -\dfrac{1}{1+x^2}$ (for all x)

(3) $\mathbf{D}(\text{arcsec}(x)) = \dfrac{1}{|x|\sqrt{x^2-1}}$ (for $|x| > 1$)

(6) $\mathbf{D}(\text{arccsc}(x)) = -\dfrac{1}{|x|\sqrt{x^2-1}}$ (for $|x| > 1$)

The proof of each of these differentiation formulas follows from what we already know about the derivatives of the trigonometric functions and the Chain Rule for Derivatives. Formula (2) is the most commonly used of these formulas, and it is proved below. The proofs of formulas (1), (4), and (5) are very similar and are left as problems. The proof of formula (3) is slightly more complicated and is included in an Appendix after the problems.

Proof of formula (2): The proof relies on two results from previous sections, that

$\mathbf{D}(\tan(f(x))) = \sec^2(f(x))\cdot\mathbf{D}(f(x))$ (using the Chain Rule) and that $\tan(\arctan(x)) = x$.

Differentiating each side of the equation $\tan(\arctan(x)) = x$, we have

$\mathbf{D}(\tan(\arctan(x))) = \mathbf{D}(x) = 1.$

Evaluating each derivative in the last equation,

$\mathbf{D}(\tan(\arctan(x))) = \sec^2(\arctan(x))\cdot\mathbf{D}(\arctan(x))$ and $\mathbf{D}(x) = 1$ so

$\sec^2(\arctan(x))\cdot\mathbf{D}(\arctan(x)) = 1$.

Finally, we can divide each side by $\sec^2(\arctan(x))$ to get

$$\mathbf{D}(\arctan(x)) = \frac{1}{\sec^2(\arctan(x))} = \frac{1}{\sec(\arctan(x))\cdot\sec(\arctan(x))}$$

$$= \frac{1}{\sqrt{1+x^2}\sqrt{1+x^2}} = \frac{1}{1+x^2}\,.$$

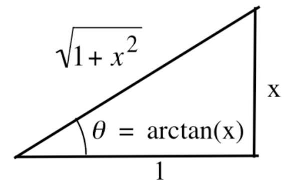

Fig. 1

Example 1: Calculate $\mathbf{D}(\arcsin(e^x))$, $\mathbf{D}(\arctan(x-3))$, $\mathbf{D}(\arctan^3(5x))$, and $\mathbf{D}(\ln(\arcsin(x)))$.

Solution: Each of the functions to be differentiated is a composition, so we need to use the Chain Rule.

$$\mathbf{D}(\arcsin(e^x)) = \frac{1}{\sqrt{1-(e^x)^2}}\,\mathbf{D}(e^x) = \frac{e^x}{\sqrt{1-e^{2x}}}\,.$$

$$\mathbf{D}(\arctan(x-3)) = \frac{1}{1+(x-3)^2}\,\mathbf{D}(x-3) = \frac{1}{1+(x-3)^2} = \frac{1}{x^2-6x+10}\,.$$

$$\mathbf{D}(\arctan^3(5x)) = 3\arctan^2(5x)\,\mathbf{D}(\arctan(5x)) = 3\arctan^2(5x)\cdot\frac{1}{1+(5x)^2}\cdot\mathbf{5}\,.$$

$$\mathbf{D}(\ln(\arcsin(x))) = \frac{1}{\arcsin(x)}\,\mathbf{D}(\arcsin(x)) = \frac{1}{\arcsin(x)}\cdot\frac{1}{\sqrt{1-x^2}}\,.$$

Practice 1: Calculate $\mathbf{D}(\arcsin(5x))$, $\mathbf{D}(\arctan(x+2))$, $\mathbf{D}(\text{arcsec}(7x))$, and $\mathbf{D}(e^{\arctan(7x)})$.

A Classic Application

Mathematics is the study of patterns, and one of the pleasures of mathematics is that the same pattern can appear in unexpected places. The version of the classical Museum Problem below was first posed in 1471 by the mathematician Johannes Muller and is one of the oldest known maximization problems.

Museum Problem: The lower edge of a 5 foot painting is 4 feet above your eye level (Fig. 2). At what distance should you stand from the wall so your viewing angle of the painting is maximum?

