14.5 SURFACE AREAS USING DOUBLE INTEGRALS

surface of revolution

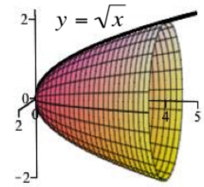

Fig. 1

In Section 5.2 we determined a method for calculating the area of a surface of revolution (Fig. 1). Here we will build a way to calculate the area of a surface of the form z = f(x,y) over a region R (Fig. 2). Sometimes both methods can be used and in that case they both give the same result. There is another type of surface called a parametric surface (Fig. 3) that is even more general, and we will build a way to determine those surface areas in Section 15.7.

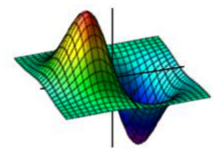

Fig. 2

parametric surface

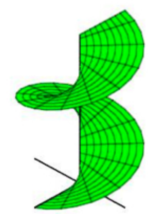

Fig. 3

To make the derivation and figures simpler, we assume that R is a rectangle in the xy-plane and that z=f(x,y)≥0 in R. As we have done before, we start by partitioning the domain into small Δx by Δy rectangles and note that the area ΔS of the tangent plane (Fig. 4) above one of these little ΔA rectangles has approximately the same area as the surface area above the ΔA rectangle. If we can represent the sides of the tilted tangent plane above the ΔA rectangle as vectors, then we can use the cross product of those vectors to determine the area of the rectangle.

Starting at one corner (x,y) of a little rectangle in the domain, and moving Δx in the x direction, the vector A along the tangent plane is $A = \langle \Delta x,\ 0,\ f_x(x,y)\cdot\Delta x\rangle$.

Similarly, starting at (x,y) and moving in the y direction, the vector B along the tangent plane is

$B = \langle 0,\ \Delta y,\ f_y(x,y)\cdot\Delta y\rangle$. Then the area of the tangent plane above the ΔA rectangle is the magnitude of the cross product of A and B:

$$\boxed{\Delta S = \{\text{tangent plane area}\} = |\,AxB\,|\ .}$$

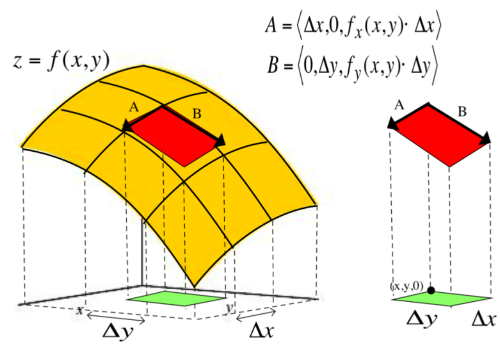

Fig. 4

Example 1: The function in Fig, 4 is $f(x,y) = 7 - \frac{1}{2}x^2 - y^2$. Determine the area of the little rectangle at the point (2,1,4) with $\Delta x = 0.3$ and $\Delta y = 0.1$.

Solution: $f_x(2,1) = -2$ and $f_y(2,1) = -2$ so $A = \langle 0.3,\ 0,\ -0.6\rangle$ and $B = \langle 0,\ 0.1,\ -0.2\rangle$.

$$AxB = \begin{vmatrix} i & j & k \\ 0.3 & 0 & -0.6 \\ 0 & 0.1 & -0.2 \end{vmatrix} = \langle 0.06,\ 0.06,\ 0.03\rangle \ \text{ so area} = |\,AxB\,| = 0.09\ ,$$

Practice 1: Determine the area of the little rectangle for the Example 1 function at the point (1, 1, 5.5) with $\Delta x = 0.2$ and $\Delta y = 0.3$.

