8.4 PARTIAL FRACTION DECOMPOSITION

Rational functions (polynomials divided by polynomials) and their integrals are important in mathematics and applications, but if you look through a table of integral formulas, you will find very few formulas for their integrals. Partly that is because the general formulas are rather complicated and have many special cases, and partly it is because they can all be reduced to just a few cases using the algebraic technique discussed in this section, Partial Fraction Decomposition.

In algebra you learned to add rational functions to get a single rational function. Partial Fraction Decomposition is a technique for reversing that procedure to "decompose" a single rational function into a **sum** of simpler rational functions. Then the integral of the single rational function can be evaluated as the sum of the integrals of the simpler functions.

Example 1: Use the algebraic decomposition $\dfrac{17x-35}{2x^2-5x} = \dfrac{7}{x} + \dfrac{3}{2x-5}$ to evaluate $\displaystyle\int \frac{17x-35}{2x^2-x}\,dx$.

Solution: The decomposition allows us to exchange the original integral for two much easier ones:

$$\int \frac{17x-35}{2x^2-5x}\,dx = \int \frac{7}{x} + \frac{3}{2x-5}\,dx = \int \frac{7}{x}\,dx + \int \frac{3}{2x-5}\,dx$$

$$= 7\ln|\,x\,| + \frac{3}{2}\ln|\,2x-5\,| + C\,.$$

Practice 1: Use the algebraic decomposition $\dfrac{7x-11}{3x^2-8x-3} = \dfrac{4}{3x+1} + \dfrac{1}{x-3}$ to evaluate $\displaystyle\int \frac{7x-11}{3x^2-8x-3}\,dx$.

The Example illustrates how to use a "decomposed" fraction with integrals, but it does not show how to achieve the decomposition. The algebraic basis for the Partial Fraction Decomposition technique is that every polynomial can be factored into a product of linear factors $ax + b$ and irreducible quadratic factors $ax^2 + bx + c$ (with $b^2 - 4ac < 0$). These factors may not be easy to find , and they will typically be more complicated than the examples in this section, but every polynomial has such factors. Before we apply the Partial Fraction Decomposition technique, the fraction must have the following form:

(i) (the degree of the numerator) **<** (degree of the denominator)

(ii) The denominator has been factored into a product of linear factors and irreducible quadratic factors.

If assumption (i) is not true, we can use polynomial division until we get a remainder which has a smaller degree than the denominator. If assumption (ii) is not true, we simply cannot use the Partial Fraction Decomposition technique.

Example 2: Put each fraction into a form for Partial Fraction Decomposition:

(a) $\dfrac{2x^2 + 4x - 6}{x^2 - 2x}$ (b) $\dfrac{3x^3 - 3x^2 - 9x + 8}{x^2 - x - 6}$ (c) $\dfrac{7x^2 + 12x - 12}{x^3 - 4x}$

Solution: (a) $\dfrac{2x^2 + 4x - 6}{x^2 - 2x} = 2 + \dfrac{8x - 6}{x^2 - 2x} = 2 + \dfrac{8x - 6}{x(x - 2)}$

(b) $\dfrac{3x^3 - 3x^2 - 9x + 8}{x^2 - x - 6} = 3x + \dfrac{9x + 8}{x^2 - x - 6} = 3x + \dfrac{9x + 8}{(x + 2)(x - 3)}$

(c) $\dfrac{7x^2 + 12x - 12}{x^3 - 4x} = \dfrac{7x^2 + 12x - 12}{x(x + 2)(x - 2)}$

Distinct Linear Factors

If the denominator can be factored into a product of distinct linear factors, then the original fraction can be written as the **sum** of fractions of the form $\dfrac{\text{number}}{\text{linear factor}}$. Our job is to find the values of the numbers in the numerators, and that typically requires solving a system of equations.

Example 3: Find values for A and B so $\dfrac{17x - 35}{x(2x - 5)} = \dfrac{A}{x} + \dfrac{B}{2x - 5}$.