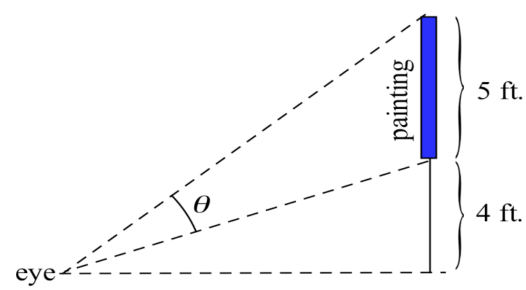

Fig. 2: Museum problem

On a typical autumn weekend, however, a lot more people would rather be watching or playing a football or soccer game than visiting a museum or solving a calculus problem about a painting. But the pattern of the Museum Problem even appears in football and soccer, sports not invented until hundreds of years after the original problem was posed and solved.

Since we also want to examine the Museum Problem in other contexts, let's solve the general version.

Example 2: **General Museum Problem.**

The lower edge of a H foot painting is A feet above your eye level (Fig. 3). At what distance x should you stand from the painting so the viewing angle is maximum?

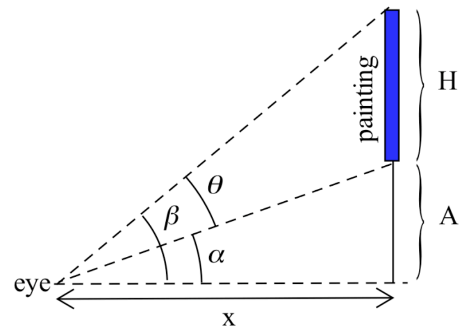

Fig. 3: General museum problem

Solution: Let B = A + H. Then $\tan(\alpha) = \frac{A}{x}$ and $\tan(\beta) = \frac{B}{x}$ so $\alpha = \arctan(A/x)$ and $\beta = \arctan(B/x)$. The viewing angle is $\theta = \beta - \alpha = \arctan(B/x) - \arctan(A/x)$. We can maximize θ by calculating the derivative $\frac{d\theta}{dx}$ and finding where the derivative is zero.

Since $\theta = \arctan(B/x) - \arctan(A/x)$,

$$\frac{d\theta}{dx} = \mathbf{D}(\arctan(B/x)) - \mathbf{D}(\arctan(A/x))$$

$$= \frac{1}{1 + \left(\frac{B}{x}\right)^2}\left(-\frac{B}{x^2}\right) - \frac{1}{1 + \left(\frac{A}{x}\right)^2}\left(-\frac{A}{x^2}\right) = \frac{-B}{x^2 + B^2} + \frac{A}{x^2 + A^2} .$$

Setting $\frac{d\theta}{dx} = 0$ and solving for x, we have $x = \sqrt{AB} = \sqrt{A(A+H)}$. (We can disregard the endpoints since we clearly do not have a maximum viewing angle with our noses pressed against the wall or from infinity far away from the wall.)

Now the Original Museum Problem and the Football and Soccer versions below are straightforward.

In our original Museum Problem, A = 4 and H = 5, so the maximum viewing angle occurs when $x = \sqrt{4(4+5)} = 6$ feet. The maximum angle is $\theta = \arctan(9/6) - \arctan(4/6) \approx 0.983 - 0.588 \approx .395$ or about 22.6° .

Practice 2: **Football.** A kicker is attempting a field goal by kicking the football between the goal posts (Fig. 4). At what distance from the goal line should the ball be spotted so the kicker has the largest angle for making the field goal? (Assume that the ball is "spotted" on a "hash mark" that is 53 feet 4 inches from the edge of the field and is actually kicked from a point about 9 yards further from the goal line than where the ball is spotted.)

