8.1 Improper Integrals

Our original development of the definite integral $\int_a^b f(x)\,dx$ used Riemann sums and assumed that

- the length of the interval of integration $[a, b]$ was finite and
- that $f(x)$ was defined and bounded at every point of the interval $[a, b]$ (including the endpoints).

Sometimes, however, we need the value of an integral which does not satisfy one or both of these assumptions. In this section we extend the ideas of the definite integral to evaluate two types of **improper definite integrals**:

(1) the length of the interval of integration is not finite

(2) the integrand function is not bounded at a point of the interval of integration.

Example 1: Represent each area as an improper definite integral.

(a) The area of the infinite region between $f(x) = 1/x^2$ and the x–axis for $x \ge 1$ (Fig. 1)

(b) The area between $g(x)$ and the x–axis for $0 \le x \le 4$ (Fig. 2)

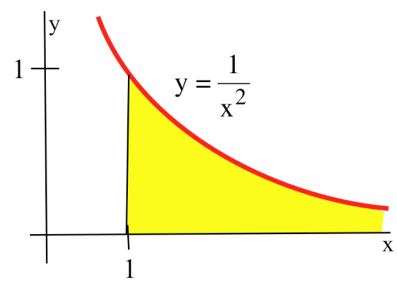

Fig. 1

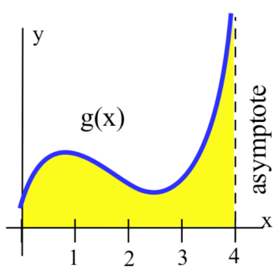

Fig. 2

Solution: (a) $\int_1^\infty \frac{1}{x^2}\,dx$ (b) $\int_0^4 g(x)\,dx$

Practice 1: Represent each quantity as an improper definite integral.

(a) The volume swept out when the infinite region between $f(x) = 1/x$ and the x–axis is revolved about the x–axis for $x \ge 4$ (Fig. 3)

(b) The area between the curves in Fig. 4 for $1 \le x \le 3$.

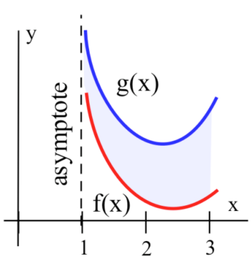

Fig. 4

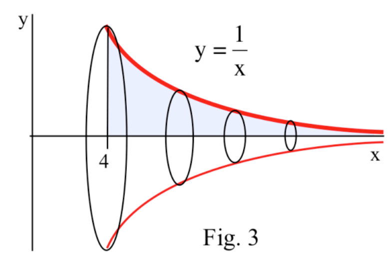

Fig. 3

General Strategy For Improper Integrals

Our general strategy for evaluating improper integrals is to shrink the interval of integration so we have a definite integral we can evaluate. Then as we let the interval grow to approach the interval of integration we want, the value of the integral on the growing intervals approaches the value of the improper integral. The value of the improper integral is the limiting value of the definite integrals as the interval grows to the interval we want, provided that the limit exists.

Infinitely Long Intervals of Integration

We evaluate an improper integral on an infinitely long interval by

- replacing the inifinitely long interval with a finite interval,
- evaluating the integral on the finite interval, and, finally,
- letting the finite interval grow longer and longer, approaching the interval we want.

Example 2: Evaluate $\int_1^{\infty} \frac{1}{x^2}\,dx$.

