14.7 TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

"In physics everything is straight, flat or round." Statement by a physics teacher

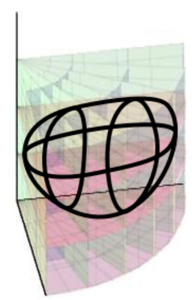
Maybe not, but lots of applications have pieces of round or spherical domains, and using cylindrical or spherical coordinates can make many triple integrals much easier to evaluate.

$\iiint$ f dV **in Cylindrical Coordinates**

If our domain of integration is round or is easily described using polar coordinates (r, θ), then a triple integral in cylindrical coordinates (r, θ, z) is often the best method, and it begins with the form of the ΔV. Fig. 1 illustrates that if we partition each of r, θ and z, then the volume of each little cell is $\Delta V = (r \cdot \Delta\theta) \cdot \Delta r \cdot \Delta z$. Then the triple Riemann sum is

$\sum \sum \sum f(r^*,\theta^*,z^*) \cdot r \cdot \Delta r \cdot \Delta\theta \cdot \Delta z$. If f is continuous, then the limit of this Riemann sum as $\Delta r,\ \Delta\theta,\ \Delta z \to 0$ is the triple integral in cylindrical coordinates

$$\iiint_R f\, dV = \int_z \int_\theta \int_r f(r,\theta,z)\ r\ dr\ \ d\theta\ \ dz\ .$$



Partition of r, θ and z

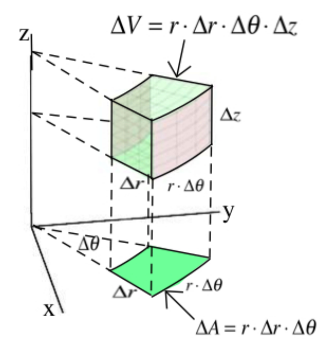


Fig. 1

As before, we can alter the order of the integrals as long as we accurately describe the domain of integration. Also, the outer integral endpoints must be constants, the middle integral endpoints can have only one variable, and the inner integral endpoints can have two variables.

(Note: Recall that $x = r \cdot \cos(\theta)$, $y = r \cdot \sin(\theta)$ so $x^2 + y^2 = r^2$ and z=z .)

Example 1: Evaluate (a) $\iiint_R f\, dV$ for $f(r,\theta,z) = r \cdot z$ with $R = \{1 \le r \le z,\ 0 \le \theta \le \pi,\ 0 \le z \le 2\}$

and (b) $\displaystyle\int_{r=0}^{3} \int_{\theta=0}^{2\pi} \int_{z=0}^{e^{-r^2}} 1\ r\ dz\ \ d\theta\ \ dr$

Solution: (a) $$\iiint_R f\, dV = \int_{z=0}^{2} \int_{\theta=0}^{\pi} \int_{r=1}^{z} (r \cdot z)\ r\ dr\ \ d\theta\ \ dz = \int_{z=0}^{2} \int_{\theta=0}^{\pi} \left(\frac{1}{3} r^3\right) \Big|_{r=1}^{z}\ d\theta\ \ dz = \int_{z=0}^{2} \frac{1}{3}\left(z^4 - z\right)\pi\ \ dz = \frac{22}{15}\pi$$

(b) $$\int_{r=0}^{3} \int_{\theta=0}^{2\pi} \int_{z=0}^{e^{-r^2}} 1\ r\ dz\ \ d\theta\ \ dr = \int_{r=0}^{3} \int_{\theta=0}^{2\pi} \left(e^{-r^2}\right)\ r\ \ d\theta\ \ dr = \int_{r=0}^{3} 2\pi\left(e^{-r^2}\right)\ r\ \ dr = -\pi e^{-r^2} \Big|_{r=0}^{3} = \pi\left(1 - e^{-9}\right)$$

Practice 1: Evaluate (a) $\iiint\limits_R dV$ for $R = \{0 \le r \le 2, 0 \le \theta \le \pi/2, 0 \le z \le 8 - r^3\}$ and

(b) the volume of the solid cylinder above the disk $x^2 + y^2 \le 4$ and below the plane z=4-y.

