14.6 TRIPLE INTEGRALS AND APPLICATIONS

Sometimes the value of a continuous function f(x,y,z) depends on the location in 3 dimensions: perhaps we know the density (kg/m^3) at each location (x,y,z) of a 3D object and want to determine the total mass (kg) of the object. Everything in this section is in rectangular coordinates. The next section considers triple integrals in cylindrical and spherical coordinates.

Our strategy is similar to that used to create double integrals, except now our region R is a 3D solid and we partition R into small rectangular cells (boxes) by cuts parallel to the coordinate planes. (Fig. 1) Then the volume of each little box is $\Delta V = \Delta x \cdot \Delta y \cdot \Delta z$. If we want the mass of the little box, it is approximately the density δ at some point (x*, y*, z*) inside each box times the volume of the box:

$\Delta M = \delta$ (x*, y*, z*)$\cdot \Delta V = \delta$ (x*, y*, z*)$\cdot \Delta x \cdot \Delta y \cdot \Delta z$. By adding the approximate masses of all of the boxes together, a triple sum, we can approximate the total mass of the solid:

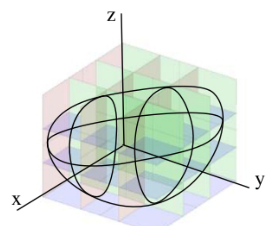
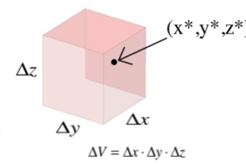


Fig. 1: Solid in 3D



sample cell

$$M \approx \sum_{\Delta z} \sum_{\Delta y} \sum_{\Delta x} \delta(x^*,y^*,z^*) \cdot \Delta x \cdot \Delta y \cdot \Delta z$$

As before, letting all of the side lengths of the boxes approach 0, we get a triple integral:

$$\underset{\Delta x,\Delta y,\Delta z \to 0}{\text{limit}} \sum_{\Delta z} \sum_{\Delta y} \sum_{\Delta x} \delta(x^*,y^*,z^*) \cdot \Delta x \cdot \Delta y \cdot \Delta z \to \iiint_R \delta(\text{x,y,z})\ \text{dV} \cdot$$

Triple integrals have all of the properties that you might expect, and these properties follow from the properties of finite sums.

1) $\iiint_R \text{k} \cdot \text{f dV} = k \cdot \iiint_R \text{f dV}$

2) $\iiint_R \{\text{f} \pm \text{g}\}\ \text{dV} = \iiint_R \text{f}\ \ dV \pm \iiint_R \text{g dV}$

3) If f≥g for all points in R, then $\iiint_R \text{f dV} \geq \iiint_R \text{g dV}$

4) If the R is partitioned into two subregions R1 and R2 by a smooth surface, then $\iiint_R \text{f dV} = \iiint_{R1} \text{f dV} + \iiint_{R2} \text{f dV}$

5) Volume of R = $\iiint_R 1\ \text{dV}$

6) Units of $\iiint_R \text{f dV}$ are (units of f) . (units of x). (units of y). (units of z)

Evaluating Triple Integrals

Triple integrals are rarely evaluated as limits of triple sums. Instead, we evaluate single integrals, working from the inside out just as we did with double integrals.

Example 1: Evaluate $\iiint_R \mathrm{f\,dV}$ for $f(x,y,z) = 2x + y + z^2$ on the solid

$$R = \{(x,y,z):\ 0 \le x \le 2,\ 1 \le y \le 4,\ 0 \le z \le 1\}$$

Solution: $\iiint_R \mathrm{f\,dV} = \int_{z=0}^{1} \int_{y=1}^{4} \int_{x=0}^{2} 2x + y + z^2 \,dx\,dy\,dz$

Starting on the inside, $\int_{x=0}^{2} 2x + y + z^2 \ dx = x^2 + xy + xz^2 \Big|_{x=0}^{2} = 4 + 2y + 2z^2$

Next, $\int_{y=1}^{4} \int_{x=0}^{2} 2x + y + z^2 \ dx \ dy = 4y + y^2 + 2yz^2 \Big|_{y=1}^{4} = 12 + 15 + 6z^2$

Finally, $\int_{z=0}^{1} 27 + 6z^2 \ dz = 27z + 2z^3 \Big|_{z=0}^{1} = 29$.

If the x, y and z units are meters, and the f units are kg/m^3 , then $\iiint_R \mathrm{f\,dV}$ = 35 kg.

> 3D Fubini's Theorem
>
> If f is continuous on the domain R,
>
> then the triple integral can be evaluated in any order that describes R.

In the case of a box [a,b]x[c,d]x[e,f], that means any order of dx, dy and dz gives the same result as long as the end points match the variable: a≤x≤b, c<y≤d and e≤z≤f.

