14.8 Changing Variables in Double and Triple Integrals

In section 4.5 we saw how a change of variables could make some integrals easier to evaluate: the integral $\int x\sqrt{3-x^2}\ dx$ is made easier by using the substitution $u = 3 - x^2$. Similarly, in section 14.3 we saw that converting some integrals from rectangular to polar coordinates can make them easier: $\iint \sqrt{x^2+y^2}\ dx\ dy$ is made easier by replacing x, y and dx dy with $r\cdot\cos(\theta)$, $r\cdot\sin(\theta)$ and $r\cdot dr\cdot d\theta$ respectively. Then $\iint_R \sqrt{x^2+y^2}\ dA = \iint_G r\cdot r\cdot dr\ d\theta$. There are other substitutions that can make double and triple integrals easier, and this section shows how to make some of those transformations.

Changing Variables in Double Integrals

Since a double integral depends on x and y, we will typically need two substitution variables, u and v, and formulas that replace x=x(u,v) and y=y(u,v). Then the domain S in the uv-plane will be mapped into a region R in the xy-plane. Reversing the transformation, we can map R from the xy-plane to S in the uv- plane. The goal is to map a complicated xy domain to an easier uv domain.

Example 1: Suppose $S = \{u,v): 0 \le u \le 2 \text{ and } 1 \le v \le 2\}$ in the uv-plane, and that the transformation T is given by $x = x(u,v) = 2u + v$ and $y = y(u,v) = u - v$. (a) What is the R= T(S) region in the xy-plane? (b) What is the inverse transformation u=u(x,y), v=v(x,y) that maps R back onto S?

Solution: (a) S is shown in Fig. 1a. The corners of S, moving counterclockwise, are (0,1), (2,1), (2,2) and (0,2) and these are mapped by T to the (x,y) points (1,–1), (5,1), (6,0) and (2,–2) respectively, and these become the corners of R = T(S) in the xy-plane as shown in Fig. 1b. Since the transformation T is linear, the straight line boundaries of S are mapped to straight lines of R. (b) Solving x=2u+v and y=u-v for u and v, we get u=(x+y)/3 and v=(x–2y)/3 . It is difficult to describe the domain of integration for R in terms of x and y, but quite easy for S in terms of u and v.

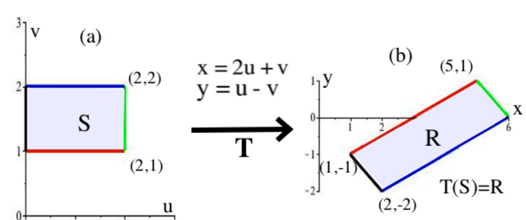


Fig. 1

Practice 1: Suppose $S = \{u,v): 1 \le u \le 3 \text{ and } 0 \le v \le 2\}$ in the uv-plane, and T is given by $x = x(u,v) = u + 2v$ and $y = y(u,v) = 2u - v$. (a) What is the R= T(S) region in the xy-plane? (b) What is the inverse transformation u=u(x,y), v=v(x,y) that maps R back onto S?

Example 2: Suppose $S = \{u,v): 0 \le u \le 1 \text{ and } 1 \le v \le 2\}$ in the uv-plane, and T is given by x=u+v and y=u/v . (a) What is the R= T(S) region in the xy-plane?

(b) What is the inverse transformation u=u(x,y), v=v(x,y) that maps R back onto S?

Solution: (a) S is shown in Fig. 2. The corners of S, moving counterclockwise, are (0,1), (1,1), (1,2) and (0,2) and these are mapped by T to the (x,y) points (1,0), (2,1), (3, 1/2) and (2,0) respectively, and these become the corners of R = T(S) in the xy-plane as shown in Fig. 2.

The boundary: If v=1 and 0≤u≤1, then y=x-1 for 1≤x≤2. If v=2 and 0≤u≤1, then y=x/2-1 for 2≤x≤3. If u=0 and 1≤v≤2, then y=0 and 1≤x≤2. Finally, the interesting boundary: if u=1 and 1≤v≤2, then y=1/(x-1) for 2≤x≤3.

(b) The inverse transformation (solving x=u+v and y=u/v for u and v) is u=xy/(1+y) and v=x/(1+y).

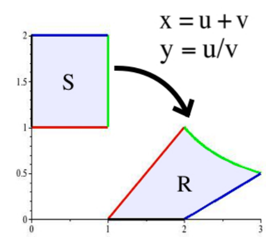


Fig. 2

Practice 2: Suppose $S = \{u,v): 0 \le u \le \pi \text{ and } 1 \le v \le 2\}$ in the uv-plane, and T is given by and $x = 1 + v \cdot \sin(u)$ and y=u . (a) What is the R= T(S) region in the xy-plane? (b) What is the inverse transformation u=u(x,y), v=v(x,y) that maps R back onto S?

Our goal in this section is not to simply map regions into other regions, but it is to do substitutions that make double integrals easier, either by making the integral domain easier or by making the integrand function easier or both. However, we need one more piece, the Jacobian.

