8.3 INTEGRATION BY PARTS

Integration by parts is an integration method which enables us to find antiderivatives of some new functions such as ln(x) and arctan(x) as well as antiderivatives of products of functions such as $x^2 \cdot \ln(x)$ and $e^x \cdot \sin(x)$. It is the method used to derive many of the general integral formulas in the Table of Integrals. The Integration By Parts Formula for integrals comes from the Product Rule for derivatives.

For functions $u = u(x)$ and $v = v(x)$, the Product Rule for derivatives is

$$\frac{d(\,uv\,)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \quad \text{or, in the form using differentials, } d(\,uv\,) = u\,dv + v\,du\ .$$

Algebraically solving for $u\,dv$, we have $u\,dv = d(\,uv\,) - v\,du$ which can then be integrated to give

$$\int u\,dv = \int d(\,uv\,) - \int v\,du = uv - \int v\,du\ .$$

This last formula is called the Integration By Parts Formula, and it enables us to find antiderivatives for many functions which we have not been able to integrate using the substitution method. In practice, the Integration By Parts Formula allows us to exchange the problem of finding one integral, $\int u\,dv$, for the problem of finding a different integral, $\int v\,du$. This trade of one integral for another may not look very useful, but we can often arrange the exchange so we trade a difficult integral for a much easier one.

INTEGRATION BY PARTS FORMULA

If u, v, u', and v' are continuous functions,

then $\int u\,dv = u \cdot v - \int v\,du\ .$

For definite integrals, the Integration By Parts Formula is

$$\int_a^b u\,dv = u \cdot v\Big|_a^b - \int_a^b v\,du = \{\,u(b)\cdot v(b) - u(a)\cdot v(a)\,\} - \int_a^b v\,du\ .$$

Example 1: Use Integration By Parts to evaluate $\int x \cdot \cos(x)\,dx$

and $\int_0^{\pi} x \cdot \cos(x)\,dx$. (Fig. 1)

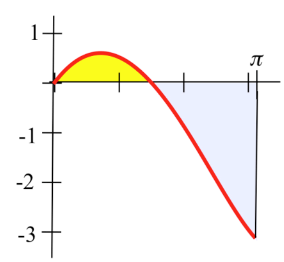


Fig. 1: $y = x \cdot \cos(x)$

Solution: Our first step is to write this integral in the form of the

Integration By Parts Formula, $\int \mathbf{u}\, dv$. If we put $\mathbf{u} = x$, then we **must** have $dv = \cos(x)\, dx$ so that $u\, dv$ completely represents the integrand $x \cdot \cos(x)$. In order to use integration by parts, we also need to calculate du and v:

Since $\mathbf{u} = x$ and $dv = \cos(x)\, dx$
then $du = dx$ and $v = \sin(x)$.

Putting these pieces into the Integration By Parts Formula, we have

$$\int x \cdot \cos(x)\, dx = x \cdot \sin(x) - \int \sin(x)\, dx = x \cdot \sin(x) + \cos(x) + C .$$

(To check this result, differentiate $x \cdot \sin(x) + \cos(x)$ to verify that its derivative is $x \cdot \cos(x)$.)

$$\int_0^{\pi} x \cdot \cos(x)\, dx = x \cdot \sin(x) + \cos(x) \Big|_0^{\pi} = \{\pi \cdot \sin(\pi) + \cos(\pi)\} - \{0 \cdot \sin(0) + \cos(0)\} = -1 - 1 = -2 .$$

The Integration By Parts formula allowed us to exchange the problem of evaluating $\int x \cdot \cos(x)\, dx$ for the much easier problem of evaluating $\int \sin(x)\, dx$.

Practice 1: Use the Integration By Parts Formula on $\int x \cdot \cos(x)\, dx$ with the choice $u = \cos(x)$ and $dv = x\, dx$. Why does this lead to a poor exchange?