In the general case $A = \langle \Delta x,\ 0,\ f_x(x,y) \rangle$ and $B = \langle 0,\ \Delta y,\ f_y(x,y) \rangle$ so

$$\text{AxB} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \Delta\mathrm{x} & 0 & \mathrm{f_x}(x,y)\cdot \Delta x \\ 0 & \Delta\mathrm{y} & \mathrm{f_y}(x,y)\cdot \Delta y \end{vmatrix} = \langle -\Delta\mathrm{y}\cdot \mathrm{f_x}\cdot \Delta x,\ -\Delta\mathrm{x}\cdot \mathrm{f_y}\cdot \Delta y,\ \Delta\mathrm{x}\cdot \Delta\mathrm{y} \rangle = \langle -\mathrm{f_x},\ -\mathrm{f_y},\ 1 \rangle \cdot \Delta A$$

and $|\,\text{AxB}\,| = \left(\sqrt{(f_x)^2 + (f_y)^2 + 1} \right) \cdot \Delta A = \Delta \text{S}$.

The total surface area is the accumulation of all of these little areas:

Surface area $A \approx \sum \sum \Delta\text{S} = \sum \sum \left(\sqrt{(f_x)^2 + (f_y)^2 + 1} \right) \cdot \Delta A \to \iint\limits_R \left(\sqrt{(f_x)^2 + (f_y)^2 + 1} \right) \cdot \text{dA}$

$$\text{For } z=f(x,y),\ \text{Surface Area} = \iint\limits_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \cdot dA$$

You should recognize the similarity of this formula to the formula for arc length from Section 5.2:

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx\ ,$$

Example 2: Determine the surface area of the plane $f(x,y) = 1 + 2x + y$ over the rectangular region $R = \{(x,y):\ 0 \le x \le 2,\ 0 \le y \le 3\}$. (Fig. 5)

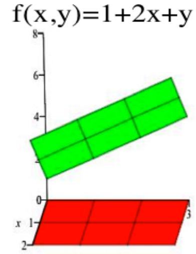

Fig. 5

Solution: $\dfrac{\partial z}{\partial x} = 2$ and $\dfrac{\partial z}{\partial y} = 1$ so

Surface area $= \int_{x=0}^{2} \int_{y=0}^{3} \sqrt{1 + (2)^2 + (1)^2}\ dy\ dx = \int_{x=0}^{2} 3\sqrt{6}\ dx = 6\sqrt{6}$

Practice 2: Determine the surface area of the plane $f(x,y) = 3 + 8x + 4y$ over the rectangular region $R = \{(x,y):\ 0 \le x \le 4,\ 0 \le y \le 3\}$.

The surface area formula also works for domains that are not rectangular, and sometimes polar coordinates make the evaluation easier.

Example 3: Determine the surface area of the plane $f(x,y) = 10 - 2x - 3y$ over the circular disk $R = \{(x,y):\ 0 \le x^2 + y^2 \le 4\}$. (Fig. 6)

f(x,y)=10-2x-3y

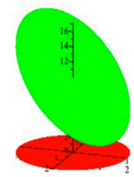

Fig. 6

Solution: $\dfrac{\partial z}{\partial x} = -2$ and $\dfrac{\partial z}{\partial y} = -3$ so Surface area = $\displaystyle\iint_R \sqrt{1 + (-2)^2 + (-3)^2} \cdot dA$

Because of the symmetry of R, this is easier to evaluate using polar coordinates:

Surface area = $\displaystyle\int_{\theta=0}^{2\pi}\int_{r=0}^{2} \sqrt{14}\ \ r \cdot dr \cdot d\theta = \int_{\theta=0}^{2\pi} 2\sqrt{14} \cdot d\theta = 4\pi\sqrt{14}$.