Solution: We can combine the two terms on the right by putting them over the common denominator x(2x – 5). Multiplying the $\dfrac{A}{x}$ term by $\dfrac{2x - 5}{2x - 5}$ and multiplying the $\dfrac{B}{2x-5}$ term by $\dfrac{x}{x}$, we have

$$\frac{A}{x} \cdot \frac{\mathbf{2x - 5}}{\mathbf{2x - 5}} + \frac{B}{2x - 5} \cdot \frac{\mathbf{x}}{\mathbf{x}} = \frac{A2x - 5A + Bx}{x(2x - 5)} = \frac{(2A + B)x - 5A}{x(2x - 5)} .$$

Since $\dfrac{(2A + B)x - 5A}{x(2x - 5)} = \dfrac{17x - 35}{x(2x - 5)}$, the coefficients of like terms in the numerators must be equal:

coefficients of x:	$2A + B$	$= 17$
constant terms:	$-5A$	$= -35$

Solving this system of two equations with two unknowns, we get $A = 7$ and $B = 3$ so

$$\frac{17x - 35}{x(2x - 5)} = \frac{7}{x} + \frac{3}{2x - 5} .$$

As a check, add $\frac{7}{x}$ and $\frac{3}{2x-5}$ and verify that the sum is $\frac{17x-35}{x(2x-5)}$.

Practice 2: Find values of A and B so $\frac{6x-7}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$.

In general, there is one unknown coefficient for each distinct linear factor of the denominator. However, if the number of distinct linear factors is large, we would need to solve a large system of equations for the unknowns.

Example 4: Find values for A, B, and C so $\frac{2x^2+7x+9}{x(x+1)(x+3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+3}$.

Solution: $\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+3} = \frac{A}{x}\frac{\mathbf{(x+1)(x+3)}}{\mathbf{(x+1)(x+3)}} + \frac{B}{(x+1)}\frac{\mathbf{x(x+3)}}{\mathbf{x(x+3)}} + \frac{C}{(x+3)}\frac{\mathbf{x(x+1)}}{\mathbf{x(x+1)}}$

$$= \frac{A(x+1)(x+3) + Bx(x+3) + Cx(x+1)}{x(x+1)(x+3)}$$

$$= \frac{(A+B+C)x^2 + (4A+3B+C)x + (3A)}{x(x+1)(x+3)} = \frac{2x^2+7x+9}{x(x+1)(x+3)} .$$

The coefficients of the like terms in the numerators must be equal:

coefficients of x^2: $A + B + C = 2$

coefficients of x: $4A + 3B + C = 7$

constant terms: $3A = 9$ so $A = 3, B = -2$, and $C = 1$.

Finally, $\frac{2x^2+7x+9}{x(x+1)(x+3)} = \frac{3}{x} + \frac{-2}{x+1} + \frac{1}{x+3}$.

Practice 3: Use the result of Example 4 to evaluate $\int \frac{2x^2+7x+9}{x(x+1)(x+3)}\, dx$.

Practice 4: How large would the system be for a Partial Fraction Decomposition of $\frac{\text{something}}{5^{\text{th}}\text{ degree polynomial}}$?

The next two subsections describe how to decompose fractions whose denominators contain irreducible quadratic factors and repeated factors. We will not discuss why the suggestions work except to note that they provide enough, but not too many, unknown coefficients for the decomposition.

Distinct Irreducible Quadratic Factors

If the factored denominator includes a distinct irreducible quadratic factor, then the Partial Fraction Decomposition **sum** contains a fraction of the form of a linear polynomial with unknown coefficients divided by the irreducible quadratic factor:

$$\frac{\text{linear polynomial}}{\text{irreducible quadratic factor}} \quad \text{or} \quad \frac{Ax + B}{\text{irreducible quadratic factor}} .$$

Once again we will solve a system of equations to find the values of the unknown coefficients A and B.