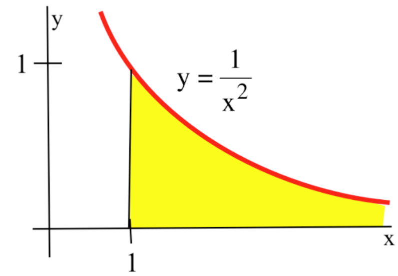

Fig. 5

Solution: The interval $[1, \infty)$ is infinitely long, but we can evaluate the integral on the finite intervals $[1, 2]$, $[1, 10]$, $[1, 1000]$, and in general $[1, C]$ (Fig. 5).

$$\int_1^{2} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^2 = \left(-\frac{1}{2}\right) - \left(-\frac{1}{1}\right) = 1 - \frac{1}{2} = \frac{1}{2} .$$

Similarly, $\int_1^{10} \frac{1}{x^2}\,dx = 1 - \frac{1}{10} = .9$, $\int_1^{1000} \frac{1}{x^2}\,dx = 1 - \frac{1}{1000} = .999$, and, in general,

$\int_1^{C} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^C = \left(-\frac{1}{C}\right) - \left(-\frac{1}{1}\right) = 1 - \frac{1}{C}$. As the value of C gets larger, the length of the interval $[1, C]$ increases, and the value of $\int_1^{C} \frac{1}{x^2}\,dx$ approaches the value of $\int_1^{\infty} \frac{1}{x^2}\,dx$.

The value of $\int_1^{\infty} \frac{1}{x^2}\,dx$ is the limit of the values of $\int_1^{C} \frac{1}{x^2}\,dx$ as C approaches infinity:

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{C\to\infty} \left\{ \int_1^{C} \frac{1}{x^2}\,dx \right\} = \lim_{C\to\infty} \left\{ 1 - \frac{1}{C} \right\} = 1 .$$

We say that the improper integral $\int_1^{\infty} \frac{1}{x^2}\,dx$ is **convergent** and **converges to 1**.

If the following limits exist,

the value of $\int_a^{\infty} f(x)\, dx$ is defined to be the value of $\lim_{C\to\infty} \left\{ \int_a^{C} f(x)\, dx \right\}$, and

the value of $\int_{-\infty}^{b} f(x)\, dx$ is defined to be the value of $\lim_{C\to -\infty} \left\{ \int_C^{b} f(x)\, dx \right\}$.

In each case, first evaluate the proper integral and then take the limit.

If the limit is a finite number, we say the improper integral is **convergent**.

If the limit does not exist or if it is infinite, we say the improper integral is **divergent**.

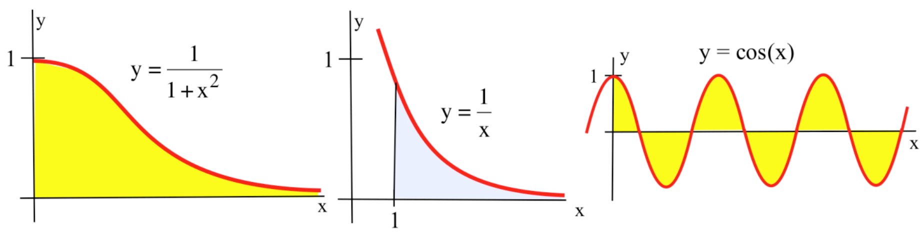

Fig. 6

Example 3: Evaluate (a) $\int_0^{\infty} \frac{1}{1+x^2}\, dx$, (b) $\int_1^{\infty} \frac{1}{x}\, dx$ and (c) $\int_0^{\infty} \cos(x)\, dx$. (Fig. 6)

Solution: (a) $\int_0^{\infty} \frac{1}{1+x^2}\, dx = \lim_{C\to\infty} \left\{ \int_0^{C} \frac{1}{1+x^2}\, dx \right\} = \lim_{C\to\infty} \left\{ \arctan(x) \Big|_0^{C} \right\}$

$= \lim_{C\to\infty} \left\{ \arctan(C) - \arctan(0) \right\} = \frac{\pi}{2} - 0 = \boldsymbol{\frac{\pi}{2}}$.

so we say that $\int_0^{\infty} \frac{1}{1+x^2}\, dx$ is **convergent**.