Example 4: Determine the surface area of the saddle $f(x,y) = 5 + x^2 - y^2$ over the circular disk $R = \{(x,y):\ 0 \le x^2 + y^2 \le 4\}$. (Fig. 7)

$f(x,y) = 5 + x^2 - y^2$

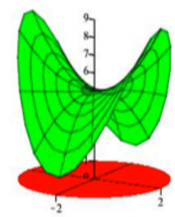

Fig. 7

Solution: $\dfrac{\partial z}{\partial x} = 2x$ and $\dfrac{\partial z}{\partial y} = -2y$ so

Surface area = $\displaystyle\iint_R \sqrt{1 + (2x)^2 + (-2y)^2} \cdot dA = \iint_R \sqrt{1 + 4(x^2 + y^2)} \cdot dA$

Again, polar coordinates make this easier:

Surface area = $\displaystyle\int_{\theta=0}^{2\pi}\int_{r=0}^{2} \sqrt{1 + 4r^2}\ \cdot r \cdot dr \cdot d\theta$

$$= \int_{\theta=0}^{2\pi} \frac{1}{12}(17)^{3/2} - \frac{1}{12}\ d\theta = \left(\frac{1}{12}(17)^{3/2} - \frac{1}{12}\right) \cdot 2\pi$$

Practice 3: Determine the surface area of the paraboloid $f(x,y) = 1 + x^2 + y^2$ over the circular disk $R = \{(x,y):\ 0 \le x^2 + y^2 \le 9\}$. (Fig. 8)

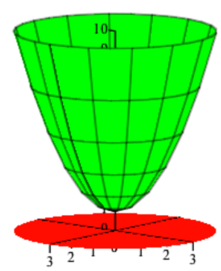

Fig. 8

These examples and practice problems were chosen so that it was possible to evaluate the integrals "by hand." Unfortunately that is rarely the situation. Once we take partial derivatives, square them, add those squares and a 1, and then take a square root the result is usually not an integral that we can evaluate by hand, at least not easily If we just change the Example 4 function slightly to be $f(x,y) = 5 + 2x - y^2$ then $f_x(x,y) = 2$ and $f_y(x,y) = -2y$ so the surface area integral is $\displaystyle\iint_R \sqrt{5 + 4y^2}\ \ dA$. This is a rather difficult antidrivative which involves the inverse hyperbolic sine function. In many cases the surface area involves integrals that do not have antiderivatives involving only elementary functions and we need to resort to software such as Maple or Mathematica.

Example 5: The formula for Fig. 2 is $f(x,y) = x\cdot e^{-x^2-y^2}$ and the graph of f is over the rectangle $-2\le x\le 2$ and $-2\le y\le 2$. Represent the surface area using double integrals.

Solution: $f_x(x,y) = x\cdot\left(e^{-x^2-y^2}\right)(-2x) + \left(e^{-x^2-y^2}\right) = (-2x^2+1)\cdot e^{-x^2-y^2}$ and

$$f_y(x,y) = x\cdot\left(e^{-x^2-y^2}\right)(-2y) = -2xy\cdot e^{-x^2-y^2} \text{ so}$$

Surface area $= \displaystyle\int_{x=-2}^{2}\int_{y=-2}^{2}\sqrt{1+\left((-2x^2+1)^2+4x^2y^2\right)\left(e^{-x^2-y^2}\right)^2}\ dy\ dx$ Yuck.

But the Maple command

int(sqrt(1+ ((1-2*x^2)^2+4*x^2*y^2)*(exp(-x^2-y^2))^2), x=-2 .. 2, y=-2 ..2);

quickly gives the result 16.72816232 .