Example 2: Use integration by parts to evaluate $\int x \cdot e^{3x}\, dx$ and $\int_0^1 x \cdot e^{3x}\, dx$. (Let $u = x$.)

Solution: Let $u = x$. Then $dv = e^{3x}\, dx$

so $du = dx$ and $v = \frac{1}{3} e^{3x}$.

Using the Integration By Parts Formula, we get

$$\int x \cdot e^{3x}\, dx = x \cdot \frac{1}{3} \cdot e^{3x} - \int \frac{1}{3} \cdot e^{3x}\, dx = \frac{x}{3} \cdot e^{3x} - \frac{1}{9} \cdot e^{3x} + C .$$

$$\int_0^1 x \cdot e^{3x}\, dx = \frac{x}{3} \cdot e^{3x} - \frac{1}{9} \cdot e^{3x} \Big|_0^1 = \left\{ \frac{1}{3} \cdot e^3 - \frac{1}{9} \cdot e^3 \right\} - \left\{ 0 - \frac{1}{9} \right\} = \frac{2}{9} \cdot e^3 + \frac{1}{9} .$$

In this Example, it is valid to choose $u = e^{3x}$ and $dv = x\, dx$, but that choice results in an integral that is more difficult than the original one. If we put $u = e^{3x}$ and $dv = x\, dx$, then $du = 3e^{3x}\, dx$ and $v = \frac{x^2}{2}$, and the Integration By Parts Formula gives

$$\int x \cdot e^{3x}\, dx = e^{3x} \cdot \frac{x^2}{2} - \int \frac{x^2}{2} 3e^{3x}\, dx .$$

We end up exchanging the integral $\int x \cdot e^{3x}\, dx$ for the more difficult integral $\int \frac{x^2}{2} 3e^{3x}\, dx$.

Practice 2: Evaluate $\int x \cdot \sin(x)\, dx$ and $\int x \cdot e^{5x}\, dx$. (In each integral, let $u = x$.)

Once we have chosen u and dv to represent the integrand as u dv , we need to calculate du and v. The "du" calculation is usually easy, but finding v from dv can be difficult for some choices of dv. In practice, you need to select u so dv is a simple enough part of the integrand so you can find v, the antiderivative of dv.

Example 3: Evaluate $\int 2x \cdot \ln(x)\, dx$.

Solution: The choice $u = 2x$ seems fine until we go a little further with the process. If $u = 2x$, then $dv = \ln(x)\, dx$ and we need to find du and v. Finding $du = 2\, dx$ is simple, but then we have the difficult problem of finding an antiderivative v for our choice $dv = \ln(x)\, dx$.

In this Example, the choice $u = \ln(x)$ results in easier calculations.

Let $u = \ln(x)$. Then $dv = 2x\, dx$
so $du = \frac{1}{x}\, dx$ and $v = x^2$.

Then the Integration By Parts Formula gives

$$\int 2x \cdot \ln(x)\, dx = \ln(x)\, x^2 - \int x^2 \cdot \frac{1}{x}\, dx$$

$$= x^2 \cdot \ln(x) - \int x\, dx = x^2 \cdot \ln(x) - \frac{x^2}{2} + C .$$

If you can not find a v for your original choice of dv, try a different choice for u and dv.

Integration by parts also enables us to evaluate the integrals of the inverse trigonometric functions and of the logarithm.

Example 4: Evaluate $\int \arctan(x)\, dx$.

Solution: Let $u = \arctan(x)$. Then $dv = dx$

so $du = \frac{1}{1 + x^2}\, dx$ and $v = x$.

Then $\int \arctan(x)\,dx = x\arctan(x) - \int x\cdot\frac{1}{1+x^2}\,dx$. We can evaluate the new integral

$\int x\cdot\frac{1}{1+x^2}\,dx$ by changing the variable using $w = 1 + x^2$. Then $dw = 2x\,dx$, so

$\int x\cdot\frac{1}{1+x^2}\,dx = \int \frac{1}{2}\frac{1}{w}\,dw = \frac{1}{2}\ln|w| = \frac{1}{2}\ln|1+x^2|$. Putting this all together,

$$\int \arctan(x)\,dx = x\arctan(x) - \frac{1}{2}\ln|1+x^2| + C .$$

Practice 3: Evaluate $\int \ln(x)\,dx$ and $\int_1^e \ln(x)\,dx$.