Definition

The **Jacobian** of the transformation T:(u,v)->(x,y) by the substitution x=x(u,v) and y=y(u,v) is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \cdot \frac{\partial x}{\partial v}$$

Note: partials are with respect to u and v

The Jacobian is the determinant of the 2x2 matrix of partial derivatives. It is required that x(u,v) and y(u,v) have continuous first partial derivatives on the region S in the uv-plane.

The Jacobian of the inverse transformation with u=u(x,y) and v=v(x,y) is

$$J(x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial y}$$

The derivation of this Jacobian formula is intricate and is given in the Appendix.

Example 3: (a) Calculate the Jacobian J(u,v) of the transformation $x = x(u,v) = 2u + v$ and $y = y(u,v) = u - v$ from Example 1.

(b) Also calculate the Jacobian J(x,y) of the inverse transformation.

Solution: (a) $\frac{\partial x}{\partial u} = 2,\ \frac{\partial x}{\partial v} = 1,\ \frac{\partial y}{\partial u} = 1$ and $\frac{\partial y}{\partial v} = -1$ so $J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$.

(b) The inverse transformation (using algebra) is u=(x+y)/3 and v=(x-2y)/3 so

$$\frac{\partial u}{\partial x} = \frac{1}{3},\ \frac{\partial u}{\partial y} = \frac{1}{3},\ \frac{\partial v}{\partial x} = \frac{1}{3} \text{ and } \frac{\partial v}{\partial y} = -\frac{2}{3} \text{ and } J(x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{vmatrix} = -\frac{1}{3}\ .$$

Practice 3: (a) Calculate the Jacobian J(u,v) of the transformation x=u+2v and y=2u-v from Practice 1.

(b) Also calculate the Jacobian J(x,y) of the inverse transformation.

Fact: You might have noticed that $J(x,y) = \frac{1}{J(u,v)}$ in the previous Examples and Practices. That is true in general and can make some computations much easier. A proof of this for the 2D Jacobian is given in the Appendix.

Now we finally get to change variables in double integrals.

Change of Variables Theorem

If the region G in the (u,v) plane is transformed into the region R in the xy-plane by x=x(u,v) and y=y(u,v), and if x(u,v) and y(u,v) have continuous first partial derivatives,

then $$\iint_R F(x,y)\,dx\,dy = \iint_S F(x(u,v),y(u,v))\ |J(u,v)|\ du\,dv$$

for any continuous function F.

Example 4: (a) Calculate the area of the region R in Example 1.

(b) Calculate the mass of the region in Example 1 when the density is $\delta(x,y) = 5 + y$.

Solution: x=2u+v and y=u-v so $J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$.

(a) $$\{\text{area of R}\} = \iint_R 1\,dA = \iint_S 1\ |J(u,v)|\ du\,dv = \int_{v=1}^{2}\int_{u=0}^{2} 1\cdot(3)\ du\ dv = \int_{v=1}^{2} 6\ dv = 6$$

(b) $$\text{Mass} = \iint_R \delta\,dA = \iint_S (5+y)\ |J(u,v)|\ du\,dv = \int_{v=1}^{2}\int_{u=0}^{2} (5+u-v)\cdot(3)\ du\ dy = \int_{v=1}^{2} 36-6v\ dv = 27$$

Practice 4: (a) Calculate the area of the region R in Practice 1.

(b) Calculate the mass of the region in Practice 1 when the density is $\delta(x,y) = x + y$.

Example 5: Let $\iint_R \frac{2x-y}{2}\, dA = \int_{y=0}^{4} \int_{x=y/2}^{(y/2)+1} \frac{2x-y}{2}\, dx\, dy$

(a) Sketch the region R.

(b) Make the substitutions $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$ and solve for x=x(u,v) and y=y(u,v).

(c) Calculate the Jacobian J(u,v) and (d) Rewrite the xy-integral in terms of u and v and evaluate this integral.

Solution: (a) Since this is a linear transformation, corners get mapped to corners and straight line boundaries get mapped to straight line boundaries. The corners in the xy-plane of R are (0,0), (2,4), (1,0) and (3,4) as shown in Fig. 3a. Under the change of variables, (0,0)-->(0,0), (2,4)-->(0,2), (1,0)-->(1,0), and (3,4)-->(1,2) so the integration region in the uv-plane is the rectangle S in Fig. 3b.

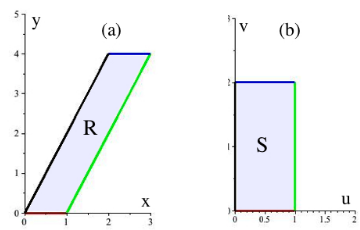


Fig. 3

(b) Simple algebra gives x=u+v and y=2v. (c) $J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$

(d) $\int_{y=0}^{4} \int_{x=y/2}^{(y/2)+1} \frac{2x-y}{2}\, dx\, dy = \int_{v=0}^{2} \int_{u=0}^{1} u\ |J(u,v)|\ du\ dv = \int_{v=0}^{2} \int_{u=0}^{1} u \cdot 2\ du\ dv = \int_{v=0}^{2} 1\ dv = 2$.

