8.5 Trigonometric Substitution — Another Change of Variable

Changing the variable is a very powerful technique for finding antiderivatives, and by now you have probably found a lot of integrals by setting u = something. This section also involves a change of variable, but for more specialized patterns, and the change is more complicated. Another difference from previous work is that instead of setting u equal to a function of x we will be replacing x with a function of θ.

The next three examples illustrate the typical steps involved making trigonometric substitutions. After these examples, we examine each step in more detail and consider how to make the appropriate decisions.

Example 1: In the expression $\sqrt{9-x^2}$ replace x with $3\sin(\theta)$ and simplify the result.

Solution: Replacing x with $3\sin(\theta)$, $\sqrt{9-x^2}$ becomes

$$\sqrt{9-(3\sin(\theta))^2} = \sqrt{9-9\sin^2(\theta)} = \sqrt{9\cdot(1-\sin^2(\theta))} = 3\cos(\theta) .$$

Example 2: Evaluate $\int \sqrt{9-x^2}\,dx$ using the change of variable $x = 3\sin(\theta)$ and then use the antiderivative to evaluate $\int_0^3 \sqrt{9-x^2}\,dx$.

Solution: If $x = 3\sin(\theta)$, then $dx = 3\cos(\theta)\,d\theta$ and $\sqrt{9-x^2} = 3\cos(\theta)$. With this change of variable, the integral becomes

$$\int \sqrt{9-x^2}\,dx = \int 3\cos(\theta)\,3\cos(\theta)\,d\theta = 9\int \cos^2(\theta)\,d\theta = 9\left\{\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right\} + C$$

$$= 9\left\{\frac{\theta}{2} + \frac{2\sin(\theta)\cos(\theta)}{4}\right\} + C = \frac{9}{2}\left\{\theta + \sin(\theta)\cos(\theta)\right\} + C$$

This antiderivative, a function of the variable θ, can be converted back to a function of the variable x. Since $x = 3\sin(\theta)$ we can solve for θ to get $\theta = \arcsin(x/3)$. Replacing θ with $\arcsin(x/3)$ in the antiderivative, we get

$$\frac{9}{2}\left\{\boldsymbol{\theta} + \sin(\boldsymbol{\theta})\cos(\boldsymbol{\theta})\right\} + C = \frac{9}{2}\left\{\mathbf{arcsin(x/3)} + \sin(\mathbf{arcsin(x/3)})\cos(\mathbf{arcsin(x/3)})\right\} + C$$

$$= \frac{9}{2}\left\{\arcsin\left(\frac{x}{3}\right) + \frac{x}{3}\,\frac{\sqrt{9-x^2}}{3}\right\} + C = \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{1}{2}x\sqrt{9-x^2} + C.$$

Using this antiderivative, we can evaluate the definite integral:

$$\int_0^3 \sqrt{9-x^2}\,dx = \frac{9}{2}\cdot\arcsin\left(\frac{x}{3}\right) + \frac{x}{2}\sqrt{9-x^2}\;\Bigg|_0^3$$

$$= \left\{\frac{9}{2}\arcsin\left(\frac{3}{3}\right) + \frac{3}{2}\sqrt{9-3^2}\right\} - \left\{\frac{9}{2}\arcsin\left(\frac{0}{3}\right) + \frac{0}{2}\sqrt{9-0^2}\right\} = \frac{9\pi}{4}\ .$$

Example 3: The definite integral $\int_0^3 \sqrt{9-x^2}\,dx$ represents the area of what region?

Solution: The area of one fourth of the circle of radius 3 which lies in the first quadrant (Fig. 1). The area of this quarter circle is

$$\frac{\text{area of whole circle}}{4} = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi 3^2 = \frac{9}{4}\pi$$

which agrees with the value found in the previous example.

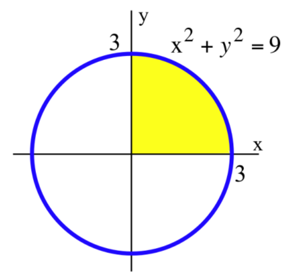

Fig. 1

Each Trigonometric Substitution involves four major steps:

1. Choose which substitution to make, x = a trigonomteric function of θ .
2. Rewrite the original integral in terms of θ and $d\theta$.
3. Find an antiderivative of the new integral.
4. Write the antiderivative in step 3 in terms of the original variable x.

The rest of this section discusses each of these steps. The first step requires you to make a decision. Then the other three steps follow from that decision. For most students, the key to success with the Trigonometric Substitution technique is to THINK TRIANGLES.

