14.3 DOUBLE INTEGRALS IN POLAR COORDINATES

As you learned in earlier classes some shapes (Fig. 1) are much simpler to describe using polar coordinates, and sometimes we need to calculate volumes of regions whose domains are those shapes. The process is straightforward, but first we need a way to partition a polar coordinate region. Since the polar variables are r and θ, it is natural to partition the domain into "polar rectangles" (Fig. 2) and then proceed as we did in Section 14.1 to build Riemann Sums and, by taking limits, get to double integrals.

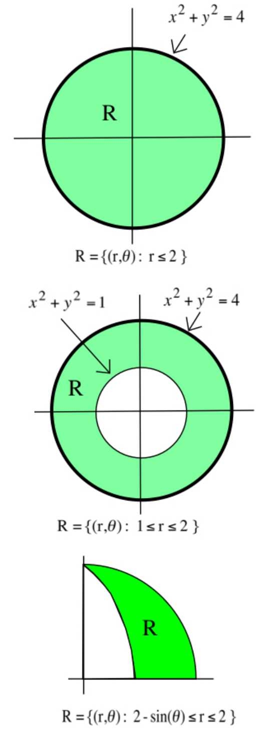


Fig. 1

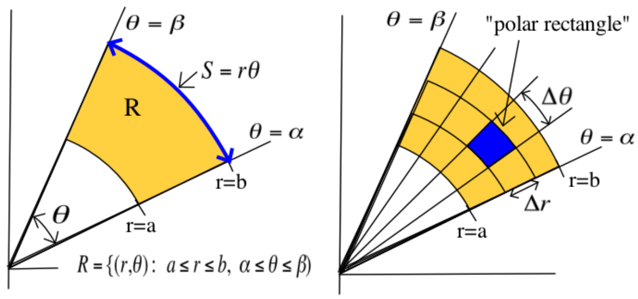


Fig. 2: Polar Rectangle

The area of each sub-rectangle is approximately $\Delta A \approx r\cdot \Delta r\cdot \Delta\theta$ (Fig. 3) so the volume above each sub-rectangle will be $\Delta V = f(x,y)\cdot \Delta A$. But we are in polar coordinates, so we need to replace x and y with their polar coordinate values $x = r\cdot \cos(\theta)$ and $y = r\cdot \sin(\theta)$.

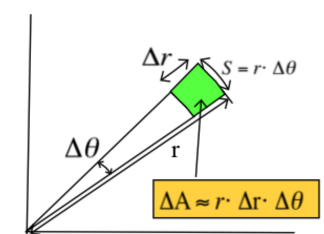


Fig. 3: Partitioned Sub-rectangle

Then the total volume is approximately the value of the double Riemann sum

$$\text{volume} \approx \sum \sum f(x,y)\cdot \Delta A = \sum_{r} \sum_{\theta} f(r\cdot \cos(\theta), r\cdot \sin(\theta))\cdot \Delta A .$$

Taking limits as Δr and $\Delta\theta$ both approach 0, the exact volume is a double integral.

Volume in Polar Coordinates

If $f(x,y) \geq 0$ is continuous on the polar rectangle $R = \{ (r,\theta) : a \leq r \leq b,\ \alpha \leq \theta \leq \beta \}$, then

The volume between the domain R and the surface $z = f(x,y)$ is

$$\text{Volume} = \iint_R f(x,y)\cdot dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=a}^{r=b} f(r\cdot\cos(\theta),\ r\cdot\sin(\theta))\cdot r\cdot dr\cdot d\theta$$

This may seem complicated but it is simply volume $=$ height$\cdot$ (base area) with height=function (in polar coordinates) and base area $= \mathrm{r} \cdot \mathrm{dr} \cdot \mathrm{d}\theta$. (Note: Don't forget the r in the base area.)