Practice 5: Suppose the area of S is $\iint_S 1\ dV = 20$, and $J(u,v) = 2$ for the transformation T(S)=R.

Determine the area of region R.

In Section 14.3 we discussed double integrals in polar coordinates and used geometry to find the conversion formula between the two types of double integrals. The conversion from polar to rectangular coordinates is simply the transformation $x = r \cdot \cos(\theta)$ and $y = r \cdot \sin(\theta)$. Then

$$J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r \cdot \sin(\theta) \\ \sin(\theta) & r \cdot \cos(\theta) \end{vmatrix} = r \cdot \cos^2(\theta) + r \cdot \sin^2(\theta) = r$$ and

$\iint_R F(x,y)\, dx\, dy = \iint_S F(r \cdot \cos(\theta), r \cdot \sin(\theta))\, r\, dr\, d\theta$, the same result we got in this special case in 14.3.

Creating the Desired Transformation for Double Integrals

So far in this section all of the transformations have been given. But often we have a "strange" xy-domain and need to pick a transformation that maps it to something nice such as a rectangle in the uv-plane. Sometimes that is very difficult, but there are a few situations and ideas that are much easier. Technology and software can help us evaluate double integrals once they are set up, but it is usually up to us to set up those integrals first.

(1) Region R is bounded by parallel lines

In Example 1 the region R (Fig. 4) is bounded by the pair of parallel lines y=6–x and y=–x which can be rewritten as x+y=**6** and x+y=**0**. This suggests that we might set x+y=**u** and let u vary from 0 to 6. Similarly with the parallel pair 2y=x–3 and 2y=x–6, rewritten as x-2y=**3** and x–2y=**6**, suggesting that we put x–2y=**v** with v going from 3 to 6. It is straight forward to verify that the transformation u=x+y, v=x-2y transforms R into the uv-plane rectangle shown in Fig. 5. (The transformation in Example 1 was u=(x+y)/3 and v=(x-2y)/3 which leads to a smaller rectangle in the uv plane.)

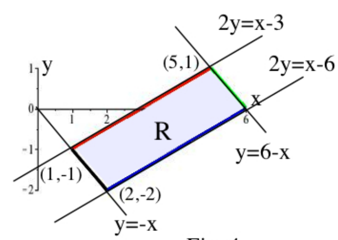


Fig. 4

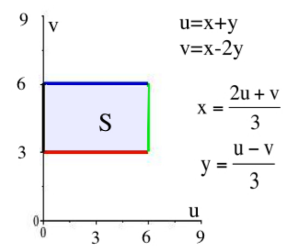


Fig. 5

Practice 6: Find a transformation of the region R bounded by the lines y=x, y=x+2, y=6–2x and y=9–2x (Fig. 6) into a rectangle S in the uv-plane.

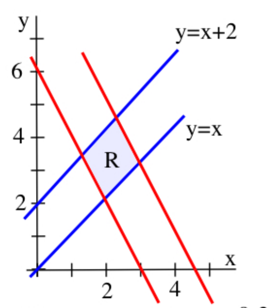


Fig. 6

(2) Region R is bounded by shifted curves

Example 6: R is the region in the xy plane bounded by the lines y=x and y=x+3 and the parabolas $y = 9 - x^2$ and $y = 16 - x^2$ (Fig. 7). Find a transformation of R into a rectangle S in the uv-plane.

Solution: y-x=0 and y-x=3 suggests putting u=y-x (so u goes from 0 to 3). The parabolas $x^2 + y = 9$ and $x^2 + y = 16$ suggests putting v= $x^2 + y$ (so v goes from 9 to 16). You can verify that this works.

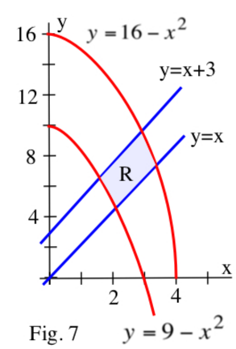


Fig. 7

Practice 7: R is the region in the xy-plane bounded by the four parabolas $y = 9 - x^2$, $y = 16 - x^2$, $y = x^2$ and $y = x^2 + 4$. Sketch the region R and find a transformation of R into a rectangle in the uv-plane.

(3) Two parameter family of curves

If the bounding curves of the region R can be written using the two parameters u and v, then we can usually transform R into a rectangle in the uv-plane.

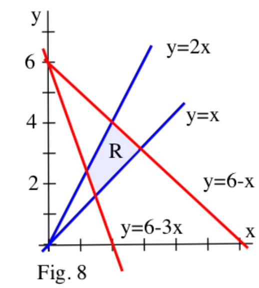


Fig. 8

Example 7: R is the region in the xy-plane bounded by the four lines y=x, y=2x, y=6–x and y=6–3x in Fig. 8. Find a transformation of R into a rectangle S in the uv-plane.