(b) $\int_1^{\infty} \frac{1}{x}\, dx = \lim_{C\to\infty} \left\{ \int_1^{C} \frac{1}{x}\, dx \right\} = \lim_{C\to\infty} \left\{ \ln(x) \Big|_1^{C} \right\} = \lim_{C\to\infty} \left\{ \ln(C) - \ln(1) \right\} = \infty$

so we say the improper integral $\int_1^{\infty} \frac{1}{x}\, dx$ is **divergent**.

(c) $\int_0^\infty \cos(x)\, dx = \lim_{C\to\infty} \left\{ \int_0^C \cos(x)\, dx \right\} = \lim_{C\to\infty} \left\{ \sin(x) \Big|_0^C \right\}$

$= \lim_{C\to\infty} \left\{ \sin(C) - \sin(0) \right\} = \lim_{C\to\infty} \sin(C)$.

As C increases the values of sin(C) oscilate between –1 and 1 and do not approach a single value so the last limit does not exist and the improper integral $\int_1^\infty \cos(x)\, dx$ is **divergent**.

Practice 2: Evaluate (a) $\int_1^\infty \frac{1}{x^3}\, dx$ and (b) $\int_0^\infty \sin(x)\, dx$.

Functions Undefined At An Endpoint Of The Interval Of Integration

If the function we want to integrate is unbounded at one of the endpoints of an interval of finite length, we can shrink the interval so the function is bounded at both endpoints of the new, smaller interval, evaluate the integral over the smaller interval, and finally, let the smaller interval grow to approach the original interval.

Example 4: Evaluate $\int_0^1 \frac{1}{\sqrt{x}}\, dx$. (Fig. 7)

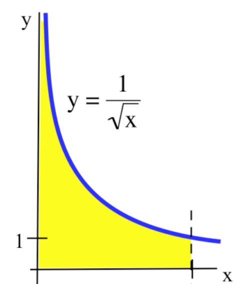

Fig. 7

Solution: The function $1/\sqrt{x}$ is not bounded at x = 0, the lower endpoint of integration, but the function is bounded on the intervals [.36, 1], [0.09, 1], and, in general, on the interval [C , 1] for any C > 0.

$\int_{.36}^1 \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x} \Big|_{.36}^1 = 2\sqrt{1} - 2\sqrt{.36} = 2 - 1.2 = 0.8$. Similarly,

$\int_{0.09}^1 \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x} \Big|_{0.09}^1 = 2\sqrt{1} - 2\sqrt{0.09} = 1.4$ and $\int_C^1 \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x} \Big|_C^1 = 2 - 2\sqrt{C}$.

As C approaches 0 from the right, the interval [C, 1] approaches the interval [0, 1] and the value of $2 - 2\sqrt{C}$ approaches 2. We say that $\int_0^1 \frac{1}{\sqrt{x}}\, dx$ converges to 2 and write $\int_0^1 \frac{1}{\sqrt{x}}\, dx = 2$.

If f(x) is not bounded (not defined) at x = b,

the value of $\int_a^b f(x)\, dx$ is defined to be the value of $\lim_{C \to b^-} \left\{ \int_a^C f(x)\, dx \right\}$

if this limit exists.

If f(x) is not bounded (not defined) at x = a,

the value of $\int_a^b f(x)\, dx$ is defined to be the value of $\lim_{C \to a^+} \left\{ \int_C^b f(x)\, dx \right\}$.

if this limit exists.

In each case, first evaluate the proper integral and then take the limit.

If the limit is a finite number, we say the improper integral is **convergent**.

If the limit does not exist or if it is infinite, we say the improper integral is **divergent**.

Practice 3: Show that (a) $\int_1^{10} \frac{1}{\sqrt{10-x}}\, dx = 6$ and (b) $\int_0^1 \frac{1}{x}\, dx$ is divergent.

If the function is unbounded at one or more points inside the interval of integration, we can split the original improper integral into several improper integrals on subintervals so the function is unbounded at one endpoint of each subinterval.