Notes: 1. Once u is chosen, then dv is completely determined: dv = rest of the integrand.

2. Since we need to find an antiderivative of dv to get v, pick u and dv so an antiderivative v can be found for the chosen dv.

3. The Integration By Parts Formula allows us to trade one integral for another one.

(a) If the new integral is more difficult than the original integral, then we have made a poor choice of u and dv. Try a different choice for u and dv or try a different technique.

(b) To evaluate the new integral $\int v\,du$ we may need to use substitution, integration by parts again, or some other technique such as the ones discussed later in this chapter.

More General Uses of Integration By Parts

The Integration By Parts Formula is also used to derive many of the entries in the Table of Integrals. For some integrands such as $x^n\cdot\ln(x)$, the result is simply a function, an antiderivative of the integrand. For some integrands such as $\sin^n(x)$, the result is a **reduction formula**, a formula which still contains an integral, but the new integrand is the sine function raised to a smaller power, $\sin^{n-2}(x)$. By repeatedly applying the reduction formula, we can evaluate the integral of sine raised to any positive integer power.

General Patterns

Example 5: Evaluate $\int x^n\cdot\ln(x)\,dx$ for $n \neq -1$.

Solution: Let $u = \ln(x)$. Then $dv = x^n\,dx$

so $du = \frac{1}{x}\,dx$ and $v = \frac{1}{n+1}x^{n+1}$.

Then $\int x^n \ln(x)\, dx = \frac{1}{n+1} \cdot x^{n+1} \cdot \ln(x) - \int \frac{1}{n+1} \cdot x^{n+1} \cdot \frac{1}{x}\, dx$.

But $\int \frac{1}{n+1} \cdot x^{n+1} \cdot \frac{1}{x}\, dx = \frac{1}{n+1} \int x^n\, dx = \frac{1}{n+1} \cdot \frac{1}{n+1} \cdot x^{n+1} = \frac{x^{n+1}}{(n+1)^2}$, so

$$\int x^n \cdot \ln(x)\, dx = \frac{x^{n+1}}{n+1} \cdot \ln(x) - \frac{x^{n+1}}{(n+1)^2} + C = \frac{x^{n+1}}{n+1} \left\{ \ln(x) - \frac{1}{n+1} \right\} + C \text{ for } n \neq -1 .$$

Practice 4: Use the **result** of Example 5 to evaluate $\int x^2 \cdot \ln(x)\, dx$ and $\int \ln(x)\, dx$.

Reduction Formulas

Sometimes the general pattern still contains an integral, but a simpler one with a smaller exponent. In that case we can reuse the reduction pattern until the resulting integral is simple enough to integrate completely.

Example 6: Evaluate $\int x^n e^x\, dx$ and use the result to evaluate $\int x^2 e^x\, dx$.

Solution: Put $u = x^n$. Then $dv = e^x\, dx$,
so $du = n\, x^{n-1}\, dx$ and $v = e^x$.

The Integration By Parts Formula gives $\int x^n e^x\, dx = x^n e^x - n \int x^{n-1} e^x\, dx$, a reduction formula since we have reduced the power of x by 1 and have succeeded in trading the integral $\int x^n e^x\, dx$ for the "reduced" integral $\int x^{n-1} e^x\, dx$.