Practice 1: Evaluate $\iiint_R x^3 + y^2 + z \ dV$ for $R = \{(x,y,z):\ 0 \le x \le 2,\ 0 \le y \le,\ 1 \le z \le 3\}$

using two different orders of integration.

Often the most difficult part of working with triple integrals is setting up the order and endpoints of integration . **Keep in mind that the outer integral must have constant endpoints, the middle integral can have at most one variable in the endpoints, and the inside integral can have at most two variables in the end points.**

If D is a region in the xy-plane, and $R = \{(x,y,z):\ (x,y) \text{ is in D and } f(x,y) \le z \le g(x,y)\}$

as in Fig. 2, then $\iiint_R \mathrm{F(x,y,z)\,dV} = \iint_D \left\{ \int_{z=f}^{z=g} F(x,y,z)\ dz \right\} dA$

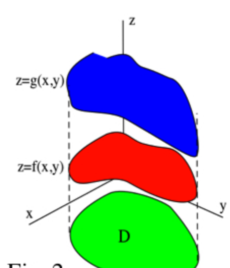


Fig 2

Example 2: Determine the volume of the region bounded by $0 \le x \le 1$, $0 \le y \le 3$ and z between plane $z = f(x,y) = 5 - 2x - y$ and the surface $z = g(x,y) = 13 - 3x^2 - y^2$ as shown in Fig. 3.

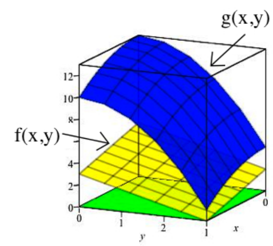


Fig. 3

Solution: The inside integral must be **dz** with z from $z = f(x,y) = 5 - 2x - y$ to $z = g(x,y) = 13 - 3x^2 - y^2$. The outer integral can be either **dx** or **dy** since each of them is bounded by constants.

$$\text{volume} = \iiint_R 1\, dV = \int_{x=0}^{1} \int_{y=0}^{3} \int_{z=5-2x-y}^{z=13-3x^2-y^2} 1 \; dz\, dy\, dx = \int_{x=0}^{1} \int_{y=0}^{3} (8 - 3x^2 - y^2 + 2x + y) \; dy\, dx$$

$$= \int_{x=0}^{1} (-9x^2 + \frac{39}{2} + 6x)\, dx = \frac{39}{2}$$

Practice 2: Write the triple integral that represents the volume of the region bounded by $0 \le x \le 1$, $0 \le y \le 3-3x$ and z between plane $z = f(x,y) = 5 - 2x - y$ and the surface $z = g(x,y) = 13 - 3x^2 - y^2$. How does this region differ from the one in Example 2?

Example 3: Write an iterated triple integral for f(x,y,z) in the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane z=4 (Fig. 4).

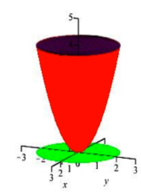


Fig. 4: Solid paraboloid

Solution: The domain in the xy-plane is the circle $x^2 + y^2 \le 4$ which can be described as $-2 \le x \le 2$ and $-\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}$. And z goes from $x^2 + y^2$ to 4. Putting this information together, the triple integral is

$$\iiint_R f(x,y,z)\; dV = \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=x^2+y^2}^{4} f(x,y,z)\; dz\; dy\; dx$$

Note that the endpoints of the outside integral had no variables, the middle endpoints have one variable, and the inside endpoints have two variables.

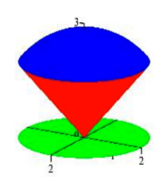


Fig. 5: Cone topped by sphere

Practice 3: (a) Write an iterated triple integral for f(x,y,z) in the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 8$ (Fig. 5).

(b) Write an iterated triple integral for this function and solid that is only in the first octant.

Applications of Triple Integrals

These are similar to the applications of single and double integrals and are very useful in some applications.

Volume of a solid region R = $\iiint_R 1 \, dV$

Average value of f on a solid region R $= \dfrac{1}{\text{volume of R}} \cdot \iiint_R f \; dV$

If $\delta = \delta(x,y,z)$ is the density of a solid region R at the location (x,y,z) then Mass = M = $\iiint_R \delta \, dV$

First moments about the coordinate planes:

$$M_{yz} = \iiint_R x \cdot \delta \, dV \qquad M_{xz} = \iiint_R y \cdot \delta \, dV \qquad M_{xy} = \iiint_R z \cdot \delta \, dV$$

Center of Mass: $\bar{x} = \dfrac{M_{yz}}{M}$ $\quad \bar{y} = \dfrac{M_{xz}}{M}$ $\quad \bar{z} = \dfrac{M_{xy}}{M}$