Testing For Convergence: The Comparison Test and P–Test

Sometimes the only thing that matters about an improper integral is whether or not it converges to a finite number. There are ways to determine its convergence even though we may not be able to or may not want to determine the exact value of the integral. In the remainder of this section we consider two methods for testing the convergence of an improper integral. Neither method gives us the value of the improper integral, but each enables us to determine whether some improper integrals are convergent. The Comparison Test For Integrals enables us to determine the convergence (or divergence) of some integrals by comparing them with some integrals we already know converge or diverge. The Comparison Test, however, requires that we know the convergence or divergence of the integrals we compare against, and the P–Test provides examples of known convergent and divergent integrals to use for this comparison.

We start with the P–Test in order to have some examples to use when we consider the Comparison Test.

P–Test for Integrals

For any $a > 0$, the improper integral $\int_a^{\infty} \frac{1}{x^p}\, dx$ $\begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \le 1 \end{cases}$

Proof: It is easiest to consider three cases: $p = 1$, $p > 1$, and $p < 1$.

Case **p = 1**: Then $\int_a^{\infty} \frac{1}{x^p}\, dx = \int_a^{\infty} \frac{1}{x}\, dx = \lim_{C \to \infty} \left\{ \int_a^C \frac{1}{x}\, dx \right\}$

$$= \lim_{C \to \infty} \left\{ \ln(x) \Big|_a^C \right\} = \lim_{C \to \infty} \left\{ \ln(C) - \ln(a) \right\} = \infty \text{ so } \int_a^{\infty} \frac{1}{x^p}\, dx \text{ diverges.}$$

For the other two cases, $p \ne 1$, so $\int_a^{\infty} \frac{1}{x^p}\, dx = \lim_{C \to \infty} \left\{ \int_a^C \frac{1}{x^p}\, dx \right\} = \lim_{C \to \infty} \left\{ \int_a^C x^{-p}\, dx \right\}$

$$= \lim_{C \to \infty} \left\{ \frac{1}{1-p} \cdot x^{1-p} \Big|_a^C \right\} = \lim_{C \to \infty} \left\{ C^{1-p} - a^{1-p} \right\}.$$

Case **p > 1**: Then $1 - p < 0$ so $\lim_{C \to \infty} C^{1-p} = 0$ and

$$\int_a^{\infty} \frac{1}{x^p}\, dx = \lim_{C \to \infty} \frac{1}{1-p} \left\{ C^{1-p} - a^{1-p} \right\} = -\frac{a^{1-p}}{1-p} = \frac{a^{1-p}}{p-1} \text{ , a finite number.}$$

Case **p < 1**: Then $1 - p > 0$ so $\lim_{C \to \infty} C^{1-p} = \infty$ and $\int_a^{\infty} \frac{1}{x^p}\, dx$ diverges.

Example 5: Determine the convergence or divergence of (a) $\int_5^{\infty} \frac{1}{x^2}\, dx$, (b) $\int_1^{\infty} \frac{1}{\sqrt{x}}\, dx$ and (c) $\int_1^{8} \frac{1}{x^{1/3}}\, dx$.

Solution: (a) $\int_5^{\infty} \frac{1}{x^2}\, dx$ matches the form for the P–Test with $p = 2 > 1$, so the integral is convergent.

(b) $\int_1^{\infty} \frac{1}{\sqrt{x}}\, dx = \int_1^{\infty} \frac{1}{x^{1/2}}\, dx$ matches the form for the P–Test with $p = 1/2 < 1$ so the integral is divergent.

(c) $\int_1^{8} \frac{1}{x^{1/3}}\, dx$ does not match the form for the P–Test because the interval $[1, 8]$ is finite.

$$\int_1^{8} \frac{1}{x^{1/3}}\, dx = \int_1^{8} x^{-1/3}\, dx = \frac{3}{2} x^{2/3} \Big|_1^{8} = \frac{3}{2} \left\{ 8^{2/3} - 1^{2/3} \right\} = \frac{3}{2} \left\{ 4 - 1 \right\} = \frac{9}{2} .$$

The following Comparison Test enables us to determine the convergence or divergence of an improper integral of a new positive function by comparing the new function with functions whose improper integrals we already know converge or diverge.