Step 1: Choosing the substitution

The first step requires that you make a decision, and the pattern of the familiar Pythagorean Theorem can help you make the correct choice.

Pythagorean Theorem: $(\text{side})^2 + (\text{side})^2 = (\text{hypotenuse})^2$ or $(\text{side})^2 = (\text{hypotenuse})^2 - (\text{side})^2$.

The pattern $\mathbf{3^2 + x^2}$ matches the Pythagorean pattern if 3 and x are sides of a right triangle. For a right triangle with sides 3 and x (Fig. 2), we know $\tan(\theta) = \text{opposite/adjacent} = x/3$ so $\mathbf{x = 3\tan(\theta)}$.

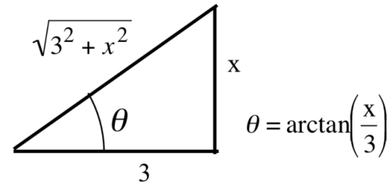

Fig. 2

The pattern $\mathbf{3^2 - x^2}$ matches the Pythagorean pattern if 3 is the hypotenuse and x is a side of a right triangle (Fig. 3). Then $\sin(\theta)$ = opposite/hypotenuse = x/3 so $\mathbf{x = 3\sin(\theta)}$.

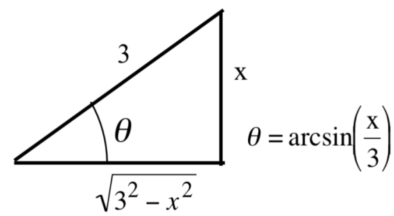

Fig. 3

The pattern $\mathbf{x^2 - 3^2}$ matches the Pythagorean pattern if x is the hypotenuse and 3 is a side of a right triangle (Fig. 4). Then $\sec(\theta)$ = hypotenuse/adjacent = x/3 so $\mathbf{x = 3\sec(\theta)}$.

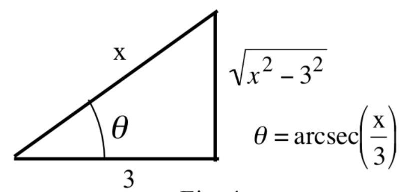

Fig. 4

Once the choice has been made for the substitution, then several things follow automatically:

dx can be calculated by differentiating x with respect to θ,

θ can be found by solving the substitution equation for θ,

(if $x = 3\tan(\theta)$ then $\tan(\theta) = x/3$ so $\theta = \arctan(x/3)$), and

the patterns $3^2 + x^2$, $3^2 - x^2$, and $x^2 - 3^2$ can be simplified using algebra and the trigonometric identities $1 + \tan^2(\theta) = \sec^2(\theta)$, $1 - \sin^2(\theta) = \cos^2(\theta)$, and $\sec^2(\theta) - 1 = \tan^2(\theta)$.

These results are collected in the table below.

$\mathbf{3^2 + x^2}$ (Fig. 2)	$\mathbf{3^2 - x^2}$ (Fig. 3)	$\mathbf{x^2 - 3^2}$ (Fig. 4)
Put $\mathbf{x = 3\tan(\theta)}$.	Put $\mathbf{x = 3\sin(\theta)}$.	Put $\mathbf{x = 3\sec(\theta)}$.
Then $dx = 3\sec^2(\theta)\,d\theta$	Then $dx = 3\cos(\theta)\,d\theta$	Then $dx = 3\sec(\theta)\tan(\theta)\,d\theta$
$\theta = \arctan(\frac{x}{3})$	$\theta = \arcsin(\frac{x}{3})$	$\theta = \operatorname{arcsec}(\frac{x}{3})$
$3^2 + x^2$ $= 3^2 + 3^2\tan^2(\theta)$ $= 3^2(1 + \tan^2(\theta))$ $= 3^2\sec^2(\theta)$	$3^2 - x^2$ $= 3^2 - 3^2\sin^2(\theta)$ $= 3^2(1 - \sin^2(\theta))$ $= 3^2\cos^2(\theta)$	$x^2 - 3^2$ $= 3^2\sec^2(\theta) - 3^2$ $= 3^2(\sec^2(\theta) - 1)$ $= 3^2\tan^2(\theta)$

Example 4: For the patterns $16 - x^2$ and $5 + x^2$, (a) decide on the appropriate substitution for x, (b) calculate dx and θ, and (c) use the substitution to simplify the pattern.

