14.1 DOUBLE INTEGRALS OVER RECTANGLES

Volume of a Solid Region in 3D

To calculate the area between the curve y = f(x) and an interval on the x-axis (Fig. 1) we partitioned the interval [a,b] (Fig. 2), created an approximation of the area using a Riemann sum, and then took the limit of the Riemann sum as the widths of the subintervals approached 0 to get the area as a definite integral:

$$\text{Area} = \int_a^b f(x)\ dx$$

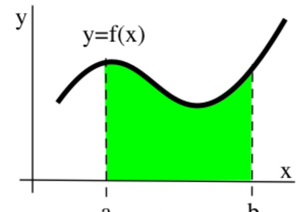


Fig. 1

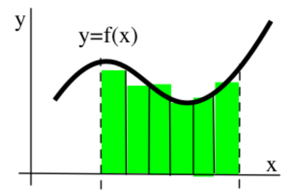


Fig. 2

In order to calculate the volume (Fig. 3) between a surface z = f(x,y) ≥ 0 and a rectangular R = $\{(x,y):\ a \le x \le b\ \ and\ \ c \le y \le d\}$ of the xy-plane, we will do something similar, but now the domain of integration is a region in the xy-plane, and our result will be a double integral:

$$\text{Volume} = \iint_R \text{f(x,y)}\ dA = \int_a^b \int_c^d f(x,y)\ dy\ dx\ ,$$

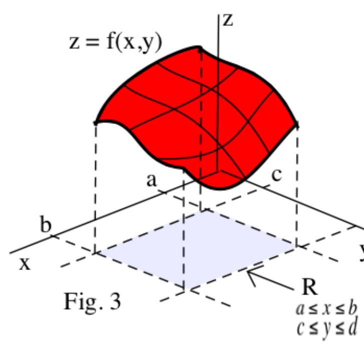


Fig. 3

For the rectangular region R we begin by partitioning along both the x-axis and along the y-axis (Fig. 4) to create small rectangles with areas

$\Delta A_{ij} = \Delta x_i \cdot \Delta y_j$ and we select any point $(x_i^*,\ y_j^*)$ in this rectangle. Then the volume of the box above this ij-rectangle with height $f(x_i^*,\ y_j^*)$ is Volume$_{ij}$ $= f(x_i^*,\ y_j^*)\cdot \Delta x_i \cdot \Delta y_j = f(x_i^*,\ y_j^*)\cdot \Delta A_{ij}$.

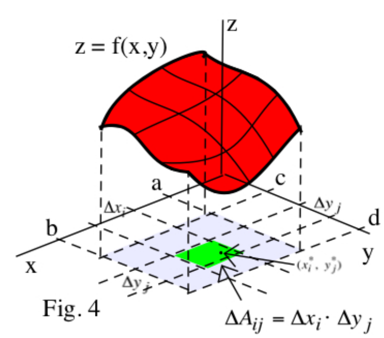


Fig. 4

Then the total approximate volume the sum of all of these little volumes:

$$\text{Approximate total volume} = \sum_j \sum_i f(x_i^*,\ y_j^*)\cdot \Delta x_i \cdot \Delta y_j\,.$$

Taking the limit as all of the Δx_i and Δy_j approach 0,

$$\text{Exact total volume} = \int_a^b \int_c^d f(x,y)\ dy\ dx = \iint_R \text{f(x,y)}\ dA\ .$$

Calculating the value of a double integral is no more difficult (nor any easier) than calculating the value of a single integral except we need to integrate twice.

Example 1: For $f(x,y) = 20 - x^2 y$ and $R = \{(x,y):\ 0 \le x \le 3,\ 1 \le y \le 2\}$ evaluate $\iint_R \text{f(x,y)}\ dA$.

Solution: In this example (Fig. 5) $\iint\limits_R f(x,y)\ dA = \int_0^3 \int_1^2 20 - x^2 y\ dy\ dx$.