Example 1: Find the volume between the plane f(x,y)=5+x+2y and the circular region $R = \{(x,y) : x^2 + y^2 \le 4\}$. (Fig. 4)

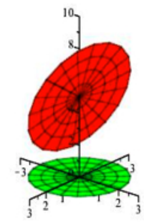


Fig. 4

Solution: Translating the problem into polar coordinates we have

$\mathrm{f(x,y)} = \mathrm{f(r\cdot \cos(\theta), r\cdot \sin(\theta))} = 5 + \mathrm{r}\cdot \cos(\theta) + 2\cdot \mathrm{r}\cdot \sin(\theta)$

and $\mathrm{R} = \{(r,\theta):\ 0 \le r \le 2,\ 0 \le \theta \le 2\pi\ \}$ so

$$\text{volume} = \iint_R \mathrm{f(x,y)} \cdot \mathrm{dA} = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \{5 + \mathrm{r}\cdot \cos(\theta) + 2\cdot \mathrm{r}\cdot \sin(\theta)\}\cdot r\cdot dr\cdot d\theta$$

which can be evaluated in the usual way starting with the inside integral.

$$\int_{r=0}^{r=2} \{5 + \mathrm{r}\cdot \cos(\theta) + 2\cdot \mathrm{r}\cdot \sin(\theta)\}\cdot r\cdot dr = \int_{r=0}^{r=2} \{5r + \mathrm{r}^2\cdot \cos(\theta) + 2\cdot \mathrm{r}^2\cdot \sin(\theta)\}\ dr$$

$$= \frac{5}{2}\mathrm{r}^2 + \frac{1}{3}\mathrm{r}^3\cdot \cos(\theta) + \frac{2}{3}\mathrm{r}^3\cdot \sin(\theta)\ \Big|_{r=0}^{r=2} = \frac{5}{2}(4) + \frac{1}{3}(8)\cdot \cos(\theta) + 2\cdot \frac{1}{3}(8)\cdot \sin(\theta)$$

Then $$\int_{\theta=0}^{\theta=2\pi} 10 + \frac{8}{3}\cdot \cos(\theta) + \frac{16}{3}\cdot \sin(\theta)\ \mathrm{d}\theta = 10\theta + \frac{8}{3}\cdot \sin(\theta) - \frac{16}{3}\cdot \cos(\theta)\Big|_{\theta=0}^{\theta=2\pi}$$

$$= \{20\pi + 0 - \frac{16}{3}\} - \{0 + 0 - \frac{16}{3}\} = 20\pi \approx 62.83 \ .$$

The volume is $20\pi \approx 62.83$ **.**

Practice 1: If we restrict the domain to $R = \{(x,y) : x^2 + y^2 \le 4,\ 0 \le y\}$ then the polar coordinate form of the domain is $\mathrm{R} = \{(r,\theta):\ 0 \le r \le 2,\ 0 \le \theta \le \pi\ \}$. Verify that the volume for the same plane f(x,y)=5+x+2y over this new domain is $\frac{32}{3} + 10\pi \approx 42.08$. Since the domain R in Practice 1 is half of the domain R in Example 1, why isn't the new volume half of the previous volume?

Example 2: Find the volume between the surface $\mathrm{f(x,y)} = \mathrm{e}^{-(\mathrm{x}^2+\mathrm{y}^2)}$ and the circular domain $R = \{(x,y) : x^2 + y^2 \le 4\}$. (Fig. 5)

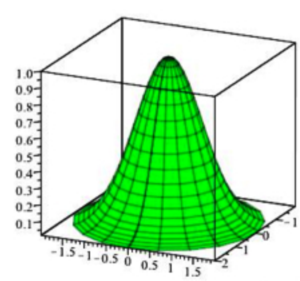


Fig. 5: $\mathrm{f(x,y)} = \mathrm{e}^{-(\mathrm{x}^2+\mathrm{y}^2)}$

Solution: This function looks complicated, but when we translate into polar coordinates, things get much easier.

$x = r\cdot \cos(\theta)$ and $y = r\cdot \sin(\theta)$ so $x^2 + y^2 = r^2$. And R becomes $R = \{(r,\theta):\ 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$. Then

$$\text{Volume} = \iint_R f \cdot dA = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} e^{-r^2} \cdot r \cdot dr \cdot d\theta \cdot$$

The inside integral $\displaystyle\int_{r=0}^{r=2} e^{-r^2} \cdot r \cdot dr = -\frac{1}{2} e^{-r^2} \Big|_{r=0}^{r=2} = \frac{1}{2}\left(1-e^{-4}\right)$ (using the substitution $u=-r^2$)

$$\text{Volume} = \iint_R f \cdot dA = \int_{\theta=0}^{\theta=2\pi} \frac{1}{2}\left(1-e^{-4}\right) \cdot d\theta = \frac{1}{2}\left(1-e^{-4}\right) \cdot \theta \Big|_{\theta=0}^{\theta=2\pi}$$

$$= \frac{1}{2}\left(1-e^{-4}\right) \cdot \theta \Big|_{\theta=0}^{\theta=2\pi} = \left(1-e^{-4}\right)\pi \approx 0.98\pi \text{ .}$$