Example 2: R is the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane z=4 (Fig. 2).

(a) Write and evaluate an iterated triple integral for the volume of R.

(b) Write and evaluate an iterated triple integral for the mass of R if the density is $\delta(x,y,x) = 1 + z$.

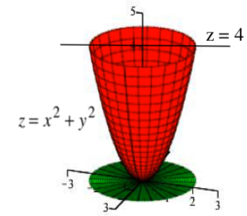


Fig. 2: Solid paraboloid

Solution: (a) The domain of this integral is the circle $x^2 + y^2 \le 4$ so $r^2 \le 4$ and $0 \le \theta \le 2\pi$:

$$\iiint\limits_R f\,dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=r^2}^{4} (1)\ r\ dz\ dr\ d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} (z\cdot r)\Big|_{z=r^2}^{4}\ dr\ d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} \left(4r - r^3\right)\ dr\ d\theta = 8\pi$$

(b) $$\text{mass} = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=r^2}^{4} (1+z)\cdot r\ dz\ dr\ d\theta = \frac{88}{3}\pi$$

Practice 2: R is the solid hemisphere $x^2 + y^2 + z^2 \le 4$ with $z \ge 0$ (Fig. 3).

(a) Write and evaluate an iterated triple integral for the volume of R.

(b) Write and evaluate an iterated triple integral for the mass of R if the density is $\delta(x,y,x) = 1 + z$.

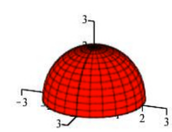


Fig. 3: Solid hemisphere

Example 3: Find the centroid of the solid that is bounded below by the disk $x^2 + y^2 \le 9$ and above by the paraboloid $z = x^2 + y^2$.

Solution: $x^2 + y^2 \le 9$ means $0 \le r \le 3$, and $z = x^2 + y^2$ means $z = r^2$, the domain is

$$R = \{(r,\theta,z):\ 0 \le r \le 3,\ 0 \le \theta \le 2\pi,\ 0 \le z \le r^2\}$$

$$\text{mass} = \int_{\theta=0}^{2\pi}\int_{r=0}^{3}\int_{z=0}^{r^2} r\cdot dz\cdot dr\cdot d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{3} r\cdot z\Big|_{z=0}^{r^2}\ dr\cdot d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{3} r^3\ dr\cdot d\theta = \frac{81}{2}\pi$$

$$M_{xy} = \int_{\theta=0}^{2\pi}\int_{r=0}^{3}\int_{z=0}^{r^2} z\cdot r\ \ dz\cdot dr\cdot d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{3} r\cdot\frac{z^2}{2}\Big|_{z=0}^{r^2}\ dr\cdot d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{3} \frac{r^5}{2}\ dr\cdot d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{3} \frac{r^6}{12}\Big|_{t=0}^{3}\ d\theta = 2\pi\left(\frac{729}{12}\right) = \frac{243}{2}\pi\ .$$

Then $\bar{z} = \dfrac{(243/2)\pi}{(81/2)\pi} = 3$. Because of the symmetry about the z-axis, $\bar{x}$ and $\bar{y}$ are both 0 so the centroid is (0, 0, 3).

$\iiint$ f dV **in Spherical Coordinates**

This development is very similar to what was done for cylindrical coordinates. First we partition our domain R into $(\Delta\rho,\Delta\theta,\Delta\varphi)$ cells, pick a representative point $(\rho^*,\theta^*,\varphi^*)$ in each cell, form the triple Riemann sum $\sum\sum\sum f(\rho^*,\theta^*,\varphi^*)\ \Delta V$, and, finally, take the limit as all of the cell dimensions approach 0 in order to form a triple integral: $\lim\limits_{\Delta\to 0}\sum\sum\sum f(\rho^*,\theta^*,\varphi^*)\ \Delta V \to \iiint\limits_R f(\rho,\theta,\varphi)\cdot dV$

