15.1 Vector Fields

2D Vector Fields

> A **2D vector field** is a function **F** that assigns a 2D vector **F**(x,y) to each point (x,y) in the domain of the field. Since F(x,y) is a 2D vector we can write
>
> **F**(x,y) = P(x,y)**i** + Q(x,y) **j** = < P(x,y), Q(x,y) > or simply **F** = P **i** + Q **j**

Unlike our earlier work with vectors, these F(x,y) vectors have assigned locations. The vector field consists of an infinite number of vectors (one at each point) so we typically just graph enough of these vectors to make the pattern clear. The convention is to put the tail of the **F**(x,y) vector at the point (x,y).

Example 1: Plot the vectors **F**(x,y)=< y, -x> at the points (1,0), (1,1), (0,1), (-1,1), (-1,0) and (-1,-1).

Solution: **F**(1,0)=<0,-1>, **F**(1,1)=<-1,1>, **F**(0,1)=<1,0>, **F**(-1,1)=<1,1>, **F**(-1,0)=<0,1>, **F**(-1,-1)=<-1,1>. These are shown in Fig. 1.

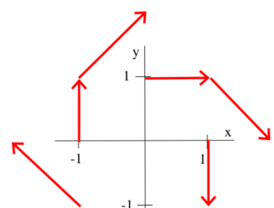


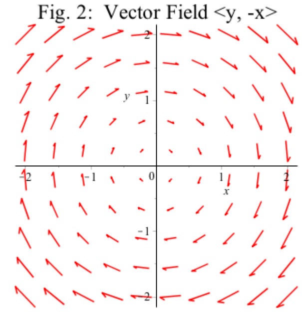
Fig. 1:vector field <y, -x>

It is very tedious to draw a vector field by hand, and some computer programs can do a very nice job. Fig. 2 is the same vector field **F**(x,y)=< y, -x> drawn by the program Maple with the command

```
with(plots): fieldplot([y, -x], x = -2 .. 2, y = -2 .. 2, arrows = THIN,
color=red, thickness=2,grid=[11,11],title = "Vector Field <y, -x>",
titlefont = ["ROMAN", 18]);
```

Maple automatically scaled the arrows to fit into the figure.

Fig. 2: Vector Field <y, -x>



Practice 1: Plot the vectors **F**(x,y)=< x, y/2> at the same points as in Example 1.

A few vector fields are very common and you should be able to recognize them after plotting just a few representative vectors.

Radial Fields <x,y> and <-x,-y>. These are shown in Fig. 3a and 3b. These vectors all point toward the origin or all vectors point away from the origin (except at the origin).

Fig. 3a <x, y>

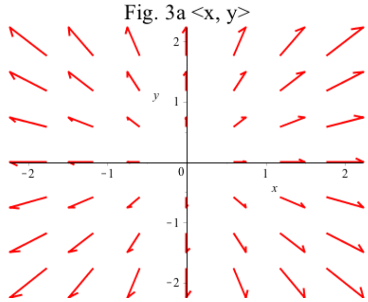
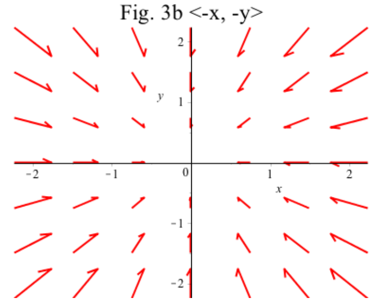


Fig. 3b <-x, -y>



Rotational Fields: < -y, x> , < y, -x> and $\left\langle \frac{y}{\sqrt{x^2+y^2}}, \frac{-x}{\sqrt{x^2+y^2}} \right\rangle$. These are shown in Fig. 4.

The first and second of these fields have vectors that increase in magnitude the farther they are from the origin. The third field has vectors of constant magnitude 1.

Fig. 4a <-y, x>

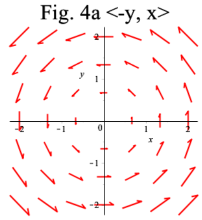


Fig. 4b <y, -x>

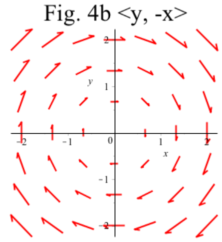
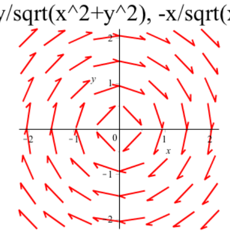


Fig. 4c <y/sqrt(x^2+y^2), -x/sqrt(x^2+y^2)>



The chapter Appendix gives the Maple commands for creating 2D and 3D vector fields.