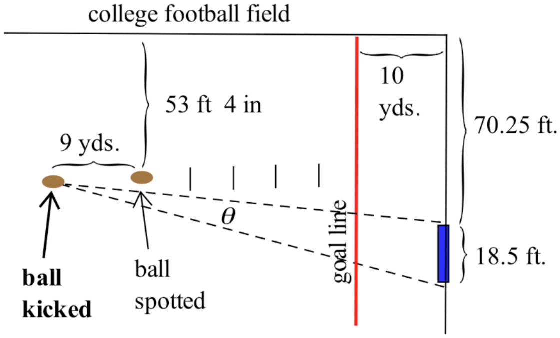

Fig. 4: Kicking a field goal

Practice 3: **Soccer.** Kelcey is bringing the ball down the middle of the soccer field toward the 25 foot wide goal which is defended by a goalie (Fig. 5). The goalie is positioned in the center of the goal and can stop a shot that is within four feet of the center of the goalie. At what distance from the goal should Kelcey shoot so the scoring angle is maximum?

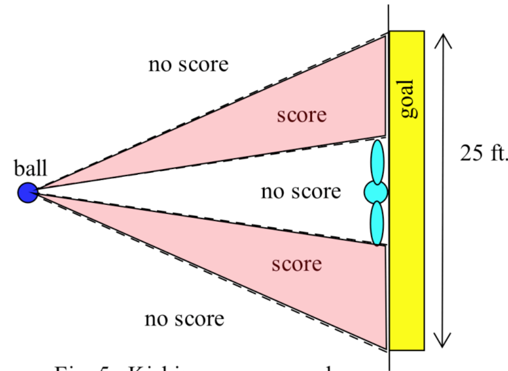

Fig. 5: Kicking a soccer goal

Antiderivative Formulas

Despite the Museum Problem and its sporting variations, the primary use of the inverse trigonometric functions in calculus is their use as antiderivatives. Each of the six differentiation formulas at the beginning of this section gives us an integral formula, but there are only three essentially different patterns:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C \qquad (\text{for } |x| < 1)$$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C \qquad (\text{for all } x)$$

$$\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \text{arcsec}(x) + C. \qquad (\text{for } |x| > 1)$$

Most of the integrals we need are variations of the basic patterns, and usually we have to transform the integrand so that it exactly matches one of the basic patterns.

Example 3: Evaluate $\int \frac{1}{16+x^2}\,dx$.

Solution: We can transform this integrand into the arctangent pattern by factoring 16 from the denominator and changing to the variable $u = x/4$:

$$\frac{1}{16+x^2} = \frac{1}{16}\cdot\frac{1}{1+\frac{x^2}{16}} = \frac{1}{16}\cdot\frac{1}{1+(x/4)^2} = \frac{1}{16}\cdot\frac{1}{1+u^2} .$$

If $u = x/4$ then $du = \frac{1}{4}\,dx$ and $\mathbf{dx = 4\,du}$ so

$$\int \frac{1}{16+x^2}\,\mathbf{dx} = \int \frac{1}{16}\cdot\frac{1}{1+u^2}\cdot\mathbf{4\,du}$$

$$= \frac{1}{4}\int \frac{1}{1+u^2}\,du = \frac{1}{4}\arctan(u) + C = \frac{1}{4}\arctan\left(\frac{x}{4}\right) + C .$$

Practice 4: Evaluate $\int \frac{1}{1+9x^2}\,dx$ and $\int \frac{1}{\sqrt{25-x^2}}\,dx$.

The most common integrands contain patterns with the forms $a^2 - x^2$, $a^2 + x^2$, and $x^2 - a^2$ where a is constant, and it is worthwhile to have general integral patterns for these forms.

$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx$	$= \arcsin(\frac{x}{a}) + C$	(for $\lvert x \rvert < \lvert a \rvert$)
$\int \frac{1}{a^2 + x^2}\,dx$	$= \frac{1}{a}\arctan(\frac{x}{a}) + C$	(for all x and for $a \neq 0$)
$\int \frac{1}{\lvert x \rvert \sqrt{x^2 - a^2}}\,dx$	$= \frac{1}{a}\operatorname{arcsec}(\frac{x}{a}) + C$	(for all $\lvert x \rvert > \lvert a \rvert > 0$)

These general formulas can be derived by factoring the a^2 out of the pattern and making a suitable change of variable. The final results can be checked by differentiating. **The arctan pattern is, by far, the most commonly needed.** The arcsin pattern appears occasionally, and the arcsec pattern only rarely.