Note: The x^2+y^2 in the rectangular function is often a signal that the polar version may be easier since $x^2+y^2=r^2$. Then the substitution $u=r^2$ means $du = 2 \cdot r \cdot dr$ so $dA = r \cdot dr \cdot d\theta = \frac{1}{2} d\theta$.

Practice 2: (a) Find the volume between the surface $z=f(x,y) = e^{-(x^2+y^2)}$ and the circular domain $R = \{(x,y) : x^2+y^2 \le A^2\}$, a circle of radius A centered at the origin.

(b) Show that as $A \to \infty$ then Volume $\to \pi$. This means the volume between the surface $z=f(x,y) = e^{-(x^2+y^2)}$ and the entire xy-plane is π.

Areas and Average Values with Double Integrals in Polar Coordinates

If the z=f(x,y) is always equal to 1, then $\displaystyle\text{Volume} = \iint_R 1 \cdot dA = \{\text{base area}\}\{\text{height} = 1\} = \text{base area}$.

Area in Polar Coordinates

If R is a closed and bounded region in the polar coordinate plane, then

$$\text{Area of R} = \iint_R 1 \cdot dA = \iint_R 1\, r \cdot dr \cdot d\theta \qquad \text{In polar coordinates } dA = r \cdot dr \cdot d\theta$$

The average value of a continuous function z=f(x,y) over a polar coordinate region is the same as we have used for rectangular coordinate regions: Average Value = $\frac{1}{\text{area}} \cdot$ volume.

Average Value of a Function in Polar Coordinates

If R is a closed and bounded region in the polar coordinate plane, and fx,y) is continuous on R, then

$$\text{Average Value of f on R} = \frac{1}{\text{area of R}} \cdot \{\text{volume between f and R}\} = \frac{\iint_R f\, dA}{\iint_R 1\, dA}$$

Example 3: Let $z = f(x,y) = e^{-(x^2+y^2)}$ be the height of a solid ice sculpture over the circular base $R = \{(x,y) : x^2 + y^2 \le 4\}$. (This is Example 2.) If this sculpture is in a water-tight cylinder and then melts, how high will the resulting water be in the cylinder? (Fig. 6)

Solution: We have already done the calculus for this volume in Example 2: volume = $0.98\,\pi$ so we just need to divide this volume by the area of the circular base, $4\,\pi$. The average height is approximately $0.98/4 = 0.245$.

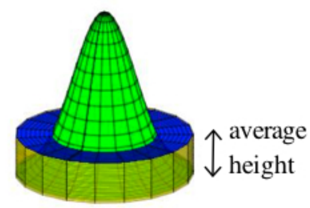


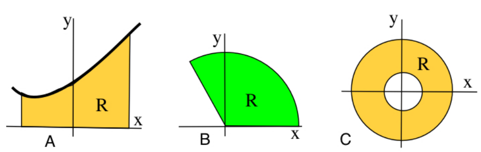
Fig. 6

Practice 3: Find the average value of $z=f(x,y)=5+x+2y$ on the domain $R = \{(x,y) : x^2 + y^2 \le 4\}$. (This is Example 1.)

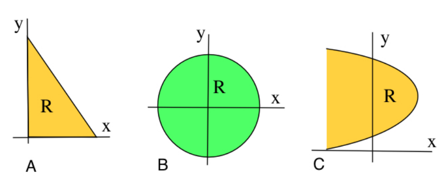
Problems

For each given region R, decide whether to use polar or rectangular coordinates to evaluate a double integral with domain R.

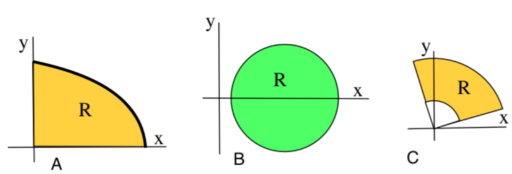
1.