But before we can actually use this idea, we first need to determine dV in terms of the variables ρ, θ and φ. That is a bit complicated and is derived in the Appendix of this section as well as in the next section when Jacobeans are introduced . In either case, $dV = \rho^2\cdot\sin(\varphi)\cdot d\rho\cdot d\theta\cdot d\varphi$, and then

$$\iiint\limits_R f(\rho,\theta,\varphi)\cdot dV = \iiint\limits_R f(\rho,\theta,\varphi)\cdot\rho^2\cdot\sin(\varphi)\cdot d\rho\cdot d\theta\cdot d\varphi$$

As before, the we can use any order of integration that describes the domain as long as the outside integral has constant endpoints, and the middle integral has at most one variable endpoint The inside integral can have two variable endpoints.

Example 4: Represent each domain R using iterated triple integrals.

(a) R is shown in Fig. 4a, (b) R is shown in Fig. 4b.

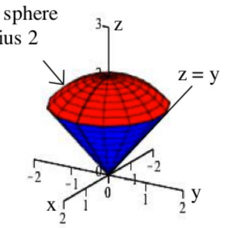


Fig. 4a

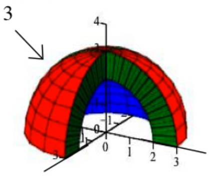


Fig. 4b

Solution: (a) $R = \{(\rho,\theta,\varphi):\ 0\le\rho\le 2,\ 0\le\theta\le 2\pi,\ 0\le\varphi\le\pi/4\}$

$$\iiint\limits_R f\ \ dV = \int_{\varphi=0}^{\pi/4}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{2} f(\rho,\theta,\varphi)\ \ \rho^2\cdot\sin(\varphi)\cdot d\rho\cdot d\theta\cdot d\varphi$$

(b) $R = \{(\rho,\theta,\varphi):\ 2\le\rho\le 3,\ \pi/2\le\theta\le 2\pi,\ 0\le\varphi\le\pi/2\}$

$$\iiint\limits_R f\ \ dV = \int_{\varphi=0}^{\pi/2}\int_{\theta=\pi/2}^{2\pi}\int_{\rho=2}^{3} f(\rho,\theta,\varphi)\ \ \rho^2\cdot\sin(\varphi)\cdot d\rho\cdot d\theta\cdot d\varphi$$

All of these integral endpoints are constants so the integrals can done be in any order.

Practice 3: Represent each domain R using iterated triple integrals.

(a) R is shown in Fig. 5a, (b) R is shown in Fig. 5b.

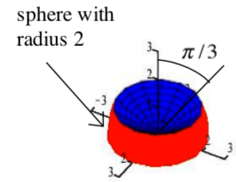


Fig. 5a

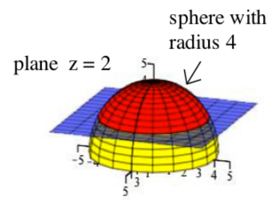


Fig. 5b

Example 5: Determine the mass and $\bar{z}$ for the solid hemisphere with radius 2 that is above the xy-plane and has density $\delta(x,y,x) = 1 + z$.

Solution: $R = \{(\rho,\theta,\varphi): \ 0 \le \rho \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi/2\}$ so

$$\text{mass} = \int_{\varphi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{2} (1+z)\ \rho^2 \cdot \sin(\varphi) \cdot d\rho \cdot d\theta \cdot d\varphi$$

$$= \int_{\varphi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{2} (1+\rho\cdot\cos(\varphi))\ \rho^2 \cdot \sin(\varphi) \cdot d\rho \cdot d\theta \cdot d\varphi$$

$$= \int_{\varphi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{2} \left(\rho^2 \cdot \sin(\varphi) + \rho^3 \cdot \cos(\varphi) \cdot \sin(\varphi)\right)\ d\rho \cdot d\theta \cdot d\varphi$$

$$= \int_{\varphi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \left(\frac{8}{3} \cdot \sin(\varphi) + 4 \cdot \cos(\varphi) \cdot \sin(\varphi)\right) \cdot d\theta \cdot d\varphi$$