Comparison Test for Integrals of Positive Functions

(a) If the new integral is smaller than one we know converges, then the new integral converges (Fig. 8):

$$\text{if } 0 \le f(x) \le g(x) \text{ and } \int_a^\infty g(x)\,dx \text{ converges, then } \int_a^\infty f(x)\,dx \text{ converges.}$$

(b) If the new integral is larger than one which diverges, then the new integral diverges (Fig. 9):

$$\text{if } f(x) \ge g(x) \ge 0 \text{ and } \int_a^\infty g(x)\,dx \text{ diverges, then } \int_a^\infty f(x)\,dx \text{ diverges.}$$

(c) If the new integral is larger than a convergent integral or smaller than a divergent integral, then we can draw no immediate conclusion about the new integral — the new integral may converge or diverge (Fig. 10).

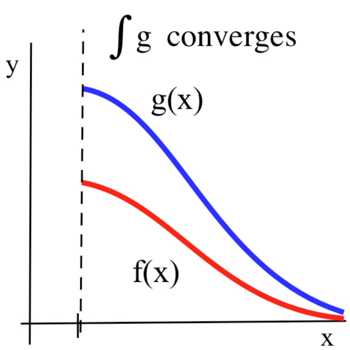

Fig. 8

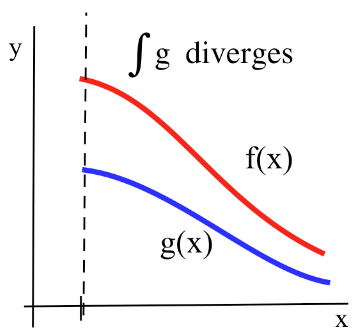

Fig. 9

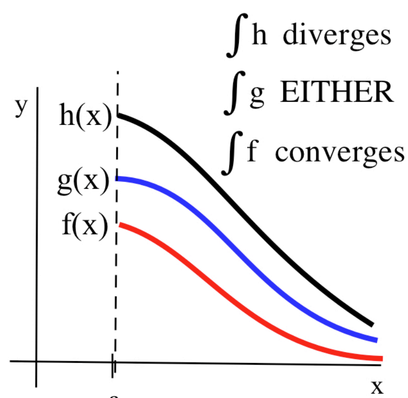

Fig. 10

The proof is a straightforward application of the definition of the value of an improper integral and facts about limits.

Example 6: Determine whether each of these integrals is convergent by comparing it with an appropriate integral which you know converges or diverges.

(a) $\displaystyle\int_1^\infty \frac{7}{x^3+5}\,dx$ (b) $\displaystyle\int_1^\infty \frac{3+\sin(x)}{x^2}\,dx$ (c) $\displaystyle\int_6^\infty \frac{9}{\sqrt{x-5}}\,dx$

Solution: (a) The dominant power of $\frac{7}{x^3+5}$ is $\frac{1}{x^3}$ so we should compare with $\frac{1}{x^3}$.

$$\int_1^\infty \frac{7}{x^3+5}\,dx < \int_1^\infty \frac{7}{x^3}\,dx = 7\int_1^\infty \frac{1}{x^3}\,dx.$$ We know $\int_1^\infty \frac{1}{x^3}\,dx$ is convergent by the P–Test ($p = 3 > 1$), so we can conclude that $\int_1^\infty \frac{7}{x^3+5}\,dx$ is convergent.

(b) We know $0 < 3 + \sin(x) \le 4$ and the dominant term is $\frac{1}{x^2}$ so we should compare with $\frac{1}{x^2}$.

$$\int_1^\infty \frac{3+\sin(x)}{x^2}\,dx \le \int_1^\infty \frac{4}{x^2}\,dx = 4\int_1^\infty \frac{1}{x^2}\,dx.$$ We know $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by the P–Test ($p = 2 > 1$), so we can conclude that $\int_1^\infty \frac{3+\sin(x)}{x^2}\,dx$ is convergent.