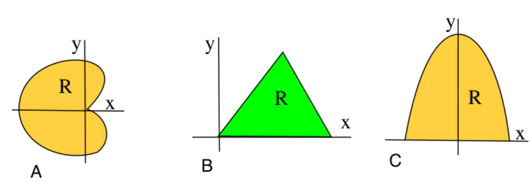
2.



3.



4.



For Problems 5 to 8 find the volume between the surface $z=f(x,y)$ and the given region R.

5. $f(x,y) = 7 + 3x + 2y$ with $R = \left\{(x,y) :\ x^2 + y^2 \le 9\right\}$.

6. $f(x,y) = 9 - 2x + 4y$ with $R = \left\{(x,y) :\ x^2 + y^2 \le 5\right\}$.

7. $f(x,y) = A + Bx + Cy$ with $R = \left\{(x,y) :\ x^2 + y^2 \le D^2\right\}$. (A, B, C, D are positive constants.)

8. $f(x,y) = A + Bx + Cy$ with $R = \left\{(x,y) :\ E^2 \le x^2 + y^2 \le D^2\right\}$. (A, B, C, D, E are positive constants.)

For Problems 9 to 12 sketch the domain of integration, use polar coordinates to evaluate each double integral.

9. Evaluate $\iint_R \sqrt{9-x^2-y^2}\ dA$ where $R=\{(x,y):\ x^2+y^2\le 9\}$.

10. Evaluate $\iint_R \sin(x^2+y^2)\ dA$ where $R=\{(x,y):\ 0\le x,\ 0\le y,\ x^2+y^2\le 9\}$.

11. Evaluate $\iint_R 2+xy\ \ dA$ where R is the region in inside the circle $x^2+y^2=4$ and above the x-axis..

12. Evaluate $\iint_R e^{x^2+y^2}\ dA$ where R is the set of (x,y) more than 1 unit and less than 3 units from the origin.

For Problems 13 to 18 use polar coordinates to find the volume of each solid.

13. Under the plane $z=5+2x+3y$ and above the disk $x^2+y^2\le 16$.

14. Under the paraboloid $z=x^2+y^2$ and above the disk $x^2+y^2\le 9$ for (x,y) in the first quadrant.

15. Between the surfaces $z=1+x+y$ and $z=8+2x+3y$ for $x^2+y^2\le 1$ and $0\le$ y.

16. Above the paraboloid $z=6-x^2-y^2$ and under the plane z = 9 for $x^2+y^2\le 4$.

17. Between the surface $z=f(x,y)=\dfrac{1}{1+x^2+y^2}$ and the xy-plane for (x,y) in the first quadrant and $1\le x^2+y^2\le 4$.

18. (a) Between $z=\dfrac{1}{\left(1+x^2+y^2\right)^3}$ and the disk $x^2+y^2\le C^2$.

(b) Between $z=\dfrac{1}{\left(1+x^2+y^2\right)^3}$ and the entire xy-plane.

In problems 19 to 24, change the rectangular coordinate integral into an equivalent polar integral and evaluate the polar integral. It is usually helpful to sketch the domain of the integral.

19. $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\ dx$

20. $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)\ dy\ dx$

21. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} y\ dy\ dx$

22. $\displaystyle\int_{0}^{1/\sqrt{2}}\int_{y}^{\sqrt{1-y^2}} (x^2+y^2)\ dx\ dy$

23. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} e^{-(x^2+y^2)}\ dy\ dx$

24. $\displaystyle\int_{0}^{\infty}\int_{0}^{\infty} e^{-(x^2+y^2)}\ dy\ dx$

For Problems 25 to 29, find the average value of the function on the given region R.

25. $f(x,y) = 7+3x+2y$ with $R = \{(x,y):\ x^2+y^2 \le 9\}$.

26. $f(x,y) = 2+xy$ for $R = \{\text{top half of the disk } x^2+y^2 \le 4\}$.

27. $f(x,y) = 5+2x+3y$ with $R = \{(x,y):\ x^2+y^2 \le 16\}$.

28. $f(x,y) = x^2+y^2$ for R={part of the disk $x^2+y^2 \le 9$ in the first quadrant}

29. A sprinkler (located at the origin) sprays water so after one hour the depth at location (x,y) feet is $f(x,y) = K\cdot e^{-(x^2+y^2)}$ feet.

