14.0 INTRODUCTION TO DOUBLE INTEGRALS AND THEIR APPLICATIONS

Chapters 4 and 5 introduced integrals of functions of a single variable, y = f(x), as well as several of their applications. Chapter 14 introduces integrals for functions of more than one variable, shows how to calculate their values, and starts to examine some of their applications. Later chapters will use these double integrals extensively.

The applications will include calculating

- Volumes in 3D
- Total masses of solids whose densities are not constant
- Moments of solids around each axis
- Centers of mass of some solids
- Moments of inertia
- Surface areas

Moving from 2D Areas to 3D Volumes

Many of the ideas about integrals of a single variable are also important for these new integrals but things do get more complicated. In Chapter 4 we began by approximating the area under a curve by partitioning the x-axis domain, calculating the area of the rectangles over each segment of the partition, and then adding those areas together to get an approximation of the total area (Fig. 1). We begin by doing something similar to approximate the volume under a surface by portioning the 2D domain in the xy-plane into rectangles, calculating the volume of boxes over each rectangular piece, and then adding those volumes together to approximate the total volume (Fig. 2).

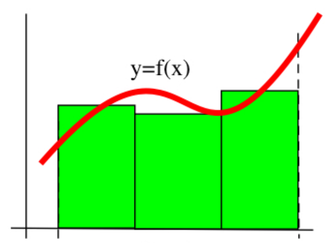

Fig. 1

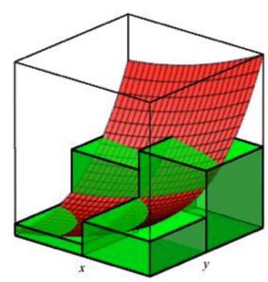

Fig. 2

The approximation of the 2D area became simply a matter of multiplying the length of the partition segment length by the height of the function and then adding those values together. The approximation of the 3D volume will be similar: multiply the area of the partition base by the height of the function, and then adding those values together.

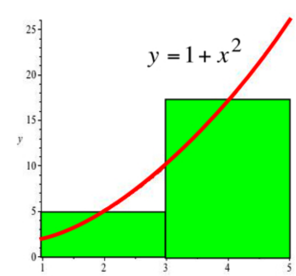

Fig. 3

Example 1: Approximate the area between the function $f(x) = 1 + x^2$ using the partition {1, 3, 5} and evaluate the function at the midpoint of each subinterval (Fig. 3).

Solution: The partition consists of the intervals [1, 3] and [3, 5] with midpoints x=2 and x=4 so the

approximation is $\sum_{i=1}^{2} f(x_i)\cdot \Delta x_i = f(2)\cdot 2 + f(4)\cdot 2 = 10 + 34 = 44$.

(Note: We know from our earlier work that the exact value of

this area is $\int_1^5 1+x^2 \; dx = \; x+\frac{1}{3}x^3 \Big|_1^5 = 45\frac{1}{3}$)

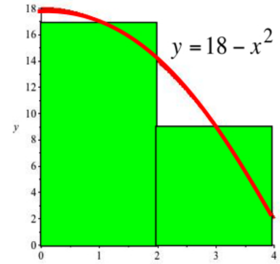

Fig. 4

Practice 1: Approximate the area between the function $g(x) = 18 - x^2$ using the partition {0, 2, 4} and evaluate the function at the midpoint of each subinterval (Fig. 4).

We can do something similar to approximate a volume.

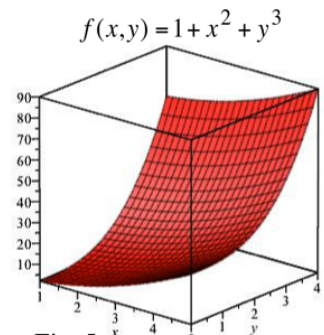

Fig. 5

Example 2: Approximate the volume of between the surface $f(x,y) = 1 + x^2 + y^3$ (Fig. 5) over the rectangle 1≤x≤5 and 0≤y≤4 by partitioning the x interval into subintervals [1,3] and [3,5] and the y interval into subintervals [0,2] and [2,4] and then evaluate the function at the midpoint of each sub-rectangle. The dots in Fig. 6 are at the midpoints of each sub-rectangle and the numbers by each dot is the value of the function at that midpoint.