$$= \int_{\varphi=0}^{\pi/2} 2\pi\left(\frac{8}{3} \cdot \sin(\varphi) + 4 \cdot \cos(\varphi) \cdot \sin(\varphi)\right) \cdot d\varphi = 2\pi\left(2 \cdot \sin^2(\varphi) - \frac{8}{3} \cdot \cos(\varphi)\right)\Bigg|_{\varphi=0}^{\pi/2}$$

$$= 2\pi(2) + 2\pi\left(\frac{8}{3}\right) = \frac{28}{3}\pi$$

$$M_{xy} = \int_{\varphi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{2} z \cdot (1+z)\ \rho^2 \cdot \sin(\varphi) \cdot d\rho \cdot d\theta \cdot d\varphi = \frac{124}{15}\pi \qquad \text{(using Maple)}$$

so $\bar{z} = \dfrac{(124/15)\pi}{(28/3)\pi} = \dfrac{32}{35} \approx 0.914$

Conclusion: Even a simple looking problem can take a long time.

These conversion formulas for cylindrical and spherical coordinates are useful.

Coordinate conversion formulas

Cylindrical to Rectangular	Spherical to Cylindrical	Spherical to Rectangular
$x = r \cdot \cos(\theta)$	$r = \rho \cdot \sin(\varphi)$	$x = \rho \cdot \sin(\varphi) \cdot \cos(\theta)$
$y = r \cdot \sin(\theta)$	$z = \rho \cdot \cos(\varphi)$	$y = \rho \cdot \sin(\varphi) \cdot \sin(\theta)$
$z = z$	$\theta = \theta$	$z = \rho \cdot \cos(\varphi)$

$$dV = dx \cdot dy \cdot dz = r \cdot dr \cdot d\theta \cdot dz = \rho^2 \cdot \sin(\varphi) \cdot d\rho \cdot d\theta \cdot d\varphi$$

Problems

In problems 1-8, evaluate the triple integrals in cylindrical coordinates.

1. $\int_0^{\pi}\int_0^{1}\int_0^{\sqrt{2-r^2}} r\ dz\cdot dr\cdot d\theta$

2. $\int_0^{4}\int_0^{\pi}\int_r^{6} r\ dz\cdot d\theta\cdot dr$

3. $\int_0^{\pi}\int_0^{\theta/\pi}\int_0^{\sqrt{4-r^2}} z\ dz\cdot dr\cdot d\theta$

4. $\int_0^{\pi}\int_0^{1}\int_{-1/2}^{1/2} \left(r^2\cdot\sin(\theta)+z^2\right)\ dz\cdot dr\cdot d\theta$

5. $\int_0^{2\pi}\int_0^{3}\int_0^{\pi/3} r^3\ dr\cdot dz\cdot d\theta$

6. $\int_{-1}^{1}\int_0^{2\pi}\int_0^{1+\sin(\theta)} 2r\ dr\cdot d\theta\cdot dz$

7. $\int_0^{2}\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi} (r\cdot\sin(\theta)+1)\cdot r\ d\theta\cdot dz\cdot dr$

8. $\int_0^{\pi}\int_r^{2r}\int_0^{\pi} r\cdot\cos(\theta)\ d\theta\cdot dz\cdot dr$

In problems 9 to 12, evaluate the integrals in cylindrical coordinates.

9. $\int_0^{4}\int_0^{\sqrt{2}/2}\int_x^{\sqrt{1-x^2}} e^{-(x^2+y^2)}\ dy\cdot dx\cdot dz$

10. $\int_0^{4}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\ dy\cdot dx\cdot dz$

11. $\int_0^{1}\int_0^{\sqrt{1-x^2}}\int_0^{4-y} 1\ dz\cdot dy\cdot dx$

12. $\int_{-2}^{2}\int_0^{\sqrt{4-x^2}}\int_0^{1} \cos(x^2+y^2)\ dz\cdot dy\cdot dx$

In problems 13 to 18, set up and evaluate the triple integrals in cylindrical coordinates.