$\int x^2 e^x\, dx$ has the form of the general pattern $\int x^n e^x\, dx$ with $n = 2$, so

$$\int x^2 e^x\, dx = x^2 e^x - 2 \int x^1 e^x\, dx .$$

Using the pattern on $\int x^1 e^x\, dx$ with $n = 1$, we have

$$\begin{aligned} \int x^2 e^x\, dx &= x^2 \cdot e^x - 2 \int x^1 e^x\, dx \\ &= x^2 \cdot e^x - 2\left\{ x \cdot e^x - \int e^x\, dx \right\} \\ &= x^2 \cdot e^x - 2x \cdot e^x + 2e^x + C \text{ or } e^x \left\{ x^2 - 2x + 2 \right\} + C . \end{aligned}$$

Practice 5: Derive the reduction formula $\int x^n \cdot \sin(x)\, dx = -x^n \cdot \cos(x) + n \int x^{n-1} \cdot \cos(x)\, dx$.

The Reappearing Integral

Sometimes the integral we are trying to evaluate shows up on both sides of the equation during our calculations in such a way that we can solve for the integral algebraically.

Example 7: Evaluate $\int e^x \cos(x)\,dx$.

Solution: Let $u = e^x$. Then $dv = \cos(x)\,dx$
so $du = e^x\,dx$ and $v = \sin(x)$.

Then $\int e^x \cos(x)\,dx = e^x \sin(x) - \int e^x \sin(x)\,dx$. The new integral does not look any easier than the original one, but lets try to evaluate the new integral using integration by parts again.

To evaluate $\int e^x \sin(x)\,dx.$, let $u = e^x$ and $dv = \sin(x)\,dx$. Then $du = e^x\,dx$ and $v = -\cos(x)$ so

$$\int e^x \sin(x)\,dx = -e^x \cos(x) + \int e^x \cos(x)\,dx.$$

Putting this result back into the original problem, we get

$$\int \mathbf{e^x \cos(x)\,dx} = e^x \sin(x) - \int e^x \sin(x)\,dx = e^x \sin(x) - \left\{-e^x \cos(x) + \int e^x \cos(x)\,dx\right\}$$

$$= e^x \sin(x) + e^x \cos(x) - \int \mathbf{e^x \cos(x)\,dx}\ .$$

The integral of $e^x \cos(x)$ appears on each side of this last equation, and we can algebraically solve for it to get

$$2\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x)\ \text{, and finally,}$$

$$\int e^x \cos(x)\,dx = \frac{1}{2}\left\{e^x \sin(x) + e^x \cos(x)\right\} + C\ .$$

Practice 6: Derive the formula $\int e^x \sin(x)\,dx = \frac{1}{2}\left\{e^x \sin(x) - e^x \cos(x)\right\} + C$.

PROBLEMS

In problems 1–6, a function u or dv is given. Find the piece u or dv which is not given, calculate du and v, and apply the Integration by Parts Formula.

1. $\int 12x\cdot\ln(x)\,dx$ $u = \ln(x)$
2. $\int x\cdot e^{-x}\,dx$ $u = x$
3. $\int x^4\ln(x)\,dx$ $dv = x^4\,dx$
4. $\int x\cdot\sec^2(3x)\,dx$ $dv = \sec^2(3x)\,dx$
5. $\int x\cdot\arctan(x)\,dx$ $dv = x\,dx$
6. $\int x\cdot(5x+1)^{19}\,dx$ $u = x$

In problems 7– 24 , evaluate the integrals.

7. $\int_0^1 \frac{x}{e^{3x}}\,dx$
8. $\int_0^1 10x\cdot e^{3x}\,dx$
9. $\int x\cdot\sec(x)\cdot\tan(x)\,dx$
10. $\int_0^{\pi} 5x\cdot\sin(2x)\,dx$
11. $\int_{\pi/3}^{\pi/2} 7x\cdot\cos(3x)\,dx$
12. $\int 6x\cdot\sin(x^2+1)\,dx$
13. $\int 12x\cdot\cos(3x^2)\,dx$
14. $\int x^2\cos(x)\,dx$
15. $\int_1^3 \ln(2x+5)\,dx$
16. $\int x^3\ln(5x)\,dx$
17. $\int_1^e (\ln(x))^2\,dx$
18. $\int_1^e \sqrt{x}\cdot\ln(x)\,dx$
19. $\int \arcsin(x)\,dx$
20. $\int x^2e^{5x}\,dx$
21. $\int x\cdot\arctan(3x)\,dx$
22. $\int x\ln(x+1)\,dx$
23. $\int_1^2 \frac{\ln(x)}{x}\,dx$
24. $\int_1^2 \frac{\ln(x)}{x^2}\,dx$