3D Vector Fields

A 3D vector field **F** assigns a 3D vector at each point in the 3D domain of **F**:

$F(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ where P, Q and R are scalar-valued functions.

Since we live in a 3D world, 3D vector (force) fields are very common in applications, but they are much more difficult to create by hand and to visualize on a 2D page. Fortunately, some software can do the work for us.

Radial Fields: **F**(x,y,z) = <x,y,z> and <-x,-y,-z>.

These are shown in Fig. 5a and 5b. In Fig. 5c all of the vectors of **F**(x,y,z) = <x,y,z> have been normalized to have the same length. The vectors in all three plots point toward or away from the origin (except at the origin).

Fig. 5a: <x,y,z>

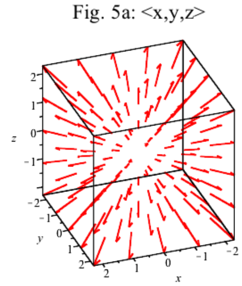


Fig. 5b: <-x,-y,-z>

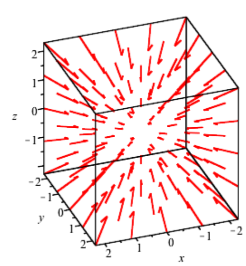
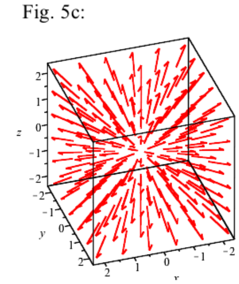


Fig. 5c:



Rotational Fields: These are more difficult to visualize because the rotation can be around any of the axes, some other line, or simply around the origin. Three views of **F**(x,y,z) = <-y, x, 0> are shown in Fig. 6.

Fig. 6a <-y,x,0>

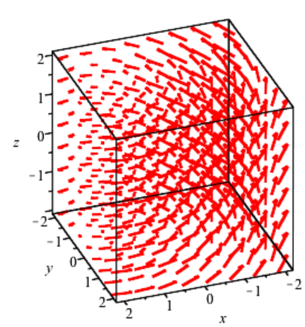


Fig. 6b <-y,x,0>

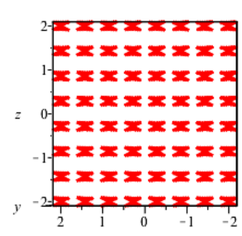


Fig. 6c <-y,x,0>

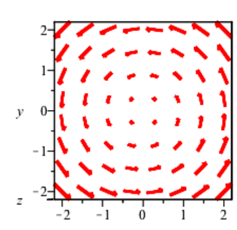


Fig. 7 shows the "swirl" field for **F**(x,y,z) = < y,z,x>, and Fig. 8 is the simple field **F**(x,y,z) = <1,3,2> in which all of the vectors have the same magnitude and direction.

Fig. 7 < y, z, x >

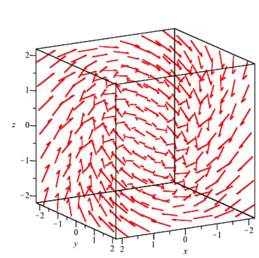
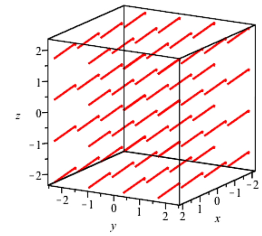


Fig. 8: <1,3,2>



Gravitation and Electric Force Fields

If, as is the situation for gravitational and electric fields, the magnitude of the force is inversely proportional to the square of the distance between the objects, then $\mathbf{F}(x,y,z) = \dfrac{k}{x^2+y^2+z^2}$ where k is a positive or negative constant. The field will look like Fig. 5a or 5b.