Solution: y=1x and y=2x can be rewritten with a single parameter u as y=ux for u from 1 to 2. y=6–1x and y=6–3x can be rewritten with the single parameter y=6–vx with v going from 1 to 3. Solving for u and v, we get the transformation u=y/x and v=(6–y)/x which takes R to a rectangle in the uv-plane. The inverse transformation is x=6/(u+v) and y=6u/(u+v) .

Changing Variables in Triple Integrals

Changing variables for triple integrals is very similar to the situation for double integrals. If T is a transformation from an uvw-space region S to an xyz-space region R (so x, y and z are each differentiable functions of u, v and w: x=g(u,v,w), y=h(u,v,w) and z=k(u,v,w)) then the

3D Jacobian is
$$J(u,v,w)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

The 3D change of variables formula is

$$\iiint_R f(x,y,z)\,dx\cdot dy\cdot dz=\iiint_S f(g(u,v,w),\ h(u,v,w),\ k(u,v,w))\cdot\left|J(u,v,w)\right|\ du\cdot dv\cdot dw$$

Example 8: Suppose f(x,y,z)=2x+4z on the box-like region (Fig. 9)

$R=\{(x,y,z):\ x\le y\le x+3,\ x\le z\le x+2 \text{ and } 1-x\le z\le 3-x\}$. Evaluate

$\iiint_R f(x,y,z)\,dx\cdot dy\cdot dz$ by using the transformation T: u=y-x (0≤u≤3),

v=z–x (0≤v≤2) w=z+x (1≤w≤3) and then evaluating the new integral.

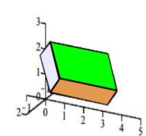


Fig. 9

Solution: We need several pieces. The inverse transformation (after a bit of algebra)

is x= –v/2+w/2, y= u-v/2+w/2 , z= v/2+w/2 , f(x,y,z)=2x+4z=v+3w , S={(u,v,w): 0≤u≤3, 0≤v≤2, 1≤w≤3}, and the Jacobian is $J(u,v,w) = \begin{vmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$

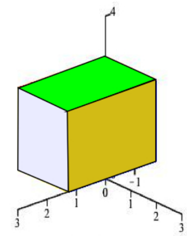


Fig. 10

The new domain in uvw-space is shown in Fug. 10. Finally,

$$\iiint\limits_R \mathrm{f(x,y,z)\ dx \cdot dy \cdot dz} \quad \int_1^3 \int_0^2 \int_0^3 (\mathrm{v}+3\mathrm{w})\left(\frac{1}{2}\right) \mathrm{du \cdot dv \cdot dw}$$

$$\int_1^3 \int_0^2 3(\mathrm{v}+3\mathrm{w})\left(\frac{1}{2}\right) \mathrm{dv \cdot dw} = \int_1^3 3+9\mathrm{w}\ \mathrm{dw} = 42 .$$

Practice 8: Evaluate $\iiint\limits_R (\mathrm{x}+\mathrm{z})\ \mathrm{dx \cdot dy \cdot dz}$ on the region

$\mathrm{R} = \{(\mathrm{x,y,z}) : \mathrm{x}+1 \le \mathrm{y} \le \mathrm{x}+3,\ \mathrm{x} \le \mathrm{z} \le 2\mathrm{x} \text{ and } 0 \le \mathrm{z} \le 3\}$ by using the transformation

T: u=y–x $(1 \le u \le 3)$, v=x/z $(1 \le v \le 2)$ w=z $(0 \le w \le 3)$.

Rectangular to Spherical: In section 14.7 we transformed some integrals in rectangular coordinates into ones in spherical coordinates, and we derived the $\mathrm{dV} = \rho^2 \cdot \sin(\varphi)\ \mathrm{d}\rho \cdot \mathrm{d}\theta \cdot \mathrm{d}\varphi$ formula for the transformation geometrically. Instead, we can use the Jacobian. For spherical coordinates $x = \rho \cdot \sin(\varphi) \cdot \cos(\theta)$, $y = \rho \cdot \sin(\varphi) \cdot \sin(\theta)$, and $z = \rho \cdot \cos(\varphi)$. Then the Jacobian J(u,v,w) is

$$J(\rho,\theta,\varphi) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \sin(\varphi)\cdot\cos(\theta) & -\rho\cdot\sin(\varphi)\cdot\sin(\theta) & \rho\cdot\cos(\varphi)\cdot\cos(\theta) \\ \sin(\varphi)\cdot\sin(\theta) & \rho\cdot\sin(\varphi)\cdot\cos(\theta) & \sin(\varphi)\cdot\sin(\theta) \\ \cos(\varphi) & 0 & -\rho\cdot\sin(\varphi) \end{vmatrix} = \ldots = \rho^2 \cdot \sin(\varphi)\ \mathrm{d}\rho \cdot \mathrm{d}\theta \cdot \mathrm{d}\varphi .$$

The " …" is simply a matter of carefully calculating the determinant and then using some fundamental trigonometric identities to simplify the result.