(c) The dominant power of $\frac{9}{\sqrt{x-5}}$ is $\frac{1}{\sqrt{x}}$ so we should compare with $\frac{1}{x^{1/2}}$.

$$\int_6^\infty \frac{9}{\sqrt{x-5}}\,dx > \int_1^\infty \frac{9}{\sqrt{x}}\,dx = 9\int_1^\infty \frac{1}{x^{1/2}}\,dx.$$ We know $\int_1^\infty \frac{1}{x^{1/2}}\,dx$ is divergent by the P–Test ($p = 1/2 < 1$), so we can conclude that $\int_6^\infty \frac{9}{\sqrt{x-5}}\,dx$ is divergent.

PROBLEMS

In 1–21, use the definition of an improper integral to evaluate the given integral.

1. $\int_{10}^\infty \frac{1}{x^3}\,dx$
2. $\int_e^\infty \frac{5}{x\cdot\ln(x)^2}\,dx$
3. $\int_3^\infty \frac{2}{1+x^2}\,dx$
4. $\int_1^\infty \frac{2}{e^x}\,dx$
5. $\int_e^\infty \frac{5}{x\cdot\ln(x)}\,dx$
6. $\int_0^\infty \frac{x}{1+x^2}\,dx$
7. $\int_3^\infty \frac{1}{x-2}\,dx$
8. $\int_3^\infty \frac{1}{(x-2)^2}\,dx$
9. $\int_3^\infty \frac{1}{(x-2)^3}\,dx$
10. $\int_3^\infty \frac{1}{x+2}\,dx$
11. $\int_3^\infty \frac{1}{(x+2)^2}\,dx$
12. $\int_3^\infty \frac{1}{(x+2)^3}\,dx$

13. $\int_0^4 \frac{1}{\sqrt{x}}\,dx$ 14. $\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx$ 15. $\int_0^{16} \frac{1}{\sqrt[4]{x}}\,dx$

16. $\int_0^2 \frac{1}{\sqrt{2-x}}\,dx$ 17. $\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$ 18. $\int_0^2 \frac{3x^2}{\sqrt{8-x^3}}\,dx$

19. $\int_{-2}^{\infty} \sin(x)\,dx$ 20. $\int_{\pi}^{\infty} \sin(x)\,dx$ 21. $\int_0^{\pi/2} \tan(x)\,dx$

22. Example 3(b) showed that $\int_1^C \frac{1}{x}\,dx$ grew arbitrarily large as C grew arbitrarily large, so no finite amount of paint would cover the area bounded between the x–axis and the graph of $f(x) = 1/x$ for $x > 1$ (Fig. 11a). Show that the volume obtained when the area in Fig. 11a is revolved about the x–axis (Fig. 11b) is finite so the 3–dimensional trumpet–shaped region can be filled with a finite amount of paint. Is there a contradiction here?

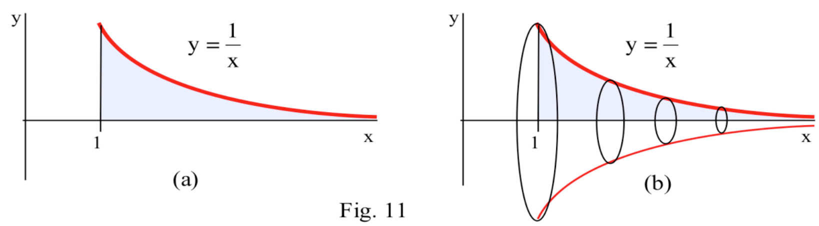

Fig. 11

23. In the **Lifting a Payload** discussion in Section 5.3, we determined that the amount of work needed to lift a payload from the surface of a moon to an altitude of A above the moon's surface was

$$\text{work} = \int_R^{R+A} \frac{R^2 P}{x^2}\,dx.$$

(a) Calculate the amount of work required to lift the payload to an altitude of R miles and 2R miles.