Example 5: Find values for A, B, and C so $\dfrac{x^2 + 3x - 15}{(x^2 + 2x + 5)x} = \dfrac{Ax + B}{x^2 + 2x + 5} + \dfrac{C}{x}$.

Solution: $$\frac{Ax + B}{x^2 + 2x + 5} + \frac{C}{x} = \frac{Ax + B}{x^2 + 2x + 5}\left(\frac{\mathbf{x}}{\mathbf{x}}\right) + \frac{C}{x}\left(\frac{\mathbf{x^2 + 2x + 5}}{\mathbf{x^2 + 2x + 5}}\right)$$

$$= \frac{Ax^2 + Bx + Cx^2 + 2Cx + 5C}{x(x^2 + 2x + 5)}$$

$$= \frac{(A + C)x^2 + (B + 2C)x + 5C}{x(x^2 + 2x + 5)} = \frac{x^2 + 3x - 15}{(x^2 + 2x + 5)x} .$$

Then $A + C = 1$, $B + 2C = 3$, and $5C = -15$ so $C = -3$, $B = 9$, and $A = 4$.

In general, there are 2 unknown coefficients for each distinct irreducible quadratic factor of the denominator. We would start the decomposition of

$$\frac{6x^3 + 36x^2 + 50x + 53}{(x^2 + 4)(x^2 + 4x + 5)} \quad \text{by writing it as the sum} \quad \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{x^2 + 4x + 5} .$$

We would finish this decomposition by solving the system of 4 equations with 4 unknowns, $A + C = 6$, $4A + B + D = 36$, $5A + 4B + 4C = 50$, and $5B + 4D = 53$ to get $A = 6$, $B = 5$, $C = 0$, and $D = 7$.

Repeated Factors

If the factored denominator contains a linear factor raised to a power (greater than one), then we need to start the decomposition with several terms. There should be one term with one unknown coefficient for each power of the linear factor. For example,

$$\frac{\text{something}}{(x + 1)(x - 2)^3} = \frac{A}{x + 1} + \frac{B}{x - 2} + \frac{C}{(x - 2)^2} + \frac{D}{(x - 2)^3} .$$

Similarly, if the factored denominator contains an irreducible quadratic factor raised to a power greater than one), then we need to start the decomposition with several terms. There should be one term with two unknown coefficients for each power of the irreducible quadratic. For example,

$$\frac{\text{something}}{x^2(x^2+9)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+9} + \frac{Ex+F}{(x^2+9)^2} + \frac{Gx+H}{(x^2+9)^3} .$$

This leads to a system of 8 equations with 8 unknowns.

Example 6: Decompose $\dfrac{-4x^2+5x+3}{x(x-1)^2}$ and evaluate $\displaystyle\int \frac{-4x^2+5x+3}{x(x-1)^2}\,dx$.

Solution:
$$\frac{-4x^2+5x+3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$= \frac{A}{x}\left(\frac{\mathbf{(x-1)^2}}{\mathbf{(x-1)^2}}\right) + \frac{B}{x-1}\left(\frac{\mathbf{x(x-1)}}{\mathbf{x(x-1)}}\right) + \frac{C}{(x-1)^2}\left(\frac{\mathbf{x}}{\mathbf{x}}\right)$$

$$= \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2}$$

$$= \frac{(A+B)x^2 + (-2A-B+C)x + A}{x(x-1)^2} = \frac{-4x^2+5x+3}{x(x-1)^2} .$$

Then $A + B = -4$, $-2A - B + C = 5$, and $A = 3$ so $A = 3, B = -7$, and $C = 4$. Finally,

$$\int \frac{-4x^2+5x+3}{x(x-1)^2}\,dx = \int \frac{3}{x} + \frac{-7}{x-1} + \frac{4}{(x-1)^2}\,dx = 3\ln|\,x\,| - 7\ln|\,x-1\,| + \frac{-4}{x-1} + C.$$

Practice 5: Decompose $\dfrac{2x^2+27x+85}{(x+5)^2}$ and evaluate $\displaystyle\int \frac{2x^2+27x+85}{(x+5)^2}\,dx$.