First we evaluate the inside integral $\int_1^2 20 - x^2 y\, dy$ **treating x as a constant**:

(this is just the inverse of partial differentiation)

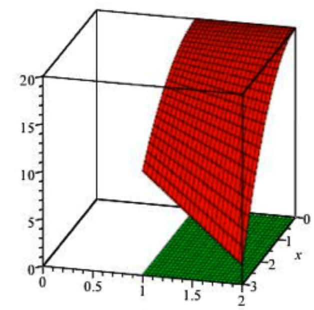


Fig. 5

$$\int_1^2 20 - x^2 y\, dy = 20y - \frac{1}{2} x^2 y^2 \Big|_{y=1}^{2}$$

$$= \frac{1}{2} x^2 (2)^2 - \frac{1}{2} x^2 (1)^2 = 20 - \frac{3}{2} x^2 .$$

Then $\displaystyle\int_0^3 \left\{ \int_1^2 20 - x^2 y\, dy \right\} dx$

$$= \int_0^3 \left\{ 20 - \frac{3}{2} x^2 \right\} dx = 20x - \frac{1}{2} x^3 \Big|_{x=0}^{3} = \frac{93}{2}.$$

Practice 1: For the same function and region R, evaluate $\int_1^2 \int_0^3 20 - x^2 y\ dx\ dy$. (This is the same solid as Example 1, but now we start by evaluating the inside integral $\int_0^3 x^2 y\ dx$, treating y as a constan.)

> Theorem:
>
> If $f(x,y) \ge 0$ and f is integrable over the rectangle R,
>
> then the volume V of the solid that lies above R and under the surface $z = f(x,y)$ is
>
> $$V = \iint\limits_R f(x,y)\ dA .$$

Note: If $f(x,y) \ge 0$ on R then the double integral gives volume. If f is sometimes negative, then the double integral gives a "signed volume" in a manner similar to how a single integral gives "signed area."

These properties of double integrals follow from the properties of summations:

(1) $\displaystyle\iint\limits_R f(x,y) + g(x,y)\ dA = \iint\limits_R f(x,y)\, dA + \iint\limits_R g(x,y)\ dA$

(2) $$\iint_R K\,f(x,y)\;dA = K\iint_R f(x,y)\,dA .$$

(3) If $f(x,y) \ge g(x,y)$ for all (x,y) in R, then $\displaystyle\iint_R f(x,y)\;dA \ge \iint_R g(x,y)\,dA$.

A few important points:

* Always work from the inside out: first evaluate the inside integral.
* For $\int f(x,y)$ **dx** integrate with respect to x and **treat y as a constant**.
* For $\int f(x,y)$ **dy** integrate with respect to y and **treat x as a constant**.

It was not an accident that the answers to Example 1 and Practice 1 were the same since both versions represented the volume of the same solid.

Fubini's Theorem:

If f is integrable over the rectangle $R = \{ (x,y) : a \le x \le b \text{ and } c \le y \le d \} = [a, b] \times [c, d]$

then $$\iint_R f(x,y)\;dA = \int_a^b \int_c^d f(x,y)\,dy\;dx = \int_c^d \int_a^b f(x,y)\,dx\;dy$$

Fubini's Theorem says that we can integrate in either order and still get the same result — sometimes one order of integration is much easier than the other order.

Example 2: Evaluate $\displaystyle\iint_R 3+y\cdot\sin(xy)\;dA$ where $R = [1,2] \times [0,\pi]$.

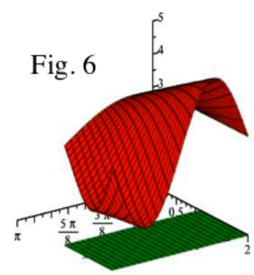


Fig. 6

Solution: The notation $R = [1,2] \times [0,\pi]$ means the rectangle $1 \le x \le 2$ and $0 \le y \le \pi$ (Fig. 6). By Fubini's Theorem we have a choice of evaluating

(a) $\displaystyle\int_0^{\pi}\int_1^2 3+y\cdot\sin(xy)\;dx\;dy$ or (b) $\displaystyle\int_1^2\int_0^{\pi} 3+y\cdot\sin(xy)\;dy\;dx$.

(a) $$\int_0^{\pi}\int_1^2 3+y\cdot\sin(xy)\;dx\;dy = \int_0^{\pi}\left\{\int_1^2 3+y\cdot\sin(xy)\;dx\right\}dy$$

$$= \int_0^{\pi}\left\{ 3x - \cos(xy)\Big|_{x=1}^{2} \right\}dy = \int_0^{\pi}\left\{ 6-\cos(2y) - 3 + \cos(1y) \right\}dy$$

$$= 3y - \frac{1}{2}\sin(2y) + \sin(y)\Big|_{y=0}^{\pi} = 3\pi$$

(b) $$\int_1^2 \int_0^{\pi} 3 + y\cdot \sin(xy)\ dy\ dx = \int_1^2 \left\{ \int_0^{\pi} 3 + y\cdot \sin(xy)\ dy \right\} dx$$ so first we need to evaluate $\int_0^{\pi} 3 + y\cdot \sin(xy)\ dy$ and that requires Integration by Parts, a more difficult situation than the method in part (a).