Solution: $\sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i, y_j)\cdot \Delta x_i \cdot \Delta y_j = f(2,1)\cdot 2\cdot 2 + f(4,1)\cdot 2\cdot 2 + f(2,3)\cdot 2\cdot 2 + f(4,3)\cdot 2\cdot 2$

$= 6\cdot 4 + 18\cdot 4 + 32\cdot 4 + 44\cdot 4 = 400$

(In the next section we will be able to calculate that the exact volume is $437\frac{1}{3}$ so our approximation was very crude.)

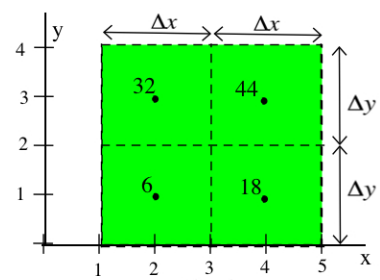

Fig. 6

Practice 2: Fig. 7 shows the depths (meters) at various locations in a backyard swimming pool. Approximate the total volume of the pool.

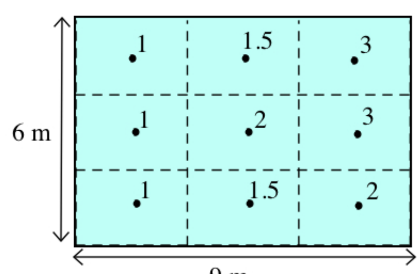

Fig. 7

If we have partitioned the domain into rectangles with dimensions Δx and Δy, then the area of each sub-rectangle is $\Delta A = \Delta x \cdot \Delta y$ and the volume of each box is $\Delta V = f(x_i, y_i) \cdot \Delta A$. But sometimes the natural partition is not rectangles.

Example 3: A gardener wants to estimate how much water a sprinkler puts out in an hour. She places several small cans at various distances from the sprinkler and measures the depth of the water in each can after one hour. Her data is given in Fig. 8. Estimate the hourly output of the sprinkler.

distance of can from sprinkler	depth of water in can
1 ft	0.3 ft
3 ft	0.4 ft
5 ft	0.3 ft
7 ft	0.2 ft

Fig. 8

Solution: In this case it is more useful to partition the circular pattern into concentric rings, and then calculate the volume of water for each ring. The area of the innermost ring (a circle) is $\Delta A_1 = \pi \cdot 2^2$, and the others are

$\Delta A_2 = \pi \cdot (4^2 - 2^2)$, $\Delta A_3 = \pi \cdot (6^2 - 4^2)$, and $\Delta A_4 = \pi \cdot (8^2 - 6^2)$.

The depths D_i are given in the table, and the total volume is

$$\sum D_i \cdot \Delta A_i = 4\pi(0.3) + 12\pi(0.4) + 20\pi(0.3) + 28\pi(0.2) \approx 55.3 \ ft^3.$$

Since each cubic foot of water is 7.5 US gallons, the sprinkler is putting out approximately 415 US gallons per hour.

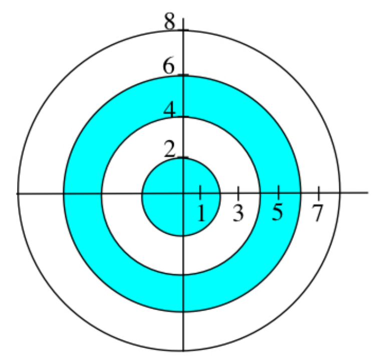

Fig. 9: Sprinkler pattern

(Note: Obviously this is a very crude estimate. She would get a much better estimate if there were more cans and the distances between the cans was smaller.)