Example 4: Derive the general formula for $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$ from the formula for $\int \frac{1}{\sqrt{1-x^2}}\,dx$.

Solution: We can algebraically transform the $a^2 + x^2$ pattern into an $1 + u^2$ pattern for an appropriate u:

$$\frac{1}{\sqrt{a^2-x^2}} = \frac{1}{\sqrt{a^2(1-x^2/a^2)}} = \frac{1}{a}\,\frac{1}{\sqrt{1-(x/a)^2}} .$$

If we put $u = x/a$, then $du = 1/a\,dx$ and $dx = a\,du$ so

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \frac{1}{a}\int \frac{1}{\sqrt{1-(x/a)^2}}\,\mathbf{dx} = \frac{1}{a}\int \frac{1}{\sqrt{1-u^2}} \cdot \mathbf{a\,du}$$

$$= \int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin(u) + C = \arcsin(\frac{x}{a}) + C.$$

Practice 5: Verify that the derivative of $\frac{1}{a} \cdot \arctan(\frac{x}{a})$ is $\frac{1}{a^2+x^2}$.

Example 5: Evaluate $\int \frac{1}{\sqrt{5-x^2}}\,dx$ and $\int_1^3 \frac{1}{5+x^2}\,dx$.

Solution: The constant a does not have to be an integer, so we can take $a^2 = 5$ and $a = \sqrt{5}$. Then

$$\int \frac{1}{\sqrt{5 - x^2}}\, dx = \arcsin\left(\frac{x}{\sqrt{5}} \right) + C \text{ , and}$$

$$\int_1^3 \frac{1}{5 + x^2}\, dx = \frac{1}{\sqrt{5}} \arctan\left(\frac{x}{\sqrt{5}} \right) \Big|_1^3 = \frac{1}{\sqrt{5}} \arctan\left(\frac{3}{\sqrt{5}} \right) - \frac{1}{\sqrt{5}} \arctan\left(\frac{1}{\sqrt{5}} \right) \approx 0.228 \text{ .}$$

The easiest way to integrate some rational functions is to split the original integrand into two pieces.

Example 6: Evaluate $\int \frac{6x + 7}{25 + x^2}\, dx$.

Solution: This integrand splits nicely into the sum of two other functions that can be easily integrated:

$$\int \frac{6x + 7}{25 + x^2}\, dx = \int \frac{6x}{25 + x^2}\, dx + \int \frac{7}{25 + x^2}\, dx \text{ .}$$

The integral of $\frac{6x}{25 + x^2}$ can be evaluated by changing the variable to $u = 25 + x^2$ and $du = 2x\, dx$.

Then $6x\, dx = 3\, du$ and $\int \frac{\mathbf{6x}}{25 + x^2}\, \mathbf{dx} = \int \frac{\mathbf{3}}{u}\, \mathbf{du} = 3 \ln| u | + C = 3 \ln(25 + x^2) + C$.

The integral of $\frac{7}{25 + x^2}$ matches the arctangent pattern with $a = 5$:

$$\int \frac{7}{25 + x^2}\, dx = \frac{7}{5} \arctan\left(\frac{x}{5} \right) + C \text{ .}$$

Finally, $\int \frac{6x + 7}{25 + x^2}\, dx = \int \frac{6x}{25 + x^2}\, dx + \int \frac{7}{25 + x^2}\, dx = 3 \ln(25 + x^2) + \frac{7}{5} \arctan\left(\frac{x}{5} \right) + C.$

The antiderivative of a linear function divided by an irreducible quadratic commonly involves a logarithm and an arctangent.

Practice 6: Evaluate $\int \frac{4x + 3}{x^2 + 7}\, dx$.

PROBLEMS

In problems 1 – 15, calculate the derivatives.