Example 3: Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes (xy, xz, and yz–planes). (Fig. 7)

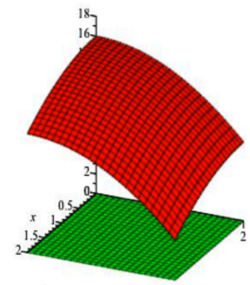


Fig.7

Solution: S lies under the surface $f(x,y) = z = 16 - x^2 - 2y^2$ and above the square $0 \le x \le 2,\ 0 \le y \le 2$. Then

$$V = \int_0^2 \int_0^2 16 - x^2 - 2y^2\, dx\ dy = \int_0^2 \left\{ \int_0^2 16 - x^2 - 2y^2\, dx \right\} dy$$

$$= \int_0^2 \left\{ 16x - \tfrac{1}{3}x^3 - 2xy^2 \Big|_{x=0}^{2} \right\} dy = \int_0^2 \left\{ \tfrac{88}{3} - 4y^2 \right\} dy = \tfrac{88}{3}y - \tfrac{4}{3}y^3 \Big|_{y=0}^{2} = 48.$$

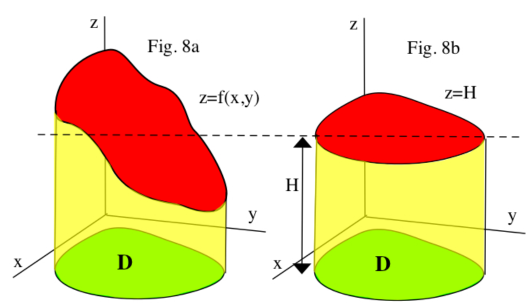
A Simple Application: Average Value of f(x,y) on R

In section 4.7 we saw that the average value of an integrable function on the interval [a,b] is $\frac{1}{b-a}\int_a^b f(x)\,dx$, that is, the average value is the integral of the function divided by the length of the domain. We have a similar result for f(x,y) on the rectangular domain D:

The average value H of an integrable function f(x,y) on domain D is $H = \frac{1}{\text{area of D}} \cdot \iint_D f(x,y)\ dA$.

The "box" (Fig. 8a) with height H={average value of f on D} has volume V ={area of D}{average height H} (Fig. 8b). But we know the volume is $V = \iint_D f(x,y)\ dA$ so

(area of D){average height H}= $\iint_D f(x,y)\ dA$ and then



$$H = \{\text{average value of f on D}\} = \frac{1}{\text{area of D}} \cdot \iint_D f(x,y)\ dA \cdot$$

Example 4: Determine the average value of the paraboloid $z = 16 - x^2 - 2y^2$ on the domain D = [0,2]x[0,2] .

Solution: From Example 3 we know $V = \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\ dx\ dy = 48$ and $\{\text{area of D}\} = 4$ so

$$H = \left(\frac{1}{4}\right)(48) = 12$$

Practice 2: Determine the average value of f(x,y) = $3 + y \cdot \sin(xy)$ on the domain $D = [1,2]\mathrm{x}[0,\pi]$. (Fig. 6)

Areas, Volumes and Double Integrals

Practice 3: Evaluate $\int_a^b \int_c^d 1\ dx\ dy$ (a<b, c<d) and interpret the meaning of the result.

So far our discussion has involved double integrals of positive functions to get volumes, but sometimes it is useful to calculate the double integral of the very simple function f(x,y) = 1 to get an area:

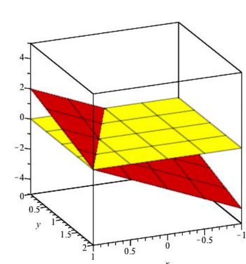
$$\{\text{area of D}\} = \iint_D 1\ dA$$

This is valid even when D is not a rectangle and is especially useful in those situations.

We began the study of single integrals by trying to find the area between a positive function y=f(x) and the x-axis, but then extended that idea to more general functions that were not always positive. In the more general case, $\int_a^b f(x)\ dx$ represented the "signed area" between f(x) and the x-axis. If the areas above and below the x-axis were equal, then $\int_a^b f(x)\ dx = 0$. In a similar manner, if z=f(x,y) is sometimes negative on D, then $\iint_D f(x,y)\ dA$ will represent the "signed volume" between f(x,y) and D in the xy-plane.