Using Averages to Estimate Volumes

Sometimes in practice we just need an approximate value for a volume. We may need an estimate of the volume of a small pond in order to know how much of a chemical to put in the pond in order to stop an algae bloom. Or we may need an estimate of the volume of a hole on a construction site in order to know how much gravel we need to order to fill the hole. In these cases a good estimate of the volume is all that we need, and an estimate of the average depth of the pond or hole can enable us to estimate the volume.

In Section 4.7 we used integrals to calculate the average value of a function f≥0 on an interval domain as

$$\{\text{ average value of f(x) on interval I}\} = \frac{1}{\text{length of I}} \cdot \{\text{area between f and I}\} = \frac{1}{\text{length of I}} \cdot \int_I f(x)\, dx .$$

A similar approach can let us estimate volume based on the average value of a function f≥0 on a domain R:

$$\{\text{average value of f(x,y) on region R}\} = \frac{1}{\text{area of R}} \cdot \{\text{volume between f and R}\}$$

so $\{\text{volume between f and R}\} = \{\text{area of R}\} \cdot \{\text{average value of f(x,y) on region R}\}$.

If we have approximate values for the area of R and for the mean value of f on R, then we simply need to multiply those values to get an approximation of the volume. It seems too easy, but it is useful.

Approximating a Volume

$$\left\{\begin{array}{l}\text{approximate volume}\\ \quad\text{of f over R}\end{array}\right\} = \left\{\begin{array}{l}\text{approximate}\\ \text{area of R}\end{array}\right\} \cdot \left\{\begin{array}{l}\text{approximate average value}\\ \text{of f(x,y) on region R}\end{array}\right\}$$

Example 4: The numbers by each dot in Fig. 6 (from Example 2) is the height of the function at that location. Use that information to approximate the volume between f and the xy-plane.

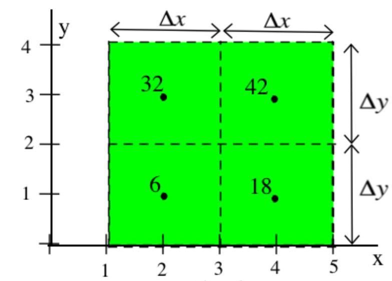

Fig. 6

Solution: The average of the four height values is (32+42+18+6)/4=24.5 and the area of the domain is (4)(4)=16 so our volume estimate is (24.5)(16)=392.

Practice 3: Use the information in Practice 2 and the average depth to estimate the volume of the pool.

Example 5: The numbers by each dot in Fig. 10 is the depth (feet) of the pond at each location. The area of the pond is approximately 12,000 square feet. Estimate the volume of the pond.

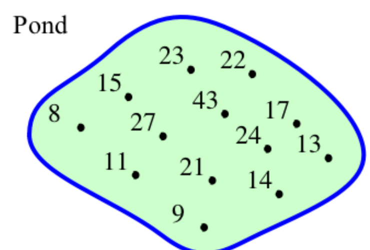

Fig. 10

Solution: The average depth is (247)/13=19 feet and the area is approximately 12,000 ft^2 so the approximate volume is (19 ft)(12,000 ft^2) = 228,000 ft^3.

In the following sections we will develop the calculus to determine volumes exactly (just as we did earlier to use integrals to calculus areas exactly), but the main ideas state as finite accumulations and lead to double integrals.

Problems

1. Fig. 11 shows the domain R of the function $z = f(x,y) = 1 + 3x + y^2$ and a partition of R into rectangles. Use the value of f at the **lower left** (x,y) point in each rectangle to approximate the volume between the graph z=f(x,y) and the xy-plane over R.

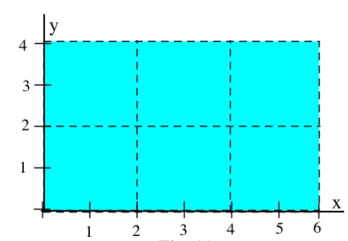

Fig. 11

2. (a) Use the same function and rectangles as in Problem 1, but evaluate the function at the **upper right** (x,y) point in each rectangle to approximate the volume between the graph z=f(x,y) and the xy-plane over R. (b) Evaluate the function at the **midpoint** (x,y) in each rectangle to approximate the volume.