The primary use of the partial fraction technique in this course is to put rational functions in a form that is easier to integrate, but this algebraic technique can also be used to simplify the differentiation of some rational functions. The next example illustrates the use of partial fractions to make a differentiation problem easier.

Example 7: For $f(x) = \dfrac{2x+13}{x^2+x-2}$, calculate $f\,'(x)$, $f\,''(x)$, and $f\,'''(x)$.

Solution: You already know how to calculate these derivatives using the quotient rule, but that process is rather tedious for the second and third derivatives here. Instead, we can use the partial fraction technique to rewrite f as $f(x) = \frac{5}{x-1} - \frac{3}{x+2} = 5(x-1)^{-1} - 3(x+2)^{-1}$. Then the derivatives are very straightforward:

$f'(x) = -5(x-1)^{-2} + 3(x+2)^{-2}$,
$f''(x) = 10(x-1)^{-3} - 6(x+2)^{-3}$, and
$f'''(x) = -30(x-1)^{-4} + 18(x+2)^{-4}$.

Practice 6: Use the partial fraction decomposition of $g(x) = \frac{9x+1}{x^2-2x-3}$ to calculate $g'(x)$, $g''(x)$, and $g^{(4)}(x)$.

PROBLEMS

In problems 1 – 12, decompose the fractions.

1. $\frac{7x+2}{x(x+1)}$
2. $\frac{7x+9}{(x+3)(x-1)}$
3. $\frac{11x+25}{x^2+9x+8}$
4. $\frac{3x+7}{x^2-1}$
5. $\frac{2x^2+15x+25}{x^2+5x}$
6. $\frac{3x^3+3x^2}{x^2+x-2}$
7. $\frac{6x^2+9x-15}{x(x+5)(x-1)}$
8. $\frac{6x^2-x-1}{x^3-x}$
9. $\frac{8x^2-x+3}{x^3+x}$
10. $\frac{9x^2+13x+15}{x^3+2x^2-3x}$
11. $\frac{11x^2+23x+6}{x^2(x+2)}$
12. $\frac{6x^2+14x-9}{x(x+3)^2}$

In problems 13 – 30, evaluate the integrals.

13. $\int \frac{3x+13}{(x+2)(x-5)}\,dx$
14. $\int \frac{2x+11}{(x-7)(x-2)}\,dx$
15. $\int_2^5 \frac{2}{x^2-1}\,dx$
16. $\int_1^3 \frac{5x^2+5x+3}{x^3+x}\,dx$
17. Integrate the functions in problems 1 – 4.
18. Integrate the functions in problems 5 – 8.
19. $\int \frac{2x^2+5x+3}{x^2-1}\,dx$
20. $\int \frac{2x^2+19x+22}{x^2+x-12}\,dx$
21. $\int \frac{3x^2+19x+24}{x^2+6x+5}\,dx$
22. $\int \frac{7x^2+8x-2}{x^2+2x}\,dx$
23. $\int \frac{3x^2-1}{x^3-x}\,dx$
24. $\int \frac{x^4+5x^3+x-15}{x^2+5x}\,dx$

25. $\displaystyle\int \frac{x^3 + 3x^2 - 4x + 30}{x^2 + 3x - 10}\,dx$ 26. $\displaystyle\int \frac{2x + 5}{(x + 1)^2}\,dx$ 27. $\displaystyle\int \frac{12x^2 + 19x - 6}{x^3 + 3x^2}\,dx$

28. $\displaystyle\int \frac{7x^3 + x^2 + 7x + 10}{x^4 + 2x^3}\,dx$ 29. $\displaystyle\int \frac{7x^2 + 3x + 7}{x^3 + x}\,dx$ 30. $\displaystyle\int \frac{7x^2 - 4x + 4}{x^3 + 1}\,dx$

31. Integrals are very sensitive to small changes in the integrand. Evaluate

(a) $\displaystyle\int \frac{1}{x^2 + 2x + \mathbf{2}}\,dx$ (b) $\displaystyle\int \frac{1}{x^2 + 2x + \mathbf{1}}\,dx$ (c) $\displaystyle\int \frac{1}{x^2 + 2x + \mathbf{0}}\,dx$.