(b) Calculate the amount of work needed to lift the payload arbitrarily high. (Calculate the limit of the work integral as "$A \to \infty$.")

In problems 24–32, determine whether the improper integral is convergent or divergent. Do not evaluate the integral.

24. $\int_3^{\infty} \frac{7}{x^2+5}\,dx$ 25. $\int_3^{\infty} \frac{1}{x^3+x}\,dx$ 26. $\int_7^{\infty} \frac{1}{x-2}\,dx$

27. $\int_3^{\infty} \frac{7}{x+\ln(x)}\,dx$ 28. $\int_3^{\infty} \frac{1}{x^2-1}\,dx$ 29. $\int_7^{\infty} \frac{1+\cos(x)}{x^2}\,dx$

30. The volume obtained when the area between the x–axis for $x \ge 1$ and the graph of $f(x) = \frac{\sin(x)}{x}$ (Fig. 12) is revolved about the x–axis.

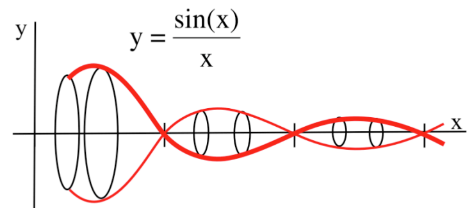

Fig. 12

31. (a) The volume obtained when the area between the positive x–axis ($x \ge 0$) and the graph of $f(x) = \frac{1}{x^2 + 1}$ (Fig. 13a) is revolved about the x–axis.

(b) The volume obtained when the area between the positive x–axis ($x \ge 0$) and the graph of $f(x) = \frac{1}{x^2 + 1}$ (Fig. 13b) is revolved about the **y–axis**. (Use the method of "tubes" from section 5.5.)

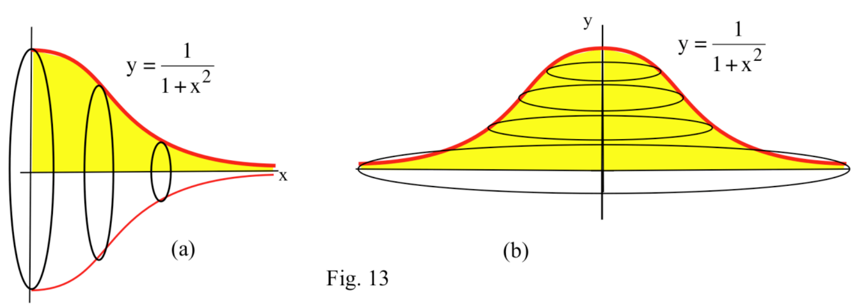

Fig. 13

32. (a) The volume obtained when the area between the positive x–axis ($x \ge 0$) and the graph of $f(x) = \frac{1}{e^x}$ is revolved about the x–axis.

(b) The volume obtained when the area between the positive x–axis ($x \ge 0$) and the graph of $f(x) = \frac{1}{e^x}$ is revolved about the **y–axis**. (Use the method of "tubes" from section 5.5.

33. (a) Use Fig. 14a to help determine which is larger: $\int_1^A \frac{1}{x}\,dx$ or $\sum_{k=1}^{A-1} \frac{1}{k}$.

(b) Use Fig. 14b to help determine which is larger: $\int_1^A \frac{1}{x}\,dx$ or $\sum_{k=2}^{A} \frac{1}{k}$.