3. Fig. 12 shows the domain R of the function $f(x,y) = x^2 + y^2$ and a partition of R into rectangles.
(a) Use the value of f at the **lower left** (x,y) point in each rectangle to approximate the volume between the graph z=f(x,y) and the xy-plane over R.
(b) Use the value of f at the **midpoint** (x,y) point in each rectangle to approximate the volume between the graph z=f(x,y) and the xy-plane over R.

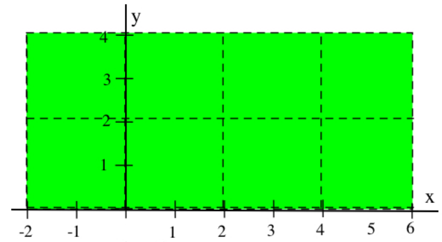

Fig. 12

4. Use the same function and rectangles as in Problem 3, but evaluate the function at the **upper right** (x,y) point in each rectangle to approximate the volume between the graph z=f(x,y) and the xy-plane over R.

5. A gardener places several small cans at a variety of locations (Fig. 13) and measures the depth of the water in each can after one hour. Estimate the hourly water output of the sprinkler.

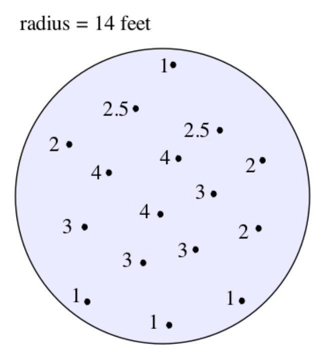

Fig. 13: water depths (inches)

6. A back-and-forth sprinkler has a rectangular distribution pattern. Fig. 14 shows the water depth (inches) at several locations after 15 minutes. Assume that the distribution is symmetric and estimate the hourly water output of the sprinkler.

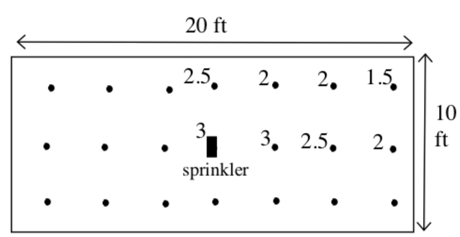

Fig. 14

7. A hole in the ground has a 4 ft. by 5 ft. rectangular opening, and the depths at the four corners are 3 ft., 4 ft. 6 ft. and 3 ft. Estimate the volume of the hole.

8. A hole in the ground has a 3 ft. by 5 ft. rectangular opening, and the depths at the four corners are 6 ft., 4 ft. 7 ft. and 6 ft. Estimate the volume of the hole.

9. A circular hole has a radius of 2 feet, and several depth measurements around the edge of the hole are 3 ft., 4 ft., 5 ft., 3 ft., and 4 ft. Estimate the volume of the hole.

10. A circular hole has a radius of 2 feet, and several depth measurements around the edge of the hole are 2 ft., 4 ft., 3 ft., 2 ft., and 3 ft. Estimate the volume of the hole.

11. Fig. 15 shows the depths (meters) at several locations in a small pond that has a surface area of 90 m^2. Approximate the volume of the pond.

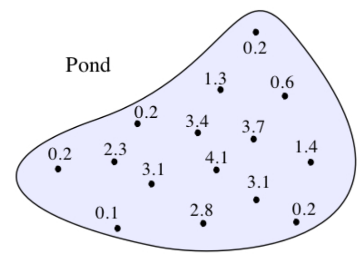

Fig. 15

12. A spacecraft has landed on a strange flat object and has measured its density ($10^3 kg/m^3$) at several locations (Fig. 16). From photographs during the approach to the object scientists have estimated the thickness of the object to be about 1.4 m thick the area to be about 3500 m^2 . Approximate the mass of the object.