Second moments (moments of inertia):

$$I_x = \iiint_R (y^2+z^2)\cdot \delta \; dV \qquad I_y = \iiint_R (x^2+z^2)\cdot \delta \; dV \qquad I_z = \iiint_R (x^2+y^2)\cdot \delta \; dV$$

about line L: $I_L = \iiint_R r^2 \cdot \delta \; dV$ where r(x,y,z) = distance of (x,y,z) from line L

Radius of gyration about a line L: $R_L = \sqrt{I_L / M}$

Example 4: A 1 cm by 1 cm (0≤y≤1, 0≤z≤1) bar along the x-axis has a length of 6 cm (0≤x≤6). The density of the bar is $\delta(x,y,z) = 1 + x$ g/ cm^3 . Determine (a) the mass M, (b) M_{yz} , (c) $\bar{x}$, (d) I_z and (e) R_z .

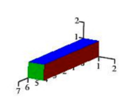


Fig. 6: Bar along the x-axis

Solution: (a) mass $= \displaystyle\int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{6} (1+x) \; dx\, dy\, dz = 24$ g

(b) $M_{yz} = \displaystyle\iiint_R x \cdot \delta \, dV = \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{6} x\cdot(1+x) \; dx\, dy\, dz = 90 \text{ g}\cdot\text{cm}$

(c) $\bar{x} = \dfrac{M_{yz}}{M} = \dfrac{90 \;\; g\cdot cm}{24} = \dfrac{15}{4} = 3.75 \quad cm$ The bar balances on a fulcrum at location x=3.75 cm.

(d) $I_z = \displaystyle\iiint_R (x^2+y^2)\cdot \delta \; dV = \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{6} (x^2+y^2)\cdot(1+x) \; dx\, dy\, dz = 404 \quad g\cdot cm^2$

(e) $R_z = \sqrt{I_z / M} = \sqrt{404/24} \approx 4.10 \quad cm$ The kinetic energy of this bar rotating around the z-axis is the same as a point mass of 24 g located at x=4.1 cm rotating around the z-axis with the same angular speed.

Based on each integral in the example, you should be able to justify the units attached to the numerical result.

Practice 4: A cube ($0 \le x \le 2$, $0 \le y \le 2$, $0 \le z \le 2$ cm) has density $\delta(x,y,z) = 1 + x + y + z^2$ g/ cm^3 .

Determine (a) the mass M, (b) the moment about each coordinate plane, and (c) the center of mass of the cube.

Example 5: $\iiint\limits_R \sqrt{x^2+y^2}\ dV$ where R is the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane z=4 (Fig. 4).

Solution: This is the solid from Example 2 so

$$\iiint\limits_R f(x,y,z)\ dV = \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=x^2+y^2}^{4} \sqrt{x^2+y^2}\ dz\ dy\ dx$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-(x^2+y^2))\cdot\sqrt{x^2+y^2}\ \ dy\ dx$$

But the domain of this remaining double integral is the disk $0 \le x^2 + y^2 \le 2$ so it is useful to switch to polar coordinates. Then we have

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \left((4-r^2)\cdot\sqrt{r^2}\right) r\ dr\ d\theta = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \left(4r^2 - r^4\right) dr\ d\theta = \int_{\theta=0}^{2\pi} \frac{64}{15}\ d\theta = \frac{128}{15}\pi\ .$$

Problems

In Problems 1 to 6, set up the appropriate iterated integrals for $\iiint\limits_R f\ dV$ on the indicated domains.

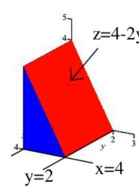


Fig. 7: Solid prism

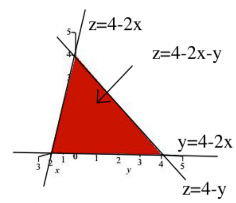


Fig. 8: Solid tetrahedron

1. R is the solid prism in Fig. 7.

2. R is the solid tetrahedron in Fig. 8.

3. R is the solid cone in Fig. 9.

4. R is the solid sliced cylinder in Fig. 10.

5. R is the solid in Fig. 11.

6. R is the solid in Fig. 12.

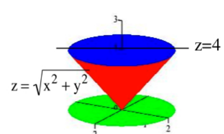


Fig. 9: Solid cone

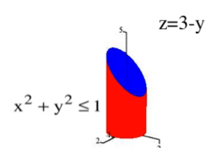


Fig. 10: Truncated cylinder

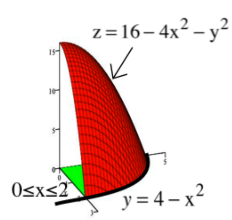


Fig. 11

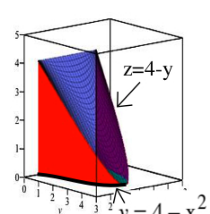


Fig. 12

In problems 7 to 18, evaluate the integrals.