Gravitational Field: Newton's Law of Gravitation says that k=GMm where m and M are the masses of the objects and G is the gravitational constant: $\mathbf{F} = \dfrac{GMm}{x^2+y^2+z^2}$. Suppose the object with mass M is located at the origin, and let $\mathbf{r} = \langle x,y,z\rangle$ be the position vector of the object of mass m. Then $|\mathbf{r}|^2 = x^2+y^2+z^2$ and the force is directed toward the origin so the direction of $\mathbf{r}$ is $-\dfrac{\mathbf{r}}{|\mathbf{r}|}$. Putting this together, the **gravitational field** is $\mathbf{F}(x,y,z) = \mathbf{F}(\mathbf{r}) = \dfrac{GMm}{|\mathbf{r}|^2}\left(-\dfrac{\mathbf{r}}{|\mathbf{r}|}\right) = \left(\dfrac{-GMm}{|\mathbf{r}|^3}\right)\mathbf{r}$. These vectors behave like those in Fig. 5b in which the vector at each point is directed toward the origin.

Electrical Field: Coulomb's Law says that the force **F** is inversely proportional to the square of the distance between the two charges. If an electric charge Q is located at the origin, and a charge q is located at (x,y,z) then $|\mathbf{F}| = \dfrac{eqQ}{|\mathbf{r}|^2}$ where $\mathbf{r} = \langle x,y,z\rangle$ and e is a constant. Finally, $\mathbf{F}(\mathbf{r}) = \left(\dfrac{eqQ}{|\mathbf{r}|^3}\right)\mathbf{r}$ with qQ<0 for unlike charges (attracting each other) and qQ>0 for like charges (repelling each other). If we consider the force per unit q of charge then we have the **electric field** $\mathbf{E}(\mathbf{r}) = \dfrac{1}{q}\mathbf{F}(\mathbf{r}) = \left(\dfrac{eQ}{|\mathbf{r}|^3}\right)\mathbf{r}$.

Gradient Fields

In section 13.5 the gradient vector, $\nabla f(x,y) = \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right\rangle$, was introduced, and this very naturally assigns a vector at each (x,y) location where f has partial derivatives.

Example 2: For $z = f(x,y) = x^2+y^2$, what is the gradient vector field for f?

Solution: $\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 2y$ so the vector field is $\mathbf{F} = \langle 2x, 2y \rangle$ .which looks similar to Fig. 3a.

Practice 2: $z = f(x,y) = xy$. Determine the gradient vector field for f(x,y) and sketch several vectors from this field.

One of the important properties of the gradient vector $\nabla f(x,y)$ is that it is perpendicular to the level curve of f(x,y) at the point (x,y) and points "uphill." So if the level curves of z=f(x,y) are known, then it is easy to sketch the gradient vector field. Fig. 9 shows level curves and gradient vectors of the paraboloid $z = f(x,y) = x^2 + y^2$.

Fig. 9: level curves and gradient vectors for z=x^2+y^2

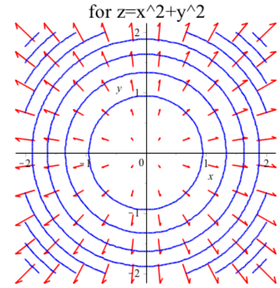


Fig. 10a shows the surface $f(x,y) = \frac{-5x}{1+x^2+y^2}$. Fig. 10b shows level curves for f(x,y) as well as gradient vectors. The gradient vectors have been drawn to all have the same length to better show their directions and that they are perpendicular to the level curves.

Fig. 10a: z=-5*x/(1+x^2+y^2)

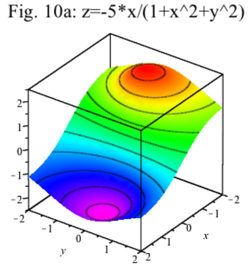
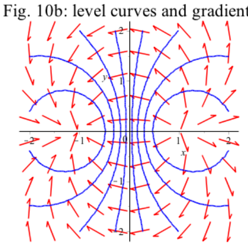
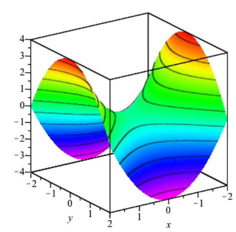


Fig. 10b: level curves and gradients

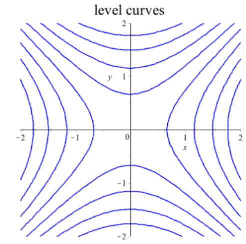


Practice 3: Fig. 11 shows the surface and the level curves for the saddle $f(x,y) = x^2 - y^2$. Sketch the gradient field.

Fig. 11: z=x^2-y^2



level curves



In Section 15.4 we will start with a vector field **F**(x,y) and determine if that field **F** is the gradient field of some function z=f(x,y). Such a field **F** is called **conservative field** and the function z=f(x,y) that gives rise to the field is called the **potential function** for **F**. Conservative fields have a number of important properties.