Solution: $16 - x^2$: This matches the Pythagorean pattern if 4 is a hypotenuse and x is the side of a right triangle. Then $\sin(\theta)$ = opposite/hypotenuse = x/4 so $\mathbf{x = 4 \sin(\theta)}$. For $x = 4\sin(\theta)$, $dx = 4\cos(\theta)\,d\theta$ and $\theta = \arcsin(x/4)$. Finally,

$16 - x^2 = 16 - (4\sin(\theta))^2 = 16 - 16\sin^2(\theta) = 16(1 - \sin^2(\theta)) = 16\cos^2(\theta)$.

$5 + x^2$: This matches the Pythagorean pattern if x and $\sqrt{5}$ are the sides of a right triangle. Then $\tan(\theta)$ = opposite/adjacent = $x/\sqrt{5}$ so $\mathbf{x = \sqrt{5}\tan(\theta)}$. For $x = \sqrt{5}\tan(\theta)$, $dx = \sqrt{5}\sec^2(\theta)\,d\theta$ and $\theta = \arctan(x/\sqrt{5})$. Finally,

$5 + x^2 = 5 + (\sqrt{5}\tan(\theta))^2 = 5 + 5\tan^2(\theta) = 5(1 + \tan^2(\theta)) = 5\sec^2(\theta)$.

Practice 1: For the patterns $25 + x^2$ and $x^2 - 13$, (a) decide on the appropriate substitution for x, (b) calculate dx and θ, and (c) use the substitution to simplify the pattern.

Step 2: Rewriting the integral in terms of θ and $d\theta$

Once we decide on the appropriate substitution, calculate dx , and simplify the the pattern, then the second step is very straightforward.

Example 5: Use the substitution $x = 5\tan(\theta)$ to rewrite the integral $\int \frac{1}{\sqrt{25 + x^2}}\,dx$ in terms of θ and $d\theta$.

Solution: Since $x = 5\tan(\theta)$, then $dx = 5\sec^2(\theta)\,d\theta$ and

$25 + x^2 = 25 + (5\tan(\theta))^2 = 25 + 25\tan^2(\theta) = 25\{1 + \tan^2(\theta)) = 25\sec^2(\theta)$. Finally,

$$\int \frac{1}{\sqrt{25 + x^2}}\,dx = \int \frac{1}{\sqrt{25\sec^2(\theta)}}\, 5\sec^2(\theta)\,d\theta = \int \frac{5\sec^2(\theta)}{5\sec(\theta)}\,d\theta = \int \sec(\theta)\,d\theta .$$

Practice 2: Use the substitution $x = 5\sin(\theta)$ to rewrite the integral $\int \frac{1}{\sqrt{25 - x^2}}\,dx$ in terms of θ and $d\theta$.

Steps 3 & 4: Finding an antiderivative of the new integral & writing the answer in terms of x

After changing the variable, the new integral typically involves trigonometric functions and we can use any of our previous methods (a change of variable, integration by parts, a trigonometric identity, or the integral tables) to find an antiderivative.

Once we have an antiderivative, usually a trigonometric function of θ, we can replace θ with the appropriate inverse trigonometric function of x and simplify. Since the antiderivatives commonly contain trigonometric functions, we frequently need to simplify a trigonometric function of an inverse trigonometric function, and it is **very** helpful to refer back to the right triangle we used at the beginning of the substitution process.

$\theta = \arctan\left(\frac{x}{5}\right)$

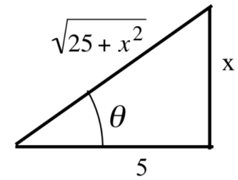

Fig. 5

Example 6: By replacing x with $5\tan(\theta)$, $\int \frac{1}{\sqrt{25+x^2}}\,dx$ becomes $\int \sec(\theta)\,d\theta$. Evaluate $\int \sec(\theta)\,d\theta$ and write the resulting antiderivative in terms of the variable x.

Solution: $x = 5\tan(\theta)$ so $\theta = \arctan(x/5)$ (Fig. 5). Then

$$\int \sec(\theta)\,d\theta = \ln|\sec(\theta)+\tan(\theta)| + C = \ln|\sec(\boldsymbol{\arctan(x/5)}) + \tan(\boldsymbol{\arctan(x/5)})| + C$$

By referring to the right triangle in Fig. 5, we see that

$$\sec(\arctan(x/5)) = \frac{\sqrt{25+x^2}}{5} \quad \text{and} \quad \tan(\arctan(x/5)) = \frac{x}{5} \quad \text{so}$$

$$\ln|\sec(\arctan(x/5)) + \tan(\arctan(x/5))| + C = \ln\left|\frac{\sqrt{25+x^2}}{5} + \frac{x}{5}\right| + C.$$

Putting these pieces together, we have $\int \frac{1}{x\sqrt{25+x^2}}\,dx = \ln\left|\frac{\sqrt{25+x^2}}{5} + \frac{x}{5}\right| + C$.