Practice 9: Calculate $\sin(\varphi)\cdot\cos(\theta)\cdot \begin{vmatrix} \rho\cdot\sin(\varphi)\cdot\cos(\theta) & \sin(\varphi)\cdot\sin(\theta) \\ 0 & -\rho\cdot\sin(\varphi) \end{vmatrix}$

Other triple integral transformations are possible, but rectangular to cylindrical or spherical are the most common.

Problems

In problems 1 to 4, (a) sketch the given set S and the image of S under the given transformation, (b) calculate the Jacobians, J(x,y) and J(u,v), and rewrite $\iint\limits_R f(x,y)\,dx\,dy$ as $\iint\limits_S \_\_\_\_\ du\,dv$.

1. $S=\{(u,v):\ 0\le u\le 2,\ 1\le v\le 4\}$ under $x=u+v$ and $y=2u-v$.
2. $S=\{(u,v):\ 1\le u\le 2,\ 0\le v\le 2\}$ under $x=2u-3v$ and $y=u+2v$.
3. $S=\{(u,v):\ 0\le u\le 1,\ 0\le v\le 1\}$ under $x=au+bv$ and $y=cu+dv$.
4. $S=\left\{(u,v):\ 2\le u\le 4,\ \frac{\pi}{6}\le v\le\frac{\pi}{2}\right\}$ under $x=u\cdot\cos(v)$ and $y=u\cdot\sin(v)$.

In problems 5 to 10, (a) sketch the given set S and the image of S under the given transformation, and (b) calculate the Jacobians, J(x,y) and J(u,v).

5. $S=\{(x,y):\ 0\le x\le 2,\ 1\le y\le 3\}$ under $u=\frac{3x-3y}{4}$ and $v=\frac{y}{3}$.
6. $S=\{(x,y):\ 0\le x\le 2,\ 1\le y\le 3\}$ under $u=x+y$ and $v=\frac{x}{y}$.
7. $S=\{(x,y):\ 0\le x\le 1,\ 0\le y\le 1\}$ under $u=x^2-y^2$ and $v=2xy$.
8. $S=$ {region bounded by the hyperbolas $y=1/x$ and $y=4/x$ and the lines $y=x$ and $y=4x$ } with $x=u/v$ and $y=uv$.
9. $S=$ {trapezoid with vertices (x,y) = (1,0), (2,0), (0,-2) and (0,-1)} under u=x–y and v=x+y .
10. $S=$ {triangle with vertices (x,y) = (1,0), (3,1) and (0,4) } under u=x–y and v=x+y .

11. Suppose $\iint\limits_S 1\ du\,dv=14$, and $J(u,v)=7$ for the transformation T(S)=R. Evaluate $\iint\limits_R 1\ dV$.
12. Suppose $\iint\limits_S 1\ du\,dv=30$, and $J(u,v)=5$ for the transformation T(S)=R. Evaluate $\iint\limits_R 1\ dA$.
13. Suppose $\iint\limits_S 1\ du\,dv=15$, and $J(x,y)=3$ for the transformation T(S)=R. Evaluate $\iint\limits_R 1\ dA$.
14. Suppose $\iint\limits_S 1\ du\,dv=24$, and $J(x,y)=4$ for the transformation T(S)=R. Evaluate $\iint\limits_R 1\ dA$.

In problems 15-24, use the given transformation to evaluate the integral.

15. $\iint\limits_R (x+3y)\,dA$. R is the triangular region with vertices (0,0), (2,1) and (1,2). Use x=2u+v and y=u+2v (so u=(2x–y)/3 and v=(2y–x)/3).
16. $\iint\limits_R x\,dA$. R is the ellipse $\frac{x^2}{9}+\frac{y^2}{4}\le 1$. Use x=3u and y=2u.
17. $\iint\limits_R \frac{x+y}{3}\,dA$ where R is the region shown in Fig. 4.

18. $\iint_R (x-2y)\, dA$ where R is the region shown in Fig. 4.

19. $\iint_R (3x+6y)\, dA$ where R is the region shown in Fig. 6.

20. $\iint_R (6x-3y)\, dA$ where R is the region bounded by the lines y=x, y=2x, y=6-2x and y=9-2x (Fig. 6).

21. $\iint_R \sqrt{\frac{y}{x}} + \sqrt{xy}\, dA$ where R={region bounded by the hyperbolas y=1/x and y=4/x and the lines y=x and y=4x} Use the substitution x=u/v and y=uv.

22. $\iint_R (6x-3y)\, dA$ where R is the region bounded by the lines y=x, y=2x, y=6-x and y=6-3x (Fig. 8). Use u=y/x and v=(6-y)/x.

23. The area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1$ can be found using earlier methods, but the integral requires a trigonometric substitution. Instead, transform the ellipse into a circle using the substitution x=au and y=bv, write the new uv integral and evaluate it to show that the area of the ellipse is πab .

24. Evaluate $\iint_R xy\, dA$ where R is the square with vertices at (0,0), (1,1), (2,0) and (1,-1).

Practice Solutions

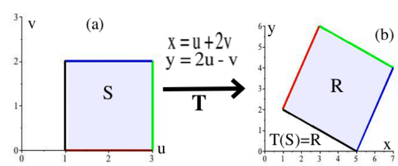


Fig. P1

Practice 1: See Fig. P1 for the S and R regions. u=(x+2y)/5 and v=(2x–y)/5 .