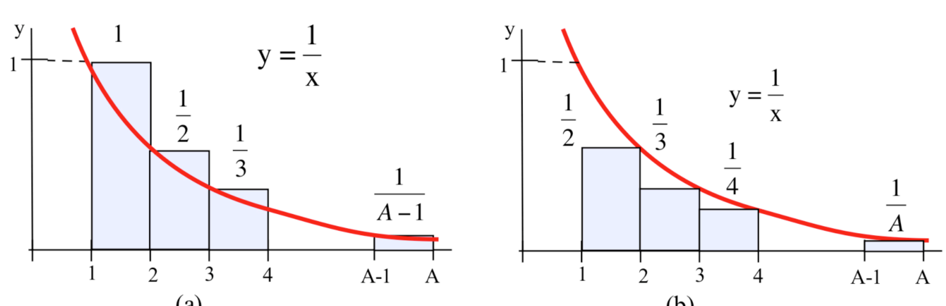

Fig. 14

34. (a) Use Fig. 15a to help determine which is larger: $\int_1^A \frac{1}{x^2}\,dx$ or $\sum_{k=1}^{A-1} \frac{1}{k^2}$.

(b) Use Fig. 15b to help determine which is larger: $\int_1^A \frac{1}{x^2}\,dx$ or $\sum_{k=2}^{A} \frac{1}{k^2}$.

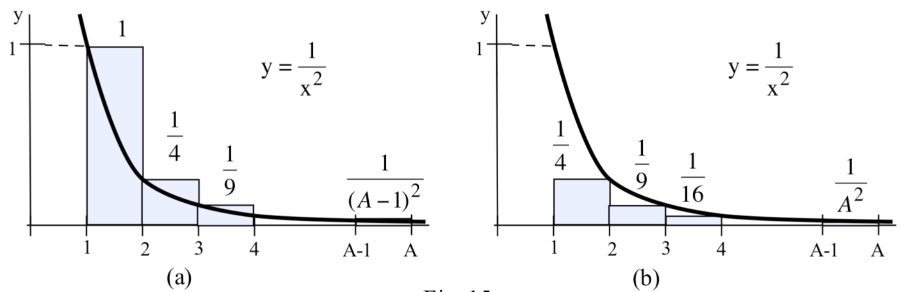

Fig. 15

Section 8.1 Practice Answers

Practice 1: (a) $V = \int_4^\infty \pi \left(\frac{1}{x}\right)^2 dx$ (b) $A = \int_1^3 g(x) - f(x)\,dx$

Practice 2: (a) $\int_1^\infty \frac{1}{x^3}\,dx = \lim_{A\to\infty} \int_1^A \frac{1}{x^3}\,dx = \lim_{A\to\infty} \left\{ -\frac{1}{2x^2} \right\} \Big|_1^A = \lim_{A\to\infty} \frac{1}{2} - \frac{1}{2A^2} = \frac{1}{2}$.

(b) $\int_0^\infty \sin(x)\,dx = \lim_{A\to\infty} \int_1^A \sin(x)\,dx = \lim_{A\to\infty} \{ -\cos(x) \} \Big|_0^A = \lim_{A\to\infty} \{ 1 - \cos(A) \}$ DNE.

$\int_0^\infty \sin(x)\,dx$ is DIVERGENT (or DIVERGES).

Practice 3: (a) $\int_1^{10} \frac{1}{\sqrt{10-x}}\,dx = \lim_{A\to 10^-} \int_1^A \frac{1}{\sqrt{10-x}}\,dx = \lim_{A\to 10^-} \{ -2(10-x)^{1/2} \} \Big|_1^A$

$= \lim_{A\to 10^-} \{ -2(10-A)^{1/2} + 2(10-1)^{1/2} \} = \lim_{A\to 10^-} \{ 6 + 2(10-A)^{1/2} \} = 6.$

(b) $\int_0^1 \frac{1}{x}\,dx = \lim_{B\to 0^+} \int_B^1 \frac{1}{x}\,dx = \lim_{B\to 0^+} \{ \ln(x) \} \Big|_B^1 = \lim_{B\to 0^+} \{ \ln(1) - \ln(B) \} = -(-\infty) = \infty.$

$\int_0^1 \frac{1}{x}\,dx$ is DIVERGENT (or DIVERGES).