32. Evaluate (a) $\displaystyle\int \frac{1}{x^2 - 6x + \mathbf{8}}\,dx$ (b) $\displaystyle\int \frac{1}{x^2 - 6x + \mathbf{9}}\,dx$ (c) $\displaystyle\int \frac{1}{x^2 - 6x + \mathbf{10}}\,dx$.

33. Use the partial fraction decomposition of the functions in problems 1 and 2 to calculate their first and second derivatives.

34. Use the partial fraction decomposition of the functions in problems 3 and 4 to calculate their first and second derivatives.

35. Use the partial fraction decomposition of the functions in problems 5 and 6 to calculate their first and second derivatives.

The following two applications involve a type of differential equation which can be solved by separating the variables and using a partial fraction decomposition to help calculate the antiderivatives. The same type of differential equation is also used to model the spread of rumors and diseases as well as some populations and chemical reactions.

Logistic Growth: The growth rate of many different populations depends not only on the number of individuals (leading to exponential growth) but also on a "carrying capacity" of the environment. If x is the population at time t and the growth rate of x is proportional to the **product** of the population and the carrying capacity M minus the population, then the growth rate is described by the differential equation

$$\frac{dx}{dt} = k\cdot x\cdot(M - x)$$

where k and M are constants for a given species in a given environment.

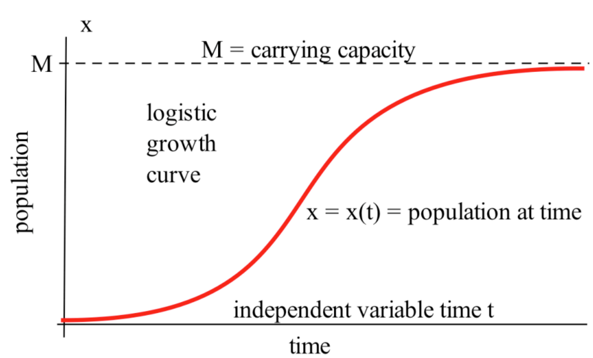

Fig. 1: Logistic growth curve

36. Let $k = 1$ and $M = 100$, and assume the initial population is $x(0) = 5$.

   (a) Solve the differential equation $\frac{dx}{dt} = x(100 - x)$ for x .

   (b) Graph the population $x(t)$ for $0 \le t \le 20$.

   (c) When will the population be 20? 50? 90? 100?

   (d) What is the population after a "long" time? (Find the limit, as t becomes arbitrarily large, of x .)

   (e) Explain the shape of the graph in (a) in terms of a population of bacteria.

   (f) When is the growth rate largest? (Maximize dx/dt .)

   (g) What is the population when the growth rate is largest?

37. Let $k = 1$ and $M = 100$, and assume the initial population is $x(0) = 150$.

   (a) Solve the differential equation $\frac{dx}{dt} = x(100 - x)$ for x and graph $x(t)$ for $0 \le t \le 20$.

   (b) When will the population be 120? 110? 100?

   (c) What is the population after a "long" time? (Find the limit, as t becomes arbitrarily large, of x .)

   (d) Explain the shape of the graph in (a) in terms of a population of bacteria.

38. Let k and M be positive constants, and assume the initial population is $x(0) = x_0$.

   (a) Solve the differential equation $\frac{dx}{dt} = k \cdot x \cdot (M - x)$ for x.

   (b) What is the population after a "long" time? (Find the limit, as t becomes arbitrarily large, of x .)

   (c) When is the growth rate largest? (Maximize dx/dt .)