Problems

In problems 1 to 8, plot vectors from the given field **F**(x,y) at several locations (x,y) for integer values for x and y with -2≤x≤2 and -2≤y≤2.

1. $\mathbf{F} = \langle 2, 1 \rangle$
2. $\mathbf{F} = \langle 1, x \rangle$
3. $\mathbf{F} = \langle x, x \rangle$
4. $\mathbf{F} = \langle y, y \rangle$
5. $\mathbf{F} = \langle x^2, y \rangle$
6. $\mathbf{F} = \langle 1, x \cdot y \rangle$
7. $\mathbf{F} = \langle x, -y \rangle$
8. $\mathbf{F} = \langle y, 1-x \rangle$

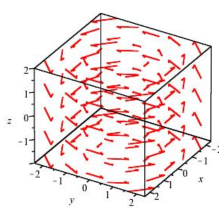
In problems 9 to 11, match the vector field F to the 3D plot.

9. (a) $\mathbf{F} = \langle 0,0,1 \rangle$

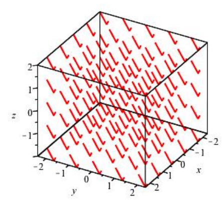
(b) $\mathbf{F} = \langle -y,x,0 \rangle$

(c) $\mathbf{F} = \langle 1,1,0 \rangle$

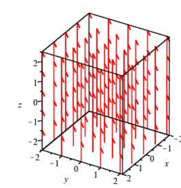
9A



9B



9C

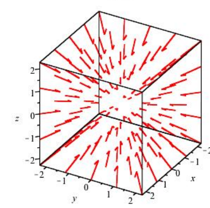


10. (a) $\mathbf{F} = \langle 0,z,-y \rangle$

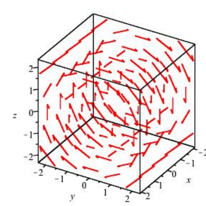
(b) $\mathbf{F} = \langle -x,-y,-z \rangle / \sqrt{x^2+y^2+z^2}$

(c) $\mathbf{F} = \langle -x,-y,-z \rangle$

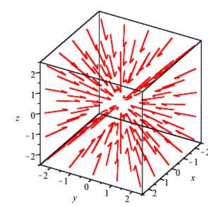
10A



10B



10C

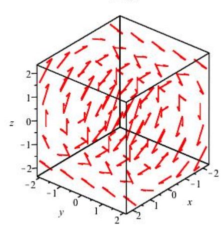


11. (a) $\mathbf{F} = \langle x,y,z \rangle$

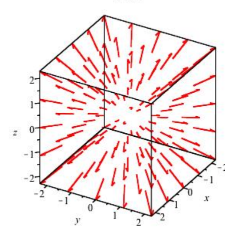
(b) $\mathbf{F} = \langle -z,0,x \rangle$

(c) $\mathbf{F} = \langle 1,0,0 \rangle$

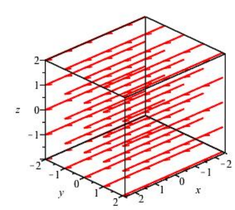
11A



11B



11C



In problems 12 to 21, find the gradient field for the given functions.

12. $f(x,y) = x^2 + y^2$

13. $f(x,y) = x^2 - y^2$

14. $f(x,y) = x/y$

15. $f(x,y) = x \cdot y^2 - x^2 \cdot y$

16. $f(x,y) = x \cdot \sin(y) + e^{x \cdot y}$

17. $f(x,y,z) = 3x + 2y - z$

18. $f(x,y,z) = x \cdot y \cdot z$

19. $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$

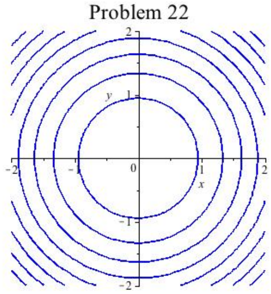
20. $f(x,y,z) = x^3 \cdot y + 2y \cdot z + 5$

21. $f(x,y,z) = \sin(x \cdot y \cdot z)$

In problems 22 to 27, some level curves of z=f(x,y) are given. On the each figure sketch the gradient vector field for this function. (Do not put heads on the arrows -- just use short line segments.)