Practice 3: Show that by replacing x with $3\sin(\theta)$, $\int \frac{1}{x^2\sqrt{9-x^2}}\,dx$ becomes $\frac{1}{9}\int \csc^2(\theta)\,d\theta$. Evaluate $\frac{1}{9}\int \csc^2(\theta)\,d\theta$ and write the resulting antiderivative in terms of the variable x.

Sometimes it is useful to "complete the square" in an irreducible quadratic to make the pattern more obvious.

Example 7: Rewrite $x^2 + 2x + 26$ by completing the square and evaluate $\int \frac{1}{\sqrt{x^2+2x+26}}\,dx$.

Solution: $x^2 + 2x + 26 = (x+1)^2 + 25$ so $\int \frac{1}{\sqrt{x^2+2x+26}}\,dx = \int \frac{1}{\sqrt{(x+1)^2+25}}\,dx$.

Put $u = x + 1$. Then $du = dx$, and

$$\int \frac{1}{\sqrt{x^2 + 2x + 26}}\, dx = \int \frac{1}{\sqrt{(x+1)^2 + 25}}\, dx$$

$$= \int \frac{1}{\sqrt{u^2 + 25}}\, du$$

$$= \ln\left| \frac{\sqrt{25 + u^2}}{5} + \frac{u}{5} \right| + C \quad \text{(using the result of Example 6)}$$

$$= \ln\left| \frac{\sqrt{25 + (x+1)^2}}{5} + \frac{x+1}{5} \right| + C = \ln\left| \frac{\sqrt{x^2 + 2x + 26}}{5} + \frac{x+1}{5} \right| + C$$

THINK TRIANGLES. The first and last steps of the method (choosing the substitution and writing the answer interms of x) are easier if you understand the triangles (Figures 2, 3, and 4) and have drawn the appropriate triangle for the problem. Of course, you also need to practice the method.

PROBLEMS

In problems 1–6, (a) make the given substitution and simplify the result, and (b) calculate dx.

1. $x = 3\cdot\sin(\theta)$ in $\dfrac{1}{\sqrt{9 - x^2}}$

2. $x = 3\cdot\tan(\theta)$ in $\dfrac{1}{\sqrt{x^2 + 9}}$

3. $x = 3\cdot\sec(\theta)$ in $\dfrac{1}{\sqrt{x^2 - 9}}$

4. $x = 6\cdot\sin(\theta)$ in $\dfrac{1}{36 - x^2}$

5. $x = \sqrt{2}\cdot\tan(\theta)$ in $\dfrac{1}{\sqrt{2 + x^2}}$

6. $x = \sec(\theta)$ in $\dfrac{1}{x^2 - 1}$

In problems 7–12, (a) solve for θ as a function of x,
(b) replace θ in $f(\theta)$ with you result in part (a), and (c) simplify.

7. $x = 3\cdot\sin(\theta)$, $f(\theta) = \cos(\theta)\cdot\tan(\theta)$

8. $x = 3\cdot\tan(\theta)$, $f(\theta) = \sin(\theta)\cdot\tan(\theta)$

9. $x = 3\cdot\sec(\theta)$, $f(\theta) = \sqrt{1 + \sin^2(\theta)}$

10. $x = 5\cdot\sin(\theta)$, $f(\theta) = \dfrac{\cos(\theta)}{1 + \sec(\theta)}$

11. $x = 5\cdot\tan(\theta)$, $f(\theta) = \dfrac{\cos^2(\theta)}{1 + \cot(\theta)}$

12. $x = 5\cdot\sec(\theta)$, $f(\theta) = \cos(\theta) + 7\cdot\tan^2(\theta)$

In problems 13–36, evaluate the integrals. (More than one method works for some of the integrals.)