These reduction formulas can all be derived using integration by parts. In problems 25–30 , use them to help evaluate the integrals. (These are entries 19, 20 and 23 in the Table of Integrals with a = 1.)

$$\int \sin^n(x)\,dx = \frac{1}{n}\left\{ -\sin^{n-1}(x)\cdot\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx \right\} + C$$

$$\int \cos^n(x)\,dx = \frac{1}{n}\left\{ \cos^{n-1}(x)\cdot\sin(x) + (n-1)\int \cos^{n-2}(x)\,dx \right\} + C$$

$$\int \sec^n(x)\,dx = \frac{1}{n-1}\left\{ \sec^{n-2}(x)\cdot\tan(x) + (n-2)\int \sec^{n-2}(x)\,dx \right\} + C$$

25. (a) $\int \sin^3(x)\,dx$ (b) $\int \sin^4(x)\,dx$ (c) $\int \sin^5(x)\,dx$

26. (a) $\int \cos^3(x)\,dx$ (b) $\int \cos^4(x)\,dx$ (c) $\int \cos^5(x)\,dx$

27. (a) $\int \sec^3(x)\,dx$ (b) $\int \sec^4(x)\,dx$ (c) $\int \sec^5(x)\,dx$

28. $\int \sin^3(5x-2)\,dx$ 29. $\int \cos^3(2x+3)\,dx$ 30. $\int \sec^3(7x-1)\,dx$

31. $\int x\cdot(2x+5)^{19}\,dx$ can be evaluated using integration by parts or a change of variable. (a) Evaluate the integral using integration by parts with $u = x$ and $dv = (2x+5)^{19}\,dx$. (b) Evaluate the integral using change of variable with $u = 2x + 5$. (c) Which method is easier?

32. $\int \frac{x}{\sqrt{1+x}}\,dx$ can be evaluated using integration by parts or using a change of variable. (a) Evaluate the integral using integration by parts with $u = x$ and $dv = \frac{1}{\sqrt{1+x}}\,dx$. (b) Evaluate the integral using change of variable with $u = 1 + x$.

33. (a) Before evaluating the integrals, which do you think is larger, $\int_0^1 x\cdot\sin(x)\,dx$ or $\int_0^1 \sin(x)\,dx$? Why?

(b) Evaluate $\int_0^1 x\cdot\sin(x)\,dx$ and $\int_0^1 \sin(x)\,dx$. Was your prediction in part (a) correct?

34. (a) Before evaluating the integrals, which do you think is larger, $\int_0^\pi x\cdot\sin(x)\,dx$ or $\int_0^\pi \sin(x)\,dx$? Why?

(b) Evaluate $\int_0^\pi x\cdot\sin(x)\,dx$ and $\int_0^\pi \sin(x)\,dx$. Was your prediction in part (a) correct?

35. In Fig. 2, the volume swept out when region A is revolved about the x–axis is $\int_{x=1}^{e} \pi(\ln(x))^2\,dx$ (using the disk method), and the volume swept out when region B is revolved about the x–axis is

$$\int_{y=0}^{1} 2\pi y\cdot e^y\,dy \quad \text{(using the tube method).}$$

(a) Before evaluating the integrals, which volume do you think is larger? Why?

(b) Evaluate the integrals. Was your prediction in part (a) correct?