Practice 2: See Fig. P2 for the S and R regions. u=y and v=x/(1+sin(y).

Practice 3: $\frac{\partial x}{\partial u}=1,\ \frac{\partial x}{\partial v}=2,\ \frac{\partial y}{\partial u}=2$ and $\frac{\partial y}{\partial v}=-1$ so $J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}1 & 2\\ 2 & -1\end{vmatrix}=-5$

The inverse transformation is u=(x+2y)/5 and v=(2x-y)/5 so

$\frac{\partial u}{\partial x}=\frac{1}{5},\ \frac{\partial u}{\partial y}=\frac{2}{5},\ \frac{\partial v}{\partial x}=\frac{2}{5}$ and $\frac{\partial v}{\partial y}=-\frac{1}{5}$ and $J(x,y)=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}1/5 & 2/5\\ 2/5 & -1/5\end{vmatrix}=-\frac{1}{5}$

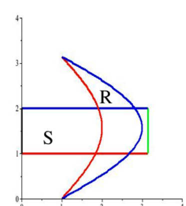


Fig. P2

Practice 4: $\delta(x,y)=x+y$. $x = x(u,v) = u+2v$ and $y = y(u,v) = u - v$ so J(u,v)= –5 (see Practice 3)

(a) $\{\text{area of R}\} = \iint_R 1\ dA = \iint_S 1\cdot|J(u,v)|\ du\ dv = \int_{v=0}^{2}\int_{u=1}^{3} (5)\ du\ dv = \int_{v=0}^{2} 10\ dv = 20$

(b) $\text{Mass} = \iint_R \delta\ dA = \iint_S (x+y)\cdot|J(u,v)|\ du\ dv = \int_{v=0}^{2}\int_{u=1}^{3} (2u+v)(5)\ du\ dv = \int_{v=0}^{2} 40+10v\ dv = 100$

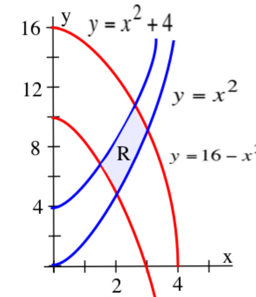


Fig. P6

Practice 5: The area of R is $\iint\limits_R 1 \ \mathrm{dV} = \iint\limits_S 1 \cdot |J(u,v)| \mathrm{dV} \iint\limits_S 2 \ \mathrm{dV} = 2(20) = 40 \cdot$

Practice 6: The first two line equations can be written as y-x=0 and y-x=2 so put u=y-x (then u goes from 0 to 2). The other two parallel lines can be written as 2x+y=6 and 2x+y=9 so put v=2x+y (then v goes from 6 to 9). This transformation takes R in the xy-plane to the 0≤u≤2 and 6≤v≤9 rectangle in the uv-plane.

Practice 7: R is shown in Fig. P6. $x^2 + y = 9$ and $x^2 + y = 16$ so put $u = x^2 + y$ (then u goes from 9 to 16).

Similarly, $x^2 - y = 0$ and $x^2 - y = 4$ so put $u = x^2 - y$ (then v goes from 0 to 4).

Practice 8: The inverse transformation is x=uv, y=vw+u and z=w. x+z=uv+w and

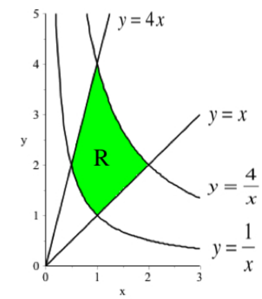
$J(u,v,w) = \begin{vmatrix} v & u & 0 \\ 1 & w & 0 \\ 0 & 0 & 1 \end{vmatrix} = vw$ so the new integral is

$$\iiint\limits_R (x+z)\ dx \cdot dy \cdot dz = \int_0^3 \int_1^2 \int_1^3 (uv+w)(vw)\ du \cdot dv \cdot dw = \int_0^3 \int_1^2 \int_1^3 (uv^2 w + vw^2)\ du \cdot dv \cdot dw$$

$$= \int_0^3 \int_1^2 \left(4v^2 w + 2vw^2\right)\ dv \cdot dw = \int_0^3 \left(\frac{28}{3} w + 3w^2\right) dw = 69$$

Practice 9: $\sin(\varphi) \cdot \cos(\theta) \cdot \begin{vmatrix} \rho \cdot \sin(\varphi) \cdot \cos(\theta) & \sin(\varphi) \cdot \sin(\theta) \\ 0 & -\rho \cdot \sin(\varphi) \end{vmatrix} = \sin(\varphi) \cdot \cos(\theta) \left[-\rho^2 \cdot \sin^2(\varphi) \cdot \cos(\theta) \right] = -\rho^2 \cdot \sin^3(\varphi) \cdot \cos^2(\theta)$

Two Transformations, Same Result



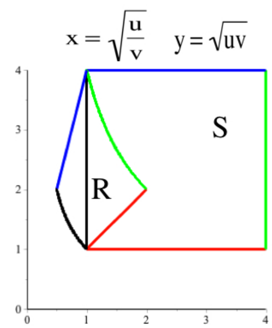
Integral of $f(x,y) = \sqrt{\frac{y}{x}} + \sqrt{xy}$ on

$R = \{(x,y)$ bounded by $y = 1/x,\ y = 4/x,\ y = x$ and $y = 4x\}$ (see figure)

(A) The "natural" transformation (pages 5 and 6):

y=vx for v=1..4 so v=y/x. y=u/x for u=1..4 so u=xy.