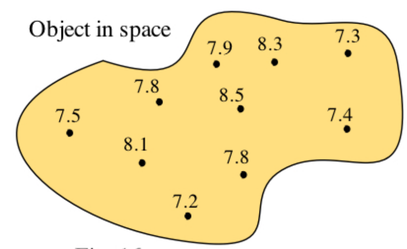

Fig. 16

13. Fig. 17 shows level elevation curves (m) for a small hill that needs to be removed for a construction project. Pick the elevations at several locations and use them to get a reasonable estimate of the volume of material to be removed. This also requires as estimate of the area of the bottom of the hill which is roughly elliptical. (ellipse area = $\frac{1}{4}\pi$(width)(length))

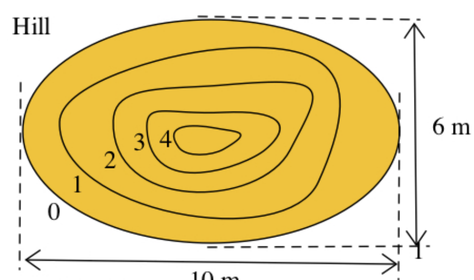

Fig. 17

14. In Example 3 (Fig. 9) we estimated the volume of the water from the sprinkler I one hour to be $55.3\ ft^3$. But if we use the (average value)(area) we get {average value} = (0.3+0.4+0.3+0.2)/4 =0.3 ft and the area is $64\ ft^2$ so our estimate for the volume is (0.3 ft)($64\ ft^2$)$\approx$60.3 ft^3 .
Which estimate do you think is better and why?

15. Fig. 18 shows the level curves for z = f(x,y) on a region R. Use the midpoint sample points to estimate the value of $\iint_R f(x,y)\ dA$.

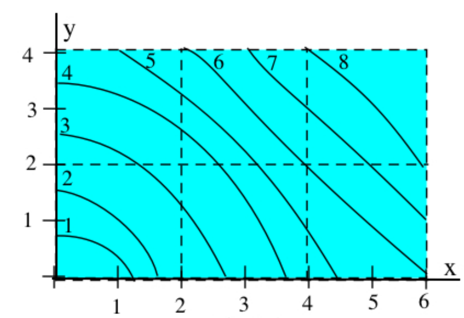

Fig. 18

16. Fig. 19 shows the level curves for z = f(x,y) on a region R. Use the midpoint sample points to estimate the value of $\iint_R f(x,y)\ dA$.

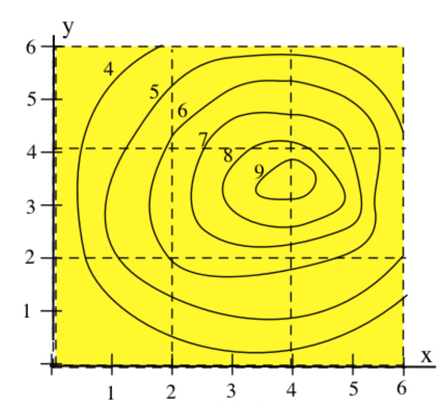

Fig. 19

Practice Answers

Practice 1: The intervals [0,2] and [2,4] have lengths 2 and 2 and midpoints x=1 and x=3. Then

$\sum_{i=1}^{2} f(x_i) \cdot \Delta x_i = f(1) \cdot 2 + f(3) \cdot 2 = 34 + 18 = 52$. (The exact area is $50\frac{2}{3}$.)

Practice 2: $\sum_{i=1}^{3} \sum_{j=1}^{3} f(x_i, y_j) \cdot \Delta x_i \cdot \Delta y_j = \sum_{i=1}^{3} \sum_{j=1}^{3} f(x_i, y_j) \cdot 2 \cdot 3 = (1+1+1+1.5+2+1.5+3+3+2) \cdot 6 = 96\ m^3$.

Practice 3: The average depth is $(1+1+1+1.5+2+1.5+3+3+2)/9 = \frac{16}{9}$ m and the area of the pool is

(6)(9)= 54 m^2 so the approximate volume is $\left(\frac{16}{9}\ m\right)\left(54\ m^2\right) = 96\ m^3$.