13. $\int \frac{1}{x^2\sqrt{9-x^2}}\,dx$

14. $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

15. $\int \frac{1}{\sqrt{x^2+49}}\,dx$

16. $\int \frac{1}{\sqrt{x^2+1}}\,dx$

17. $\int \sqrt{36-x^2}\,dx$

18. $\int \sqrt{1-36x^2}\,dx$

19. $\int \frac{1}{\sqrt{36+x^2}}\,dx$

20. $\int \frac{1}{x\sqrt{25-x^2}}\,dx$

21. $\int \frac{1}{\sqrt{49+x^2}}\,dx$

22. $\int \frac{\sqrt{25-x^2}}{x^2}\,dx$

23. $\int \frac{x}{\sqrt{25-x^2}}\,dx$

24. $\int \frac{1}{x^2+49}\,dx$

25. $\int \frac{x}{x^2+49}\,dx$

26. $\int \frac{1}{49x^2+25}\,dx$

27. $\int \frac{1}{(x^2-9)^{3/2}}\,dx$

28. $\int \frac{1}{(4x^2-1)^{3/2}}\,dx$

29. $\int \frac{5}{2x\sqrt{x^2-25}}\,dx$

30. $\int \frac{1}{x\sqrt{3-x^2}}\,dx$

31. $\int \frac{1}{25-x^2}\,dx$

32. $\int \frac{1}{a^2+x^2}\,dx$

33. $\int \frac{1}{\sqrt{a^2+x^2}}\,dx$

34. $\int \frac{1}{x\cdot\sqrt{a^2+x^2}}\,dx$

35. $\int \frac{1}{x^2\sqrt{a^2+x^2}}\,dx$

36. $\int \frac{1}{(a^2+x^2)^{3/2}}\,dx$

In problems 37–42, first complete the square, make the appropriate substitutions, and evaluate the integral.

37. $\int \frac{1}{\sqrt{(x+1)^2+9}}\,dx$

38. $\int \frac{1}{\sqrt{(x+3)^2+1}}\,dx$

39. $\int \frac{1}{x^2+10x+29}\,dx$

40. $\int \frac{1}{x^2-4x+13}\,dx$

41. $\int \frac{1}{\sqrt{x^2+4x+3}}\,dx$

42. $\int \frac{1}{\sqrt{x^2-6x-16}}\,dx$

Section 8.5 Practice Answers

Practice 1: $25 + x^2$: (a) Put $x = 5\cdot\tan(\theta)$

(b) Then $dx = 5\cdot\sec^2(\theta)\, d\theta$ and $\theta = \arctan(\ x/5\ )$

(c) $25 + x^2 = 25 + 25\cdot\tan^2(\theta) = 25(\ 1 + \tan^2(\theta)\ ) = 25\cdot\sec^2(\theta)$

$x^2 - 13$: (a) Put $x = \sqrt{13}\cdot\sec(\theta)$

(b) Then $dx = \sqrt{13}\cdot\sec(\theta)\cdot\tan(\theta)\, d\theta$ and $\theta = \operatorname{arcsec}(\ x/\sqrt{13}\ )$

(c) $x^2 - 13 = 13\cdot\sec^2(\theta) - 13 = 13(\ \sec^2(\theta) - 1\ ) = 13\cdot\tan^2(\theta)$

Practice 2: $x = 5\cdot\sin(\theta)$ so $dx = 5\cdot\cos(\theta)\, d\theta$ and $25 - x^2 = 25(\ 1 - \sin^2(\theta)\ ) = 25\cdot\cos^2(\theta)$.

Then $\displaystyle\int \frac{1}{\sqrt{25 - x^2}}\, dx = \int \frac{1}{\sqrt{25 - \sin^2(\theta)}}\, 5\cdot\cos(\theta)\, d\theta$

$$= \int \frac{1}{\sqrt{25\cdot\cos^2(\theta)}}\, 5\cdot\cos(\theta)\, d\theta = \int 1\, d\theta = \theta + C = \arcsin(\ x/5\ ) + C$$

Practice 3: $x = 3\cdot\sin(\theta)$ so $dx = 3\cdot\cos(\theta)\, d\theta$ and $9 - x^2 = 9(\ 1 - \sin^2(\theta)\ ) = 9\cdot\cos^2(\theta)$.

Then $\displaystyle\int \frac{1}{x^2\sqrt{9 - x^2}}\, dx = \int \frac{1}{9\cdot\sin^2(\theta)\sqrt{9 - \sin^2(\theta)}}\, 3\cdot\cos(\theta)\, d\theta = \int \frac{1}{9\cdot\sin^2(\theta)\sqrt{9\cdot\cos^2(\theta)}}\, 3\cdot\cos(\theta)\, d\theta$

$$= \frac{1}{9}\int \csc^2(\theta)\, d\theta = -\frac{1}{9}\cot() + C = -\frac{1}{9}\cot(\ \arcsin(x/3)\ ) + C = -\frac{1}{9}\frac{\sqrt{9 - x^2}}{x} + C.$$