13. $f(x,y)=\sqrt{x^2+y^2}$. R is the region inside the cylinder $x^2+y^2=9$ and between the planes z=3 and z=5.

14. $f(x,y)=(x^3+xy^2)$. R is the region in the first octant and under the paraboloid $z=4-x^2-y^2$.

15. $f=e^z$. R is the region enclosed by paraboloid $z=1+x^2+y^2$, the cylinder $x^2+y^2=7$, and the xy-plane.

16. $f=x^2$. R is the region inside the cylinder $x^2+y^2=4$, below the cone $z^2=9x^2+9y^2$ and above the xy-plane.

17. Find the volume of the region R in first octant below the $z=x^2+y^2$ and above $z=36-3x^2-3y^2$.

18. $\mathrm{f(x,y)}=6+4\mathrm{x}^2+4\mathrm{y}^2$. R is the region in the 2nd, 3rd and 4th octants, inside the cylinder $x^2+y^2=1$, and between the planes z=2 and z=3.

Now spherical

In problems 19 to 26 evaluate the integrals in spherical coordinates.

19. $\int_0^{\pi}\int_0^{\pi}\int_0^{2\cdot\cos(\varphi)} \rho^2\cdot\sin(\varphi)\ d\rho\cdot d\varphi\cdot d\theta$

20. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} \rho^3\cdot\sin^2(\varphi)\ d\rho\cdot d\varphi\cdot d\theta$

21. $\int_0^{\pi}\int_0^{\pi}\int_0^{1} 5\rho^3\cdot\sin^3(\varphi)\ d\rho\cdot d\varphi\cdot d\theta$

22. $\int_0^{\pi}\int_0^{\pi/3}\int_{\sec(\varphi)}^{1} 3\rho^2\cdot\sin(\varphi)\ d\rho\cdot d\varphi\cdot d\theta$

23. $\displaystyle\int_0^{\pi/2}\int_0^{\pi}\int_1^{2} \rho^2\cdot\sin(\varphi)\ d\rho\cdot d\theta\cdot d\varphi$

24. $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{1} e^{\left(\rho^2\right)}\cdot\rho\cdot\sin(\varphi)\ d\rho\cdot d\theta\cdot d\varphi$

25. $\displaystyle\int_0^{\pi}\int_0^{\pi/3}\int_0^{\cos(\varphi)} 4\rho^3\cdot\sin(\varphi)\ d\rho\cdot d\varphi\cdot d\theta$

26. $\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\int_0^{\csc(\varphi)} \sin(\varphi)\ d\rho\cdot d\varphi\cdot d\theta$

Practice Answers

Practice 1: (a) $\displaystyle\int_{r=0}^{2}\int_{\theta=0}^{\pi/2}\int_{z=0}^{8-r^3} \mathrm{r}\ dz\ d\theta\ d\mathrm{r} = \int_{z=0}^{2}\int_{\theta=0}^{\pi/2} (z\cdot r)\Big|_{z=0}^{8-r^3}\ d\theta\ d\mathrm{r} = \int_{z=0}^{2}\int_{\theta=0}^{\pi/2} 8r-r^4\ d\theta\ d\mathrm{r}$

$\displaystyle = \int_{z=0}^{2} \frac{\pi}{2}(8r-r^4)\ d\mathrm{r}\ = \frac{\pi}{2}\left(\frac{48}{5}\right)$

(b) $y = r\cdot\sin(\theta)$ so

$\displaystyle \text{volume} = \int_{r=0}^{2}\int_{\theta=0}^{2\pi}\int_{z=0}^{4-r\cdot\sin(\theta)} 1\ \mathrm{r}\ dz\ d\theta\ d\mathrm{r} = \int_{r=0}^{2}\int_{\theta=0}^{2\pi} (4-r\cdot\sin(\theta)\cdot r\ d\theta\ d\mathrm{r} = \int_{r=0}^{2} 2\pi(4r)\ d\mathrm{r} = 16\pi$

Practice 2: $z^2 \le 4-(\mathrm{x}^2+\mathrm{y}^2) = 4-r^2$, and, as in Example 1, $0\le r\le 2$ and $0\le\theta\le 2\pi$.