Problems

1. Find the area of the surface $f(x,y) = x^2 + y$ over the triangular domain bounded by the x-axis, the line x=2 and the line y=2x.
2. Find the area of the surface $f(x,y) = x^2 + 3y$ over the triangular domain bounded by the x-axis, the line x=2 and the line y=x.
3. Find the area of the surface $f(x,y) = 4x + y^2$ over the triangular domain with vertices (0,0), (0,4) and (2,4).
4. Find the area of the surface $f(x,y) = x + 3y^2$ over the triangular domain with vertices (0,0), (0,4) and (2,4).
5. Find the area of the surface $f(x,y) = 1 + 3x + 4y$ over the domain bounded by the x-axis and the parabola $y = 4 - x^2$.
6. Find the area of the surface $f(x,y) = 2 + 12x + y$ over the domain bounded by the x-axis and the parabola $y = 1 - x^2$.
7. Find the area of the surface $f(x,y) = 5 + 4x + 3y$ over the circular domain $R = \{(x,y):\ x^2 + y^2 \le 9\}$.
8. Find the area of the surface $f(x,y) = 1 + x + y$ over the circular domain $R = \{(x,y):\ x^2 + y^2 \le 4\}$.
9. Find the area of the surface $f(x,y) = 1 + x + y$ over the domain $R = \{(x,y):\ 1 \le x^2 + y^2 \le 9\}$.
10. Find the area of the cone $f(x,y) = \sqrt{x^2 + y^2}$ over the circular domain $R = \{(x,y):\ x^2 + y^2 \le 4\}$.
11. Find the area of the cone $f(x,y) = \sqrt{x^2 + y^2}$ over the domain $R = \{(x,y):\ 1 \le x^2 + y^2 \le 9\}$.
12. Find the area of the paraboloid $f(x,y) = x^2 + y^2$ over the circular domain $R = \{(x,y):\ x^2 + y^2 \le 4\}$.

In problems 13 to 16 set up the integrals representing the surface area of the given function on the given domain. (These may be too difficult to evaluate by hand.)

13. $f(x,y) = x \cdot y^2$ on the domain $R = \{(x,y):\ -2 \le x \le 2,\ x^2 \le y \le 4\}$.
14. $f(x,y) = x^2 + y^3$ on the domain $R = \{(x,y):\ -2 \le x \le 2,\ x^2 \le y \le 4\}$.
15. $f(x,y) = 2 + \sin(x) + \cos(y)$ on the domain $R = \{(x,y):\ 0 \le x \le 2,\ 0 \le y \le 3\}$.
16. $f(x,y) = 2 + x^3 - y^2$ on the domain $R = \{(x,y):\ 0 \le x \le 2,\ 0 \le y \le 1\}$.

Practice Answers

Practice 1: $f_x(1,1) = -1$ and $f_y(1,1) = -2$ so $A = \langle 0.2,\ 0,\ -0.2 \rangle$ and $B = \langle 0,\ 0.3,\ -0.6 \rangle$

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$$\text{AxB} = \begin{vmatrix} i & j & k \\ 0.2 & 0 & -0.2 \\ 0 & 0.3 & -0.6 \end{vmatrix} = \langle 0.06,\ 0.12,\ 0.06 \rangle \text{ so area} = |\text{AxB}| \approx 0.146 .$$

Practice 2: $f(x,y) = 3 + 8x + 4y$ so $\dfrac{\partial z}{\partial x} = 8$ and $\dfrac{\partial z}{\partial y} = 4$ then

$$\text{Surface area} = \int_{x=0}^{4} \int_{y=0}^{3} \sqrt{1 + (8)^2 + (4)^2}\ dy\ dx = \int_{x=0}^{4} 27\ dx = 108$$

Practice 3: $\dfrac{\partial z}{\partial x} = 2x$ and $\dfrac{\partial z}{\partial y} = 2y$ so

$$\text{Surface area} = \iint_R \sqrt{1 + (2x)^2 + (2y)^2} \cdot dA = \iint_R \sqrt{1 + 4(x^2 + y^2)} \cdot dA$$

Again, polar coordinates make this easier (with a $u = 1 + 4r^2$ substitution).

$$\text{Surface area} = \int_{\theta=0}^{2\pi} \int_{r=0}^{3} \sqrt{1 + 4r^2} \cdot r \cdot dr \cdot d\theta = \int_{\theta=0}^{2\pi} \frac{1}{12}(37)^{3/2} - \frac{1}{12}\ d\theta = \left(\frac{1}{12}(37)^{3/2} - \frac{1}{12} \right) \cdot 2\pi$$