   (d) What is the population when the growth rate is largest?

Chemical Reaction: In some chemical reactions, a new material X is formed from materials A and B, and the rate at which X forms is proportional to the **product** of the amount of A and the amount of B remaining in the solution. Let x represent the amount of material X present at time t, and assume that the reaction begins with a grams of A, b grams of B, and no material X ($x(0) = 0$). Then the rate of formation of material X can be described by the differential equation

$$\frac{dx}{dt} = k(a - x)(b - x).$$

39. Solve the differential equation $\frac{dx}{dt} = k(a - x)(b - x)$ for x if $k = 1$ and the reaction begins with (i) 7 grams of A and 5 grams of B, and (ii) 6 grams of A and 6 grams of B.

40. Solve the differential equation $\frac{dx}{dt} = k(a - x)(b - x)$ for x if $k = 1$ and the reaction begins with (i) a grams of A and b grams of B with $a \ne b$, and (ii) c grams of A and c grams of B ($c \ne 0$).

Section 8.4 **PRACTICE Answers**

Practice 1: $\int \frac{7x-11}{3x^2-8x-3}\,dx = \int \frac{4}{3x+1}\,dx + \int \frac{1}{x-3}\,dx$

$$= \mathbf{\frac{4}{3} \cdot \ln|3x+1| + \ln|x-3| + C} .$$

Practice 2: $\frac{6x-7}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2} = \frac{A(x-2)+B(x+3)}{(x+3)(x-2)} = \frac{(A+B)x+(-2A+3B)}{(x+3)(x-2)}$.

This gives us the system: $A + B = 6$ and $-2A + 3B = -7$ so (solving) $\mathbf{A = 5}$ and $\mathbf{B = 1}$.

$$\frac{6x-7}{(x+3)(x-2)} = \frac{\mathbf{5}}{x+3} + \frac{\mathbf{1}}{x-2} .$$

Practice 3: From Example 4,

$$\int \frac{2x^2+7x+7}{x(x+1)(x+3)}\,dx = \int \frac{3}{x} + \frac{-2}{x+1} + \frac{1}{x+3}\,dx$$

$$= 3\cdot\ln|x| - 2\cdot\ln|x+1| + \ln|x+3| + C .$$

Practice 4: If the 5^{th} degree polynomial can be factored into a product of 5 distinct linear terms, then we would have

$$\frac{\text{something}}{5^{th}\text{ degree polynomial}} = \frac{A}{1^{st}\text{ term}} + \frac{B}{2^{nd}\text{ term}} + \ldots + \frac{E}{5^{th}\text{ term}} .$$

Practice 5: $\frac{2x^2+27x+85}{(x+5)^2} = \frac{2x^2+27x+85}{x^2+10x+25} = 2 + \frac{7x+35}{(x+5)^2} = 2 + \frac{7}{x+5}$.

Then $\int \frac{2x^2+27x+85}{(x+5)^2}\,dx = \int 2 + \frac{7}{x+5}\,dx = \mathbf{2x + 7\cdot\ln|x+5| + C}$.

Practice 6: $g(x) = \frac{9x+1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} = \frac{A(x+1)+B(x-3)}{(x-3)(x+1)} = \frac{(A+B)x+(A-3B)}{(x-3)(x+1)}$.

This gives us the system $A + B = 9$ and $A - 3B = 1$ so (solving) $A = 7$ and $B = 2$.

$g(x) = \frac{7}{x-3} + \frac{2}{x+1} = 7(x-3)^{-1} + 2(x+1)^{-1}$. Then

$g'(x) = -7(x-3)^{-2} - 2(x+1)^{-2}$,

$g''(x) = 14(x-3)^{-3} + 4(x+1)^{-3}$,

$g'''(x) = -42(x-3)^{-4} - 12(x+1)^{-4}$, and

$g''''(x) = 168(x-3)^{-5} + 48(x+1)^{-5}$.