1. **D(** arcsin(3x) **)**
2. **D(** arctan(7x) **)**
3. **D(** arctan(x + 5) **)**
4. **D(** arcsin(x/2) **)**
5. **D(** arctan($\sqrt{x}$) **)**
6. **D(** arcsec(x^2) **)**
7. **D(** ln(arctan(x)) **)**
8. **D(** $\sqrt{\text{arcsin(x)}}$ **)**
9. **D(** (arcsec(x) $)^3$ **)**

10. **D(** $\arctan(5/x)$ **)**

11. **D(** $\arctan(\ln(x))$ **)**

12. **D(** $\arcsin(x+2)$ **)**

13. **D(** $e^x \cdot \arctan(2x)$ **)**

14. **D(** $\dfrac{\arcsin(x)}{\arccos(x)}$ **)**

15. **D(** $\arcsin(x) + \arccos(x)$ **)**

16. **D(** $x \cdot \arctan(x)$ **)**

17. **D(** $\dfrac{1}{\arcsin(x)}$ **)**

18. **D(** $(1 + \text{arcsec}(x))^3$ **)**

19. **D(** $\sin(3 + \arctan(x))$ **)**

20. **D(** $\tan(x) \cdot \arctan(x)$ **)**

21. **D(** $x \cdot \arctan(\frac{1}{x})$ **)**

22. $\displaystyle\int \frac{7}{\sqrt{9 - x^2}}\,dx$

23. $\displaystyle\int_0^1 \frac{3}{x^2 + 25}\,dx$

24. $\displaystyle\int_5^7 \frac{5}{x\sqrt{x^2 - 16}}\,dx$

25. $\displaystyle\int \frac{9}{\sqrt{49 - x^2}}\,dx$

26. $\displaystyle\int_1^4 \frac{2}{7 + x^2}\,dx$

27. $\displaystyle\int_6^{10} \frac{3}{x\sqrt{x^2 - 25}}\,dx$

28. $\displaystyle\int \frac{7}{(x-5)^2 + 9}\,dx$

29. $\displaystyle\int_{-1}^1 \frac{e^x}{1 + e^{2x}}\,dx$

30. $\displaystyle\int_1^e \frac{1}{x} \cdot \frac{3}{1 + (\ln(x))^2}\,dx$

31. $\displaystyle\int \frac{\cos(x)}{\sqrt{9 - \sin^2(x)}}\,dx$

32. $\displaystyle\int \frac{8x}{16 + x^2}\,dx$

33. $\displaystyle\int \frac{6x}{9 + x^4}\,dx$

34. (a) Use arctangents to describe the viewing angle for the sign in Fig. 6 when the observer is x feet from the entrance to the tunnel. (b) At what distance x is the viewing angle maximized?

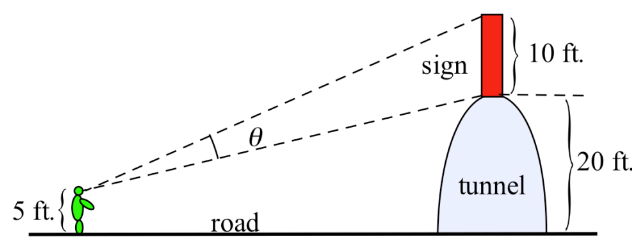

Fig. 6: Looking at a sign

35. (a) Use arctangents to describe the viewing angle for the chalk board in Fig. 7 when the student is x feet from the front wall. (b) At what distance x is the viewing angle maximized?

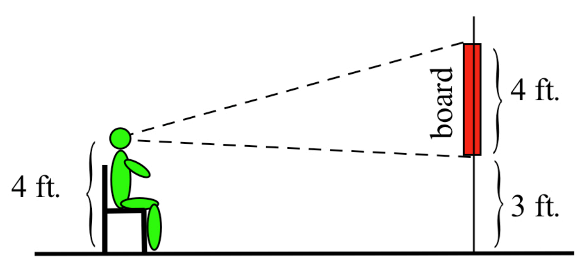

Fig. 7: In class

In problems 36 – 39, find the function y which satisfies the differential equation and goes through the given point.

36. $\frac{dy}{dx} = \frac{1}{y(1+x^2)}$ and the point (0,4).

37. $\frac{dy}{dx} = \frac{y}{\sqrt{1-x^2}}$ and $y(0) = e$.

38. $y' \cdot \sqrt{16-x^2} = y$ and $y(4) = 1$.

39. $\frac{dy}{dx} = \frac{y^2}{9+x^2}$ and the point (1,2).