Then $x = \sqrt{\frac{u}{v}}$ and $y = \sqrt{uv}$.



The Jacobian is $J(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial x} \end{vmatrix} = \begin{vmatrix} y & x \\ \frac{-y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{2y}{x} = 2v$.

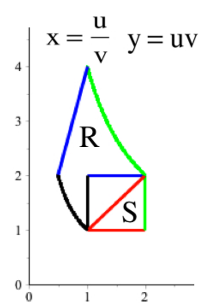
$$\iint_R \sqrt{\frac{y}{x}} + \sqrt{xy}\ dx\,dy = \int_{u=1}^{4} \int_{v=1}^{4} [\sqrt{v} + \sqrt{u}]\left(\frac{1}{2v}\right) dv\,du$$

$$= \int_{u=1}^{4} \int_{v=1}^{4} \left[\frac{1}{2\sqrt{v}} + \frac{\sqrt{u}}{2v}\right] dv\,du = \int_{u=1}^{4} \left[\sqrt{v} + \frac{\sqrt{u}}{2}\ln(v)\right] \Big|_{v=1}^{v=4} du$$

$$= \int_{u=1}^{4} \left[1 + \frac{\sqrt{u}}{2}\ln(4)\right] \Big|_{v=1}^{v=4} du = u + \frac{1}{2}\cdot\frac{2}{3}(u)^{3/2}\ln(4) \Big|_{u=1}^{u=4} = \mathbf{3 + \frac{7}{3}\ln(4)}$$

(B) The "suggested" transformation (Problem 8):

x=u/v and y=uv. Then u=xv so



$y = (xv)v = x \cdot v^2$ with $\mathbf{v^2 = 1..4}$ **and** $\mathbf{v = 1..2}$ and $v = \sqrt{\frac{y}{x}}$.

$xy = \left(\frac{u}{v}\right)(uv) = u^2$ with $\mathbf{u^2 = 1..4}$ **and** $\mathbf{u = 1..2}$ and $u = \sqrt{xy}$.

The Jacobian is $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}$.

$$\iint_R \sqrt{\frac{y}{x}} + \sqrt{xy}\ dx\,dy = \int_{u=1}^{2} \int_{v=1}^{2} [v + u]\left(\frac{2u}{v}\right) dv\,du = \int_{u=1}^{2} \int_{v=1}^{2} \left[2u + \frac{2u^2}{v}\right] dv\,du$$

$$= \int_{u=1}^{2} \left[2uv + 2u^2\ln(v)\right] \Big|_{v=1}^{v=2} du = \int_{u=1}^{2} \left[2u + 2u^2\ln(2)\right] du = u^2 + \frac{2}{3}u^3\ln(2) \Big|_{u=1}^{u=2}$$

$$= u^2 + \frac{2}{3}u^3\ln(2) \Big|_{u=1}^{u=2} = 3 + \frac{14}{3}\ln(2) = \mathbf{3 + \frac{7}{3}\ln(4)} \ .$$

Conclusion: These two transformations have different Jacobians (magnifications) and lead to different S regions ([1,4]x[1,4] verses [1,2]x[1,2]) but the resulting integral values are the same.

Appendix: Derivation of the 2D Jacobian Formula

Suppose T is a transformation from a rectangular region S in the uv-plane to a region R in the xy-plane given by x=x(u,v) and y=y(u,v) as in Fig A1.

The corners of the rectangular region S are (u,v), $(u+\Delta u,v)$, $(u,v+\Delta v)$ and $(u+\Delta u,v+\Delta v)$.

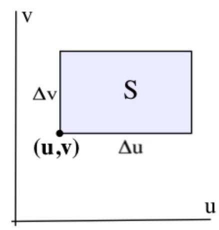


Fig. A1

Then T:(u,v)-> (x(u,v), y(u,v)) = A, a point in the xy-plane (Fig. A2)
T: $(u+\Delta u,v)$->($x(u+\Delta u,v)$, $y(u+\Delta u,v)$) = B
T: $(u,v+\Delta v)$->($x(u,v+\Delta v)$, $y(u,v+\Delta v)$) = C

