15.0 Introduction to Vector Calculus

This may seem like a strange title since we have already been doing calculus with vectors, but the topics in this chapter extend the main calculus ideas into fields of vectors such as the one illustrated in Fig. 1 (from **Modeling Waves and Currents Produced by Hurricanes Katrina, Rita, and Wilma** by Lie-Yauw Oey and Dong-Ping Wang) . First we will examine some examples of vector fields. Then on to line integrals (integrals along paths in 2D or 3D) that will enable us to calculate the work done moving an object along a path in a wind field or magnetic field. Finally we will consider surface integrals and then some important theorems that generalize the Fundamental Theorem of Calculus to vector fields.

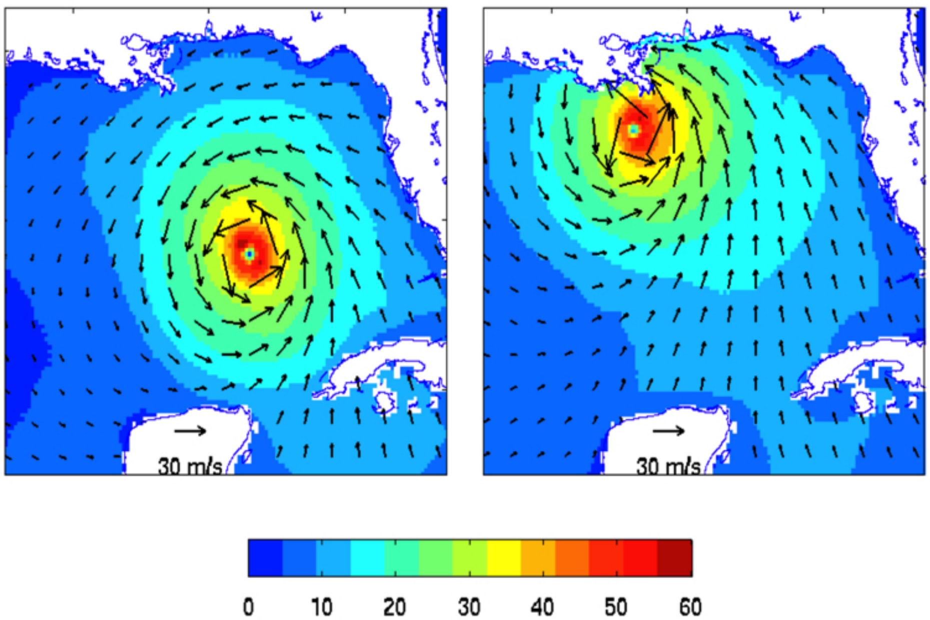


Fig. 1: Hurricane Katrina wind vectors (in m/s) on August 29, 2005

Vector fields contain a great deal of information and calculus can help us use that information. But even without calculus we can answer some questions about such fields.

Example 1: (a) Which way, clockwise or counterclockwise, where the winds blowing?

(b) Were the strongest winds greater in the left picture or in the right picture?

(c) Approximately what is the maximum wind speed in the right picture?

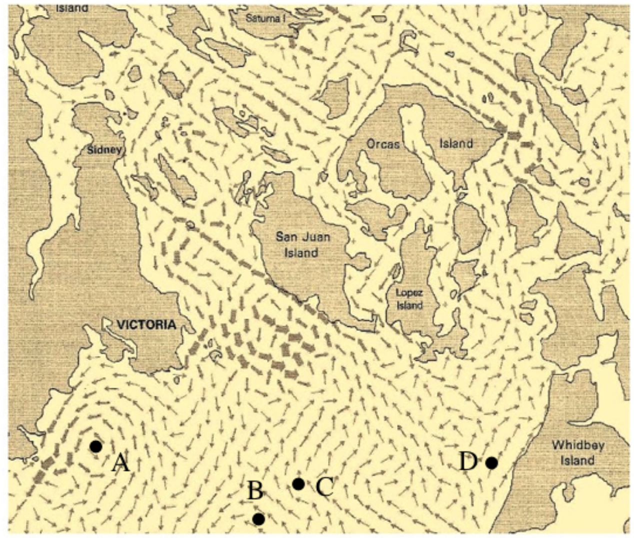
(d) In which direction was the eye of the hurricane traveling?

Solution: (a) Counterclockwise. (b) Right picture

(c) Using the "30 m/s" key at the bottom of each picture, the maximum wind speed was about 50-60 m/s (about 120 miles/hour) . (d) To the NNW.

Other common vector fields are water velocity vectors indicating currents (Fig. 2) and force fields indicating the strength and direction of attractive or repulsive forces either in 2 or 3 dimensions.

(Note: Excellent, animated wind velocity fields for various parts of the world are at http://www.wunderground.com/maps/us/WindSpeed.html)



The biggest arrows represent current velocity over 2.5 mph, the smallest, under .25 mph. The arrows point the direction of flow.

Fig. 2: Currents in the San Juan Islands

Problems

Problems 1 to 4 refer to Fig. 2. This picture uses wider arrows to represent stronger current.

1. What is happening to the current at point A?

2. If we drop small pieces of wood into the water at points A and B, do they stay close or do they drift apart?

3. Is it easier to row from point D to San Juan Island or from San Juan Island to point D?

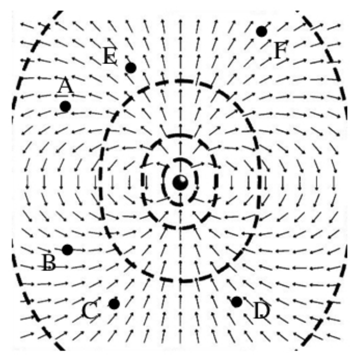


Fig. 3: Lines of Force for a bar magnet

4. Is it easier to row from point C to point D or from point D to point C?

Problems 5 to 8 refer to Fig. 3. The dotted curves each represent places of equal force. Moving from the center, the force at each dotted curve is 1/10 the force at the next closer dotted curve.

5. Is it easier to move from A to B or from B to A?

6. Is it easier to move from C to D or from D to C?

7. Is it easier to move from E to F or from F to E?

8. Imagine that this is a water current and that you dropped a cork into the water at A. Sketch the path of the cork.