40. Prove differentiation formula (1): $\mathbf{D}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$ (for $|x| < 1$).

41. Prove differentiation formula (4): $\mathbf{D}(\arccos(x)) = -\frac{1}{\sqrt{1-x^2}}$ (for $|x| < 1$).

42. Prove differentiation formula (5): $\mathbf{D}(\text{arccot}(x)) = -\frac{1}{1+x^2}$ (for all x).

43. Let $A(x) = \int_0^x \frac{1}{1+t^2}\, dt$, the area between the curve $y = \frac{1}{1+t^2}$ and the t–axis between t = 0 and x.

(a) Evaluate A(0), A(1), A(10). (b) Evaluate $\lim_{x \to \infty} A(x)$. (c) Find $\frac{d\,A(x)}{dx}$.

(d) Is A(x) an increasing, decreasing or neither? (e) Evaluate $\lim_{x \to \infty} A'(x)$.

44. Find area between the curve $y = \frac{1}{1+x^2}$ and the x–axis (a) from x = – 10 to 10 and

(b) from x = –A to A. (c) Find the area under the **whole** curve. (Calculate the limit of your answer in part (b) as $A \to \infty$.)

45. $\int \frac{8x-5}{x^2+9}\, dx$

46. $\int \frac{1-4x}{x^2+1}\, dx$

47. $\int \frac{7x+3}{x^2+10}\, dx$

48. $\int \frac{x+5}{x^2+9}$

Problems 49 – 53 illustrate how we can sometimes decompose a difficult integral into easier ones.

49. (a) $\int \frac{8}{x^2+6x+10}\, dx$ (hint: $x^2 + 6x + 10 = (x+3)^2 + 1$. Try $u = x + 3$.)

(b) $\int \frac{4x+12}{x^2+6x+10}\, dx$ (hint: Try $u = x^2 + 6x + 10$. Then $(2x+6)\,dx = 2\,du$.)

(c) $\int \frac{4x+20}{x^2+6x+10}\, dx$ (hint: $\frac{4x+20}{x^2+6x+10} = \frac{4x+12}{x^2+6x+10} + \frac{8}{x^2+6x+10}$

50. (a) $\int \frac{7}{x^2 + 4x + 5}\,dx$ (b) $\int \frac{12x + 24}{x^2 + 4x + 5}\,dx$ (c) $\int \frac{12x + 31}{x^2 + 4x + 5}\,dx$

51. $\int \frac{6x + 15}{x^2 + 4x + 20}\,dx$ 52. $\int \frac{2x + 5}{x^2 - 4x + 13}\,dx$

Section 7.3 **PRACTICE Answers**

Practice 1: **D**(arcsin(5x)) $= \frac{1}{\sqrt{1 - (5x)^2}} \cdot 5 = \frac{5}{\sqrt{1 - 25x^2}}$.

D(arctan(x+2)) $= \frac{1}{1 + (x+2)^2} = \frac{1}{x^2 + 4x + 5}$. **D**(arrcsec(7x)) $= \frac{1}{|7x|\sqrt{(7x)^2 - 1}} \cdot 7 = \frac{1}{|x|\sqrt{49x^2 - 1}}$.

D($e^{\arctan(7x)}$) $= e^{\arctan(7x)}$ **D**(arctan(7x)) $= e^{\arctan(7x)} \cdot \frac{7}{1 + (7x)^2} = e^{\arctan(7x)} \cdot \frac{7}{1 + 49x^2}$.

Practice 2: Football: A = 16.9 ft. H = 18.5 ft. (see Fig. 8) so $x = \sqrt{A(A+H)} = \sqrt{598.26} \approx 24.46$ feet from the back edge of the end zone. Unfortunately, that is still more than 5 feet into the endzone. Our mathematical analysis shows that to maximize the angle for kicking a field goal, the ball should be placed 5 feet into the end zone, a touchdown! If the ball is placed on a hash mark at the goal line, then the kicking distance is 57 feet (10 yards for the width of the endzone plus 9 yards that the ball is hiked) and the scoring angle for the kicker is $\theta = \arctan(35.4/57) - \arctan(16.9/57) \approx 15.3^o$. It is somewhat interesting to see how the scoring angle $\theta = \arctan(35.4/x) - \arctan(16.9/x)$ changes with the distance x (feet), and also to compare the scoring angle for balls placed on the hash mark with those placed in the center of the field.