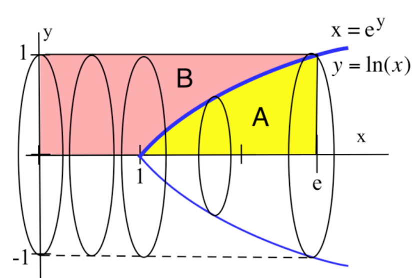


Fig. 2: Comparing volumes

36. Use the tube method to calculate the volume when the region between the x–axis and the graph of $y = \sin(x)$ for $0 \le x \le \pi$ is rotated about the y–axis.

37. We derived the Integration by Parts Formula analytically, but the formula also has a geometric interpretation. In Fig. 3, let D be the large rectangle formed by the regions A, B, and C so we have the area equation

(area of C) = (area of D) – (area of A) – (area of B).

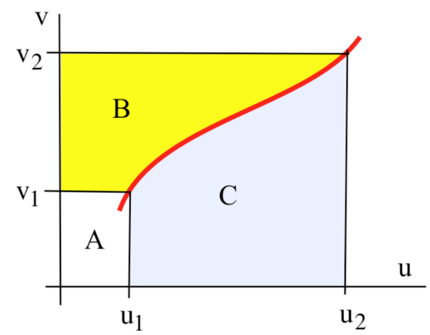
(a) Represent the area of the large rectangle D as a function of u_2 and v_2.

(b) Represent the area of the small rectangle (region A) as a function of u_1 and v_1.

(c) Represent the area of region C as an integral with respect to the variable u.

(d) Represent the area of region B as an integral with respect to the variable v.

(e) Rewrite the area equation using the representations in parts (a) – (d). This result should look very familiar.



D is the region obtained by putting together regions A, B and C

Fig. 3: Graphic version of Integration by Parts

38. $\int x \cdot (\ln(x))^2 \, dx$

39. $\int x^2 \cdot \arctan(x) \, dx$

40. $\int_0^1 e^{-x} \sin(x) \, dx$

41. $\int_0^1 \frac{\cos(x)}{e^x} \, dx$

42. $\int \sin(\ln(x)) \, dx$

43. $\int \cos(\ln(x)) \, dx$

44. $\int e^{3x} \sin(x) \, dx$

45. $\int e^x \cos(3x) \, dx$

46. Use integration by parts to evaluate $\int \sec^3(x) \, dx$.

47. Derive a reduction formula for $\int x^n e^{ax} \, dx$.

48. Derive a reduction formula for $\int x^n \sin(ax) \, dx$.

49. Derive a reduction formula for $\int x \cdot (\ln(x))^n \, dx$.

50. Suppose f and f ' are continuous and bounded on the interval $[0, 2\pi]$ ($|f(x)| < M$ and $|f'(x)| < M$ for all $0 \le x \le 2\pi$). The n^{th} Fourier Sine Coefficient of f is defined as the value of

$$S_n = \int_0^{2\pi} f(x) \cdot \sin(nx)\, dx .$$

(a) Use the Integration by Parts Formula with $u = f(x)$ and $dv = \sin(nx)\,dx$ to represent the formula for S_n in a different way.

(b) Use the new representation of S_n in part (a) to determine what happens to the values of S_n when n is very large ($n \to \infty$). (Hint: $| f'(x) \cdot \cos(nx) | \le | f'(x)| \cdot |\cos(nx)| < M \cdot 1 = M$.)

(c) What happens to the values of the n^{th} Fourier Cosine Coefficient $C_n = \int_0^{2\pi} f(x) \cdot \cos(nx)\, dx$ when n is very large.

Section 8.3 **PRACTICE Answers**

Practice 1: $\int_a^b x \cdot \cos(x)\, dx$. Put $u = \cos(x)$ and $dv = x\, dx$. Then $du = -\sin(x)\, dx$ and $v = \frac{1}{2} x^2$.

$$uv - \int v\, du = (\cos(x))(\tfrac{1}{2} x^2) - \int (\tfrac{1}{2} x^2)(-\sin(x))\, dx$$

$$= \frac{1}{2} x^2 \cos(x) + \frac{1}{2} \int x^2 \sin(x)\, dx .$$

The last integral is worse than the original integral.