Example 5: Evaluate the double integral of f(x,y) = 2x-y over the rectangle D={(x,y): -1≤x≤1 and 0≤y≤2}.

Fig. 9: f(x,y)=2x-y



Solution: $\int_0^2 \int_{-1}^1 (2x - y)\ dx\ dy = \int_0^2 (x^2 - xy)\Big|_{x=-1}^{x=1}\ dy = \int_0^2 -2y\ dy = -4$

so more of the volume between f(x,y) and D lies below the xy-plane than lies above the xy-plane. (Fig. 9)

Practice 4: Suppose f(x,y) = Ax+By is a plane through the origin and $D = [-a,a]\text{x}[-b,b]$ is a rectangle that is symmetric about the origin. Show that the volume between the plane and D that is above the xy=plane is the same of as the volume that is below the xy-plane.

This section has focused on "what is a double integral" and "how to calculate the value of a double integral over a rectangular domain." In Section 14,2 we will extend these ideas to domains that are not rectangles and in Section 14.3 we will then use these double integral calculations in several applied settings.

PROBLEMS

For problems 1 – 4, calculate $\int_0^2 f(x,y)\,dx$ and $\int_0^1 f(x,y)\,dy$

1. $f(x,y) = x^2y^3$
2. $f(x,y) = 2xy - 3x^2$
3. $f(x,y) = x \cdot e^{x+y}$
4. $f(x,y) = \dfrac{x}{y^2+1}$

In problems 5 – 9, evaluate the double integrals.

5. $\displaystyle\int_{x=0}^{x=2}\int_{y=0}^{y=3} 3+4x+2y \;\; dy \;\; dx$
6. $\displaystyle\int_{x=0}^{x=2}\int_{y=1}^{y=2} 7+3x-6y \;\; dy \;\; dx$
7. $\displaystyle\int_{x=0}^{x=4}\int_{y=0}^{y=1} 1+e^{x}+\cos(y) \;\; dy \;\; dx$
8. $\displaystyle\int_1^3\int_2^5 x\cdot\cos(y) \;\; dx \;\; dy$
9. $\displaystyle\int_0^2\int_0^3 x^2+y^2 \;\; dx \;\; dy$
10. $\displaystyle\int_0^{\pi}\int_0^{\pi} \cos(x+y) \;\; dx \;\; dy$
11. $\displaystyle\int_0^4\int_0^2 x\sqrt{y} \;\; dx\,dy$
12. $\displaystyle\int_{-1}^1\int_0^1 (x^3y^3 + 3xy^2) \;\; dy\,dx$
13. $\displaystyle\int_0^3\int_0^1 \sqrt{x+y} \;\; dx\,dy$
14. $\displaystyle\int_0^{\pi/4}\int_0^3 \sin(x) \;\; dy\,dx$
15. $\displaystyle\int_0^{\ln(2)}\int_0^{\ln(5)} e^{2x-y} \;\; dx\,dy$

In problems 16-24 decide which order of integration is easier, $\int f(x,y)\;dx$ or $\int f(x,y)\;dy$, and then calculate the easier antiderivative.

16. $f(x,y) = x\cdot e^{xy}$
17. $f(x,y) = y\cdot\sqrt{x^2+y^2}$
18. $f(x,y) = \dfrac{4x}{x^2+y^2}$
19. $f(x,y) = \sin(x)\cdot e^{x+y}$
20. $f(x,y) = \dfrac{\cos(x)}{x+y}$
21. $f(x,y) = \sqrt{x+y^2}$

22. $f(x,y) = x \cdot \ln(y+3)$ 23. $f(x,y) = e^{(y^2)}$ 24. $f(x,y) = \cos(x^2)$

In problems 25-28, evaluate the double integrals. One order of integration may be easier than the other.

25. $\iint_R (2y^2 - 3xy^3)\ dA$ where $R = \{ (x,y) : 1 \le x \le 2, 0 \le y \le 3 \}$.

26. $\iint_R x \sin(y)\ dA$ where $R = \{ (x,y) : 1 \le x \le 4, 0 \le y \le \pi/6 \}$.

27. $\iint_R x \sin(x+y)\ dA$ where $R = [0, \pi/6] \times [0, \pi/6]$.

28. $\iint_R \frac{1}{x+y}\ dA$ where $R = [1,2] \times [0,1]$.