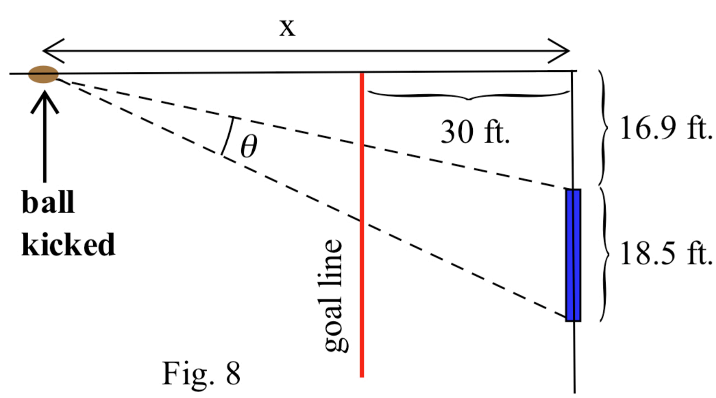

Fig. 8

Practice 3: Soccer: See Fig. 9. A = 4 ft. and H = 8.5 ft. so $x = \sqrt{A(A+H)} = \sqrt{50} \approx 7.1$ ft.

From 7.1 feet, the scoring angle on one side of the goalie is $\theta = \arctan(12.5/7.1) - \arctan(4/7.1) \approx 31^{\circ}$.

For comparison, the scoring angles $\theta = \arctan(12.5/x) - \arctan(4/x)$ are given for some other distances x from the goal.

x	θ
5	29.5°
7.1	31.0°
10	29.5°
15	24.9°
20	20.7°
30	15.0°
40	11.6°

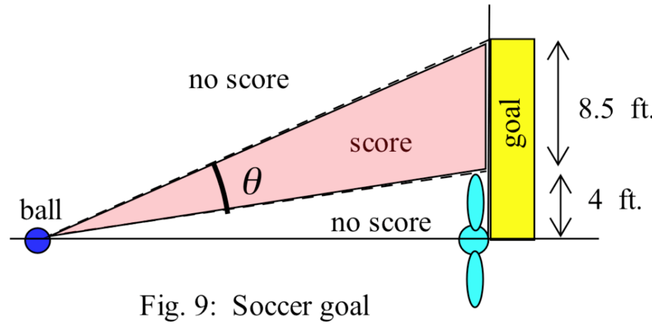

Fig. 9: Soccer goal

Practice 4: $\int \frac{1}{1+9x^2}\,dx = \int \frac{1}{1+(3x)^2}\,dx$. Put u = 3x. Then du = 3 dx and $dx = \frac{1}{3}\,du$.

$$\int \frac{1}{1+(3x)^2}\,dx = \int \frac{1}{1+(u)^2} \cdot \frac{1}{3}\,du = \frac{1}{3}\arctan(u) + C = \mathbf{\frac{1}{3}\arctan(3x) + C}.$$

$\int \frac{1}{\sqrt{25-x^2}}\,dx = \int \frac{1}{5}\frac{1}{\sqrt{1-(x/5)^2}}\,dx$. Put $u = \frac{x}{5}$. Then $du = \frac{1}{5}\,dx$ and dx = 5 du.

$$\int \frac{1}{5}\frac{1}{\sqrt{1-(x/5)^2}}\,dx = \int \frac{1}{5}\frac{1}{\sqrt{1-(u)^2}} \cdot 5\,du = \arcsin(u) + C = \mathbf{\arcsin(\tfrac{x}{5}) + C}.$$

Practice 5: $\mathbf{D}(\frac{1}{a}\arctan(\frac{x}{a})) = \frac{1}{a} \cdot \frac{1}{1+(x/a)^2}\frac{1}{a} = \frac{1}{a^2} \cdot \frac{1}{1+(x/a)^2} = \frac{1}{a^2+x^2}$.