Practice 2: (a) $\int_a^b x \cdot \sin(x)\, dx$. Put $u = x$ and $dv = \sin(x)\, dx$. Then $du = dx$ and $v = -\cos(x)\, dx$.

$$uv - \int v\, du = (x)(-\cos(x)) - \int -\cos(x)\, dx = \mathbf{-x \cdot \cos(x) + \sin(x) + C} .$$

(b) $\int_a^b x \cdot e^{5x}\, dx$. Put $u = x$ and $dv = e^{5x}\, dx$. Then $du = dx$ and $v = \frac{1}{5} e^{5x}$.

$$uv - \int v\, du = (x)(\tfrac{1}{5} e^{5x}) - \int \tfrac{1}{5} e^{5x}\, dx = \mathbf{\tfrac{1}{5} x \cdot e^{5x} - \tfrac{1}{25} e^{5x} + C} .$$

Practice 3: $\int \ln(x)\, dx$. Put $u = \ln(x)$ and $dv = dx$. Then $du = \frac{1}{x}\, dx$ and $v = x$.

$$uv - \int v\, du = \ln(x) \cdot x - \int x \cdot \frac{1}{x}\, dx = x \cdot \ln(x) - \int 1\, dx = \mathbf{x \cdot \ln(x) - x + C} .$$

$$\int_1^e \ln(x)\, dx = x \cdot \ln(x) - x \Big|_1^e = \{ e \cdot \ln(e) - e \} - \{ 1 \cdot \ln(1) - 1 \} = \{ e - e \} - \{ 0 - 1 \} = 1 .$$

Practice 4: Example 5: $\int x^n \ln(x)\,dx = \frac{x^{n+1}}{n+1}\left\{ \ln(x) - \frac{1}{n+1} \right\} + C$.

(a) $n = 2$: $\int x^2 \ln(x)\,dx = \frac{x^3}{3}\left\{ \ln(x) - \frac{1}{3} \right\} + C$.

(b) $n = 0$: $\int \ln(x)\,dx = \frac{x^1}{1}\left\{ \ln(x) - \frac{1}{1} \right\} + C = x\cdot\ln(x) - x + C$.

Practice 5: $\int x^n \sin(x)\,dx$. Put $u = x^n$ and $dv = \sin(x)\,dx$. Then $du = nx^{n-1}\,dx$ and $v = -\cos(x)$.

$$uv - \int v\,du = (x^n)(-\cos(x)) - \int -\cos(x)\,nx^{n-1}\,dx$$

$$= -x^n\cos(x) + n\int x^{n-1}\cos(x)\,dx\ .$$

Practice 6: (Similar to Example 7)

$\int e^x \sin(x)\,dx$. Put $u = e^x$ and $dv = \sin(x)\,dx$. Then $du = e^x\,dx$ and $v = -\cos(x)\,dx$.

$$uv - \int v\,du = (e^x)(-\cos(x)) - \int -\cos(x)\,e^x\,dx$$

$$= -e^x\cos(x) + \int e^x\cos(x)\,dx\ .$$

(For the last integral, put $u = e^x$, $dv = \cos(x)\,dx$, $du = e^x\,dx$, $v = \sin(x)\,dx$)

$$= -e^x\cos(x) + \int e^x\cos(x)\,dx = -e^x\cos(x) + \left\{ e^x\sin(x) - \int e^x\sin(x)\,dx \right\}$$

So $2\int e^x\sin(x)\,dx = -e^x\cos(x) + e^x\sin(x)$

and $\int e^x\sin(x)\,dx = \boldsymbol{\frac{1}{2}\left\{ -e^x\cos(x) + e^x\sin(x) \right\} + C}$.