(a) How much water reaches the annulus $2 \le r \le 4$ and the annulus $8 \le r \le 10$ in one hour?

(b) What is the average amount of water (depth per square foot) of water at (x,y) in each annulus after one hour?

(c) Why is this a poor design for a sprinkler?

30. $f(x) = K\cdot e^{-\left(\frac{x^2}{2}\right)}$ is the normal probability distribution for a population with mean 0 and standard deviation 1, and is extremely important in probability theory and applications. Unfortunately, it does not have a "nice" antiderivative in terms of elementary functions, but we can use double integrals in polar coordinates to evaluate $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\ dx$.

(a) The rectangular coordinate double integral of $f(x,y) = e^{-(x^2+y^2)} = e^{-x^2}\cdot e^{-x^2}$ with

R={square -C≤x≤C, -C≤y<C} is

$$\int_{-C}^{C}\int_{-C}^{C} e^{-(x^2+y^2)}\ dx\cdot dy = \int_{-C}^{C}\int_{-C}^{C} e^{-x^2}\cdot e^{-y^2}\ dx\cdot dy$$

$$= \int_{-C}^{C} e^{-y^2}\left(\int_{-C}^{C} e^{-x^2}dx\right)dy = \left(\int_{-C}^{C} e^{-x^2}dx\right)\left(\int_{-C}^{C} e^{-y^2}dy\right) = \left(\int_{-C}^{C} e^{-x^2}dx\right)^2.$$

So the

$$\iint_{xy-plane} e^{-(x^2+y^2)}\ dA = \lim_{C\to\infty}\int_{-C}^{C}\int_{-C}^{C} e^{-(x^2+y^2)}\ dx\cdot dy = \lim_{C\to\infty} = \left(\int_{-C}^{C} e^{-x^2}dx\right)^2 = = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2$$

But from Practice 2 we know that $\displaystyle\iint_{xy-plane} e^{-(x^2+y^2)}\ dA = \pi$ so $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.

Finally, changing the variable to $u = \dfrac{x}{\sqrt{2}}$ we get $\displaystyle\int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2}\right)}dx = \int_{-\infty}^{\infty} e^{-u^2}\sqrt{2}\cdot du = \sqrt{2\pi}$.

That was a lot of work, but this is a very important integral.

Practice Answers

Practice 1: The only difference from Example 1 is that the domain angle θ goes from 0 to π instead of from 0 to 2π so the calculation is the same until we get to the final evaluation:

$$\text{volume} = \iint_R f(x,y)\cdot dA = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2} \{5 + r\cdot\cos(\theta) + 2\cdot r\cdot\sin(\theta)\}\cdot r\cdot dr\cdot d\theta$$

$$= 10\theta + \frac{8}{3}\cdot\sin(\theta) - \frac{16}{3}\cdot\cos(\theta)\Big|_{\theta=0}^{\theta=\pi} = \left\{10\pi + 0 + \frac{16}{3}\right\} - \left\{0 + 0 - \frac{16}{3}\right\} = 10\pi + \frac{32}{3}$$

The new domain may be half the area of the original domain, but the function is larger over the new domain than over the original domain. Symmetry is very powerful, but we need to be careful and only use it when it is justified.

Practice 2: (a) The only change from Example 2 is that now the radius r goes from 0 to A instead of from 0 to 2 so most of the calculations are the same:

$$\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=A} e^{-r^2}\cdot r\cdot dr\cdot d\theta = \frac{1}{2}\left(1 - e^{-A^2}\right)\theta\ \Big|_{\theta=0}^{\theta=2\pi} = \left(1 - e^{-A^2}\right)\pi\ .$$

(b) $\lim\limits_{A\to\infty}$ Volume $= \lim\limits_{A\to\infty}\left(1 - e^{-A^2}\right)\pi = \pi$

Practice 3: From Example 1 we know the volume is 20π, and the area of the circular base is 4π so the average value is 5. (Fig. 7)

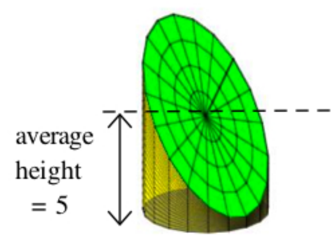


Fig. 7: f(x,y)=5+x+2y