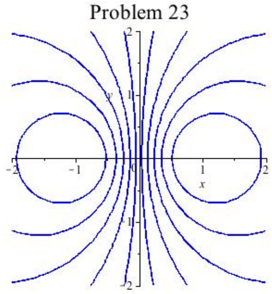
22.

Problem 22



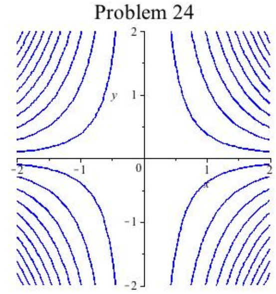
23.

Problem 23

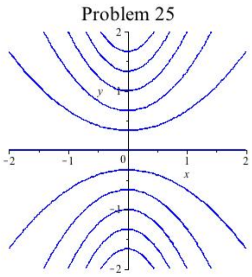


24.

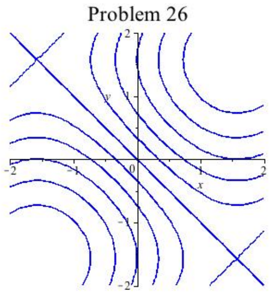
Problem 24



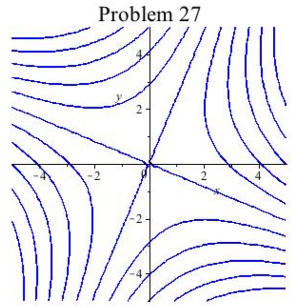
25.



26.



27.



Practice Answers

Practice 1: See Fig. P1. Fig. P1 Maple shows the < x, y/2> field for -2≤x≤2 and -2≤y≤2.

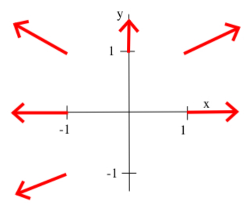


Fig. P1: vector field < x, y/2>

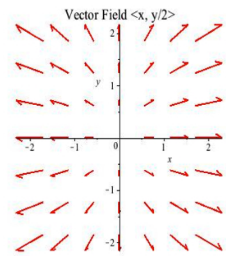
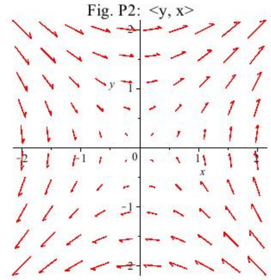


Fig. P1 Maple: < x, y/2 >

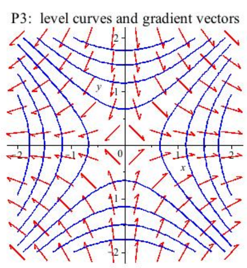
Practice 2: $z = f(x,y) = xy$. $\nabla f(x,y) = \langle y,x \rangle$.

The field is shown in Fig. P2.

Fig. P2: <y, x>



Practice 3: Fig. P3 shows gradient vectors for $f(x,y) = x^2 - y^2$. They have all been drawn the same length. The vectors closer to the origin should be shorter.



Appendix: Maple commands for plotting 2D and 3D vector fields

2D field <y,x> with(plots): fieldplot([y, x], x = -2..2, y = -2 .. 2);

This variation adds more details:

with(plots): fieldplot([y, x], x = -2..2, y = -2 .. 2, arrows = THIN, color = red, thickness=1, grid=[10,10], scaling=constrained, title = "Fig. P2: <y, x>", titlefont = ["ROMAN", 18]);

3D field "swirl" <y,z,x> with(plots): fieldplot3d([y,z,x], x = -2 .. 2, y = -2 .. 2, z = -2 .. 2);

With more details:

with(plots): fieldplot3d([y,z,x], x = -2 .. 2, y = -2 .. 2, z = -2 .. 2, arrows = THIN, color=red, thickness=1, grid=[8,8,8], title = "Fig. 6: <y,z,x>", titlefont = ["ROMAN", 18],orientation=[70,60,0]);

Note: Maple automatically centers each vector $\mathbf{F}(x,y)$ at the location (x,y).

Level curves and gradient vectors together

with(plots):

CP:=contourplot(x^2-y^2,x = -2..2, y = -2..2, color=blue, thickness=1, contours=8):

Gfld:=fieldplot([2*x,-2*y], x = -2..2, y = -2..2, arrows = THIN, color = red, thickness=1,grid=[10,10]):

display(Gfld,CP);