7. $\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}\left(x^2+y^2+z\right)\,dz\,dy\,dx$

8. $\int_{z=1}^{e}\int_{y=1}^{e}\int_{x=1}^{e}\frac{1}{xyz}\,dx\,dy\,dz$

9. $\int_{0}^{1}\int_{0}^{x-2}\int_{0}^{3-x-y}dz\,dy\,dx$

10. $\int_{0}^{1}\int_{0}^{\pi}\int_{0}^{\pi}y\cdot\cos(z)\,dx\,dy\,dz$

11. $\int_{0}^{1}\int_{0}^{2}\int_{0}^{x+z}(12xz)\,dy\,dx\,dz$

12. $\int_{0}^{2}\int_{x}^{2x}\int_{0}^{y}(6xyz)\,dz\,dy\,dx$

13. $\int_{0}^{3}\int_{0}^{1}\int_{0}^{1-z}z\cdot e^{y}\,dx\,dz\,dy$

14. $\int_{0}^{3}\int_{0}^{1}\int_{0}^{1-z}x\cdot e^{y}\,dx\,dz\,dy$

15. $\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2}}1\,dy\,dz\,dx$

16. $\int_{0}^{4}\int_{0}^{\ln(y)}\int_{\ln(y)}^{\ln(3y)}e^{x+y-z}\,dx\,dz\,dy$

17. $\int_{0}^{2}\int_{0}^{3}\int_{0}^{y}(2x+4y+6z)\,dz\,dy\,dx$

18. $\int_{0}^{3}\int_{0}^{1}\int_{0}^{y}(4xy)\,dx\,dz\,dy$

For problems 19 to 22 write the Maple command to evaluate the triple integral.

19. f(x,y)=2x+3 on the domain of Problem 19.

20. f(x,y)=sin(xy)+z on the domain of Problem 20.

21. f(x,y)=xyz on the domain of Problem 21.

22. f(x,y)=1 on the domain of Problem 22.

Practice Answers

Practice 1: Evaluate $\iiint\limits_R x^3+y^2+z \; dV$ for $R=\{(x,y,z):\; 0\le x\le 2,\; 0\le y\le ,\; 1\le z\le 3\}$.

$$\iiint\limits_R x^3+y^2+z \; dV = \int_{z=1}^{3}\int_{y=0}^{1}\int_{x=0}^{2} x^3+y^2+z \; dx \; dy \; dz$$

$$\int_{x=0}^{2} x^3+y^2+z \; dx = 4+2y^2+2z, \quad \int_{y=0}^{1} 4+2y^2+2z \; dy = 4+\frac{2}{3}+2z,$$

and finally $\displaystyle\int_{z=1}^{3} \frac{14}{3}+2z \; dz = \frac{28}{3}+8$.

Check that integrating in some other order gives the same result.

Try $\displaystyle\int_{x=0}^{2}\int_{z=1}^{3}\int_{y=0}^{1} (x^3+y^2+z)\text{ dy dz dx}$ and $\displaystyle\int_{y=0}^{1}\int_{x=0}^{2}\int_{z=1}^{3} (2x+y+z^2)\text{ dz dx dy}$

Practice 2: $\displaystyle\text{volume} = \int_{x=0}^{1}\int_{y=0}^{3-3x}\int_{z=5-2x-y}^{z=13-3x^2-y^2} 1 \text{ dz dy dx}$ (= 23/2)

The domain is now a triangle in the xy-plane (Fig. P2).

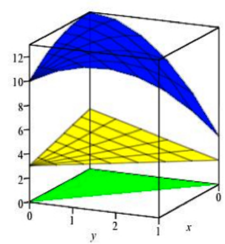


Fig. P2

Practice 3: (a) $\displaystyle\iiint\limits_R f\,dV = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} f(x,y,z)\text{ dz dy dx}$

(b) $\displaystyle\int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}}\int_{z=\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} f(x,y,z)\text{ dz dy dx}$

Practice 4: Since x, y and z have constant endpoints, the integration can be done in any order.

(a) $\displaystyle\text{Mass} = \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2}\left(1+x+y+z^2\right)\text{ dz dy dx} = \frac{104}{3}\; g$

(b) $\displaystyle M_{yz} = \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} x\cdot\left(1+x+y+z^2\right)\text{ dz dy dx} = \frac{112}{3}\; g\cdot cm,\; M_{xz}=M_{yz}$

$\displaystyle M_{xy} = \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} z\cdot\left(1+x+y+z^2\right)\text{ dz dy dx} = 40\; g\cdot cm$

(c) $\displaystyle\bar{x} = \frac{M_{yz}}{M} = \frac{14}{13}\; cm,\; \bar{y}=\bar{x}=\frac{14}{13}\; cm,\; \bar{z}=\frac{M_{xy}}{M}=\frac{15}{13}\; cm$