Next we want to approximate the area of the region R using the cross product.
Put P = vector AB $= \langle x(u+\Delta u,v) - x(u,v),\ y(u+\Delta u,v) - y(u,v)\rangle$

$$= \left\langle \frac{x(u+\Delta u,v)-x(u,v)}{\Delta u}, \frac{y(u+\Delta u,v)-y(u,v)}{\Delta u}\right\rangle \Delta u = \left\langle \frac{\Delta x}{\Delta u}, \frac{\Delta y}{\Delta u}\right\rangle \Delta u$$

and Q = vector AC = $\langle x(u,v+\Delta v) - x(u,v),\ y(u,v+\Delta v) - y(u,v)\rangle$

$$= \left\langle \frac{x(u,v+\Delta v)-x(u,v)}{\Delta v}, \frac{y(u,v+\Delta v)-y(u,v)}{\Delta v}\right\rangle \Delta v = \left\langle \frac{\Delta x}{\Delta v}, \frac{\Delta y}{\Delta v}\right\rangle \Delta v$$

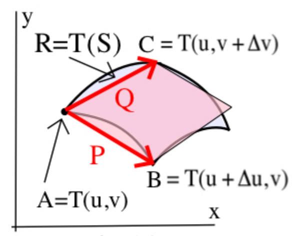


Fig. A2

Then the area of region R is approximately $|PxQ| = \begin{vmatrix} \frac{\Delta x}{\Delta u} & \frac{\Delta y}{\Delta u} \\ \frac{\Delta x}{\Delta v} & \frac{\Delta y}{\Delta v} \end{vmatrix} \Delta u \Delta v$

As $\Delta u,\ \Delta v \to 0$, the fractions in the determinant approach the partial derivatives.

So $dA = dx \cdot dy = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du \cdot dv = |J(u,v)| du \cdot dv$

Appendix: Simple proof that $J(x,y) = \frac{1}{J(u,v)}$ for a linear transformation in 2D (could be a student assignment)

Theorem; If T:(u,v)-->(x,y) is a linear transformation (x=au+bv and y=cu+dv) then J(u,v)=1/J(x,y).

Proof: $J(x,y) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Using elementary algebra and solving for u and v we get $u = \frac{dx - by}{ad - bc}$ and $v = \frac{-cx + ay}{ad - bc}$ so

$$J(u,v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{vmatrix} = \frac{ad-bc}{(ad-bc)^2} = \frac{1}{ad-bc} = \frac{1}{J(x,y)}.$$

Appendix: Proof that $J(x,y) = \frac{1}{J(u,v)}$ for a general linear transformation in 2D.

The following results are true for larger matrices and the proofs are similar.

Lemma: For 2x2 matrices S and T, $|S\cdot T| = |S|\cdot|T|$

Proof: If $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ then $|S| = ad - bc$ and $|T| = AD - BC$.

By matrix multiplication, $S\cdot T = \begin{pmatrix} aA + bC & aB + bD \\ cA + dC & cB + dD \end{pmatrix}$ so

$$|S\cdot T| = (aA + bC)(cB + dD) - (cA + dC)(aB + bD)$$

$$= (aAcB + aAdD + bCcB + bCdD) - (cAaB + cAbD + dCaB + dCbD)$$

$$= acAB + adAD + bcBC + bdCD - acAB - bcAD - adBC - bdCD$$

$$= adAD + bcBC - bcAD - adBC = (ad - bc)(AD - BC) = |S|\cdot|T|$$

Lemma: If $S : (x,y) \to (u,v)$ and $T : (u,v) \to (w,z)$ then $J(T\circ S) = J(S)\cdot J(T)$.

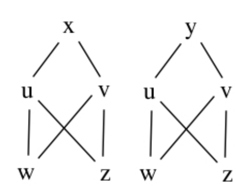
Proof. $J(S) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ and $J(S) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$.

Then $T\circ S : (x,y) \to (u,v) \to (w,z)$ so $J(T\circ S) = \begin{vmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial z} \end{vmatrix}$.

But by the Chain Rule for functions of several variables,

$$\frac{\partial x}{\partial w} = \frac{\partial x}{\partial u}\cdot\frac{\partial u}{\partial w} + \frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial w} \qquad \frac{\partial y}{\partial w} = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial w} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial w}$$

$$\frac{\partial x}{\partial z} = \frac{\partial x}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial z} \qquad \frac{\partial y}{\partial z} = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial z}$$



so $$J(T\circ S) = \begin{vmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u}\cdot\frac{\partial u}{\partial w} + \frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial w} & \frac{\partial x}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial z} \\ \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial w} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial w} & \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial z} \end{vmatrix}$$

$$= \left|\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\begin{pmatrix} \frac{\partial u}{\partial w} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial w} & \frac{\partial v}{\partial z} \end{pmatrix}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\cdot\begin{vmatrix} \frac{\partial u}{\partial w} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial w} & \frac{\partial v}{\partial z} \end{vmatrix} = J(S)\cdot J(T)$$

Theorem: If S and T are inverse transformations $\left(T\circ S = I : (x,y) \to (x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$, then $J(S)\cdot J(T) = 1$.

Proof: Since $T\circ S = I$ then $J(T\circ S) = J(I) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ and $J(S)\cdot J(T) = J(T\circ S) = 1$.