Practice 6: $\int \frac{4x+3}{x^2+7}\,dx = \int \frac{4x}{x^2+7}\,dx + \int \frac{3}{x^2+7}\,dx$.

For the first integral on the right, put $u = x^2 + 7$. Then du = 2x dx and 2 du = 4x dx. Then

$$\int \frac{4x}{x^2+7}\,dx = \int \frac{2}{u}\,du = 2 \cdot \ln|u| + C = 2 \cdot \ln(x^2+7) + C.$$

The second integral on the right matches the "arctan" pattern;

$$\int \frac{3}{x^2+7}\,dx = \frac{3}{\sqrt{7}}\arctan(\frac{x}{\sqrt{7}}) + C.$$

Therefore, $\int \frac{4x+3}{x^2+7}\,dx = \mathbf{2 \cdot \ln(x^2+7) + \frac{3}{\sqrt{7}}\arctan(\frac{x}{\sqrt{7}}) + C}$.

Appendix: Proofs of Some Derivative Formulas

Proof of formula (1): The proof relies on two results from previous sections, that

D(sin(f(x))) = cos(f(x))• **D**(f(x)) (the Chain Rule) and that sin(arcsin(x)) = x (for $|x| \le 1$).

Putting these two results together and differentiating both sides of sin(arcsin(x)) = x, we get

$$\mathbf{D}(\sin(\arcsin(x))) = \mathbf{D}(x).$$

Evaluating the derivatives,

$$\mathbf{D}(\sin(\arcsin(x))) = \cos(\arcsin(x))\cdot\mathbf{D}(\arcsin(x)) \text{ and } \mathbf{D}(x) = 1$$

so $$\cos(\arcsin(x))\cdot\mathbf{D}(\arcsin(x)) = 1.$$

Dividing each side by cos(arcsin(x)) and using the fact that $\cos(\arcsin(x)) = \sqrt{1-x^2}$, we have

$$\mathbf{D}(\arcsin(x)) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}} .$$

The derivative of arcsin(x) is defined only when $1 - x^2 > 0$, or equivalently, if $|x| < 1$.

Proof of formula (3): The proof relies on two results from previous sections, that

D(sec(f(x))) = sec(f(x))•tan(f(x))• **D**(f(x)) (Chain Rule) and sec(arcsec(x)) = x .

Differentiating each side of sec(arcsec(x)) = x , we have

$$\mathbf{D}(\sec(\text{arcsec}(x))) = \mathbf{D}(x) = 1.$$

Evaluating each derivative, we have

$$\mathbf{D}(\sec(\text{arcsec}(x))) = \sec(\text{arcsec}(x))\cdot\tan(\text{arcsec}(x))\cdot\mathbf{D}(\arctan(x)) = 1$$

so $$\mathbf{D}(\text{arcsec}(x)) = \frac{1}{\sec(\text{arcsec}(x))\cdot\tan(\text{arcsec}(x))} = \frac{1}{x\cdot\tan(\text{arcsec}(x))} .$$

To evaluate tan(arcsec(x)), we can use the identity $\tan^2(\theta) = \sec^2(\theta) - 1$ with $\theta = \text{arcsec}(x)$. Then $\tan^2(\text{arcsec}(x)) = \sec^2(\text{arcsec}(x)) - 1 = x^2 - 1$. Now, however, there is a slight difficulty: is $\tan(\text{arcsec}(x)) = +\sqrt{x^2-1}$ or is $\tan(\text{arcsec}(x)) = -\sqrt{x^2-1}$? It is clear from the graph of arcsec(x) (Fig. 8) that **D**(arcsec(x)) is positive everywhere it is defined, so we need to choose the sign of the square root to guarantee that our calculated value for **D**(arcsec(x)) is positive. An easier way to guarantee that the derivative is positive is simply to always use the positive square root and to take the absolute value of x. Then

$$\mathbf{D}(\text{arcsec}(x)) = \frac{1}{|x|\sqrt{x^2-1}} > 0 .$$

The derivative of arcsec(x) is defined only when $x^2 - 1 > 0$, or equivalently, if $|x| > 1$.