(a) $\displaystyle \text{volume} = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{\sqrt{4-r^2}} (1)\cdot\mathrm{r}\ dz\ dr\ d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} z\cdot r\Big|_{z=0}^{\sqrt{4-r^2}}\ dr\ d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} \sqrt{4-r^2}\cdot r\ dr\ d\theta$

$\displaystyle = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} \sqrt{4-r^2}\cdot r\ dr\ d\theta = \int_{\theta=0}^{2\pi}\left(-\frac{1}{3}\left(4-r^2\right)^{3/2}\right)\Big|_{r=0}^{2}\ d\theta = \int_{\theta=0}^{2\pi}\frac{8}{3}\ d\theta = \frac{16}{3}\pi$

(b) $\displaystyle \text{mass} = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{\sqrt{4-r^2}} (1+z)\cdot\mathrm{r}\ dz\ dr\ d\theta\ = \frac{28}{3}\pi$

Practice 3: (a) $R = \{(\rho,\theta,\varphi):\ 0\le\rho\le 2,\ 0\le\theta\le 2\pi,\ \pi/3\le\varphi\le\pi/2\}$

$$\int_{\varphi=\pi/3}^{\pi/2}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{2} f(\rho,\theta,\varphi)\ \rho^2\cdot\sin(\varphi)\cdot d\rho\cdot d\theta\cdot d\varphi$$

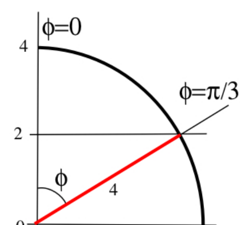


Fig. Pr4a

(b) Clearly $0\le\theta\le 2\pi$, but φ and ρ require a bit of work (Fig. Pr4):

$R = \{(\rho,\theta,\varphi):\ 2\sec(\varphi)\le\rho\le 4,\ 0\le\theta\le 2\pi,\ 0\le\varphi\le\pi/3\}$

$$\int_{\varphi=0}^{\pi/3}\int_{\theta=0}^{2\pi}\int_{\rho=2\sec(\varphi)}^{4} f(\rho,\theta,\varphi)\ \rho^2\cdot\sin(\varphi)\cdot d\rho\cdot d\theta\cdot d\varphi$$

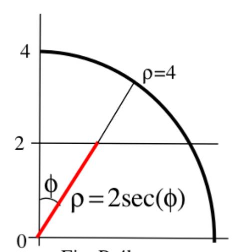


Fig. Pr4b

Appendix : Why $dV = \rho^2 \cdot \sin(\varphi) \cdot d\rho \cdot d\theta \cdot d\varphi$

The figures A1-A3 are an attempt to explain where this strange equation comes from. We need to partition the space by partitioning each of the three variables ρ , θ and φ. This results in cells as shown greatly magnified in Fig. A1. Using different views of a typical cell in Fig. A2, it is possible to determine the lengths of the sides of this cell. Putting all of this together in Fig. A3, the volume of the cell, ΔV ia the product of the lengths of the sides:

$$dV = \rho^2 \cdot \sin(\varphi) \cdot d\rho \cdot d\theta \cdot d\varphi$$

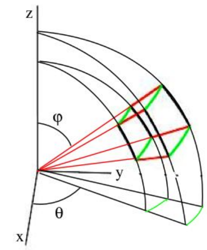


Fig. A1

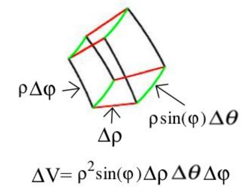


Fig. A3

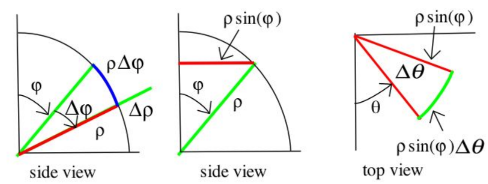


Fig. A2