In problems 29-34 determine the average value of f(x,y) on the given domain.

29. $f(x,y) = 3 + 4x + 2y$ on $R = [0,2]X[0,3]$ 30. $f(x,y) = 7 + 3x - 6y$ on $R = [0,2]X[1,2]$

31. $f(x,y) = x^2 + y^2$ on $R = [0,2]X[0,3]$ 32. $f(x,y) = \cos(x+y)$ on $R = [0,\pi]X[0,\pi]$

In problems 33 and 34 the depths (in meters) of a small rectangular pond are shown. Give a good estimate of the volume of water in the pond. You should be able to justify why your estimate is reasonable.

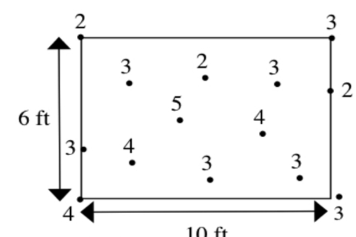


Fig. 10

33. Length = 4 m. Width = 3 m. Depths at the four corners are 2.4, 3.3, 2.1 and 3.2 m.

34. Depths (ft) are shown at various locations (Fig. 10).

In problems 35 and 36 some height contours are shown for a small hill. Give a good estimate of the volume of the hill. You should be able to justify why your estimate is reasonable.

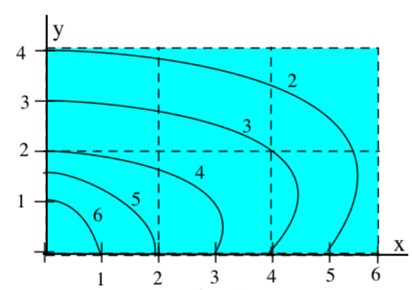


Fig. 11

35. See Fig. 11. All of the measurements are in meters.

36. See Fig. 12. All of the measurements are in meters.

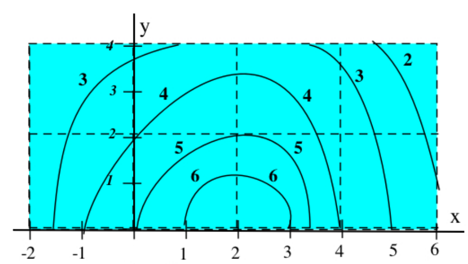


Fig. 12

Practice Answers

Practice 1: $\int_1^2 \int_0^3 20 - x^2 y \, dx \, dy$ means $\int_1^2 \left\{ \int_0^3 20 - x^2 y \, dx \right\} dy$ so first we evaluate the inside integral $\int_0^3 x^2 y \, dx$ **treating y as a constant**:

$$\int_0^3 20 - x^2 y \, dx = 20x - \tfrac{1}{3} x^3 y \Big|_{x=0}^{3} = 20(3) - \tfrac{1}{3}(3)^3 y = 60 - 9y .$$

Then $\int_1^2 \left\{ \int_0^3 x^2 y \, dx \right\} dy = \int_1^2 \left\{ 60 - 9y \right\} dy$

$$= 60y - \frac{9}{2} y^2 \Big|_{y=1}^{2} = \left(60(2) - \frac{9}{2}(2)^2\right) - \left(60(1) - \frac{9}{2}(1)^2\right) = (102) - \frac{111}{2} = \frac{93}{2}.$$

Practice 2: In Example 2 we evaluated $\int_0^\pi \int_1^2 3 + y \cdot \sin(xy) \, dx \, dy = 3\pi$.

The area of D is $(2-1)\cdot(\pi - 0) = \pi$ so the average value is $(3\pi)/\pi = 3$.

Practice 3: $\int_a^b \int_c^d 1 \, dx \, dy = \int_a^b x \Big|_{x=c}^{x=d} dy = \int_a^b (d - c) \, dy = (d - c) \cdot y \Big|_{y=a}^{y=b} = (d-c)\cdot(b-a)$

This is the area of the rectangular domain.

Practice 4: $\int_{-a}^{a} \int_{-b}^{b} Ax + By \; dy \; dx = \int_{-a}^{a} Axy + \frac{B}{2} y^2 \Big|_{y=-b}^{y=b} dx = \int_{-a}^{a} 2abx \; dx = abx^2 \Big|_{x=-a}^{x=a} = 0$

So the volume between the plane f(x,y)=Ax+By and D that is above the xy=plane is the same as the volume that is below the xy-plane.