## 15.10 Gauss/Divergence Theorem

The Gauss/Divergence Theorem is the final fundamental theorem of calculus and the final mathematical piece needed to create Maxwell's equations. Like each of the previous fundamental theorems, it relates an accumulation (integral) in some dimension to the values of a related function in a lower dimension.

The Fundamental Theorem of Calculus (section 4.5) and the Fundamental Theorem of Line Integrals (section 15.4) said that the accumulation (integral) of a function over an interval or a line is equal to a related function (the antidarivative or potential function) evaluated at the boundary (endpoints) of the interval or line.

Stoke's Theorem (15.9), like the curl-circulation form of Green's Theorem (section 15.5), said that the accumulation of a function (curl) on a 2D surface is equal to a related function (the circulation) evaluated on the boundary curve of the surface.

Finally, the Gauss/Divergence Theorem, like the divergence-flux form of Green's Theorem, says that the accumulation of the divergence in a solid 3D region is equal to a related function (flux) evaluated on the boundary surface of the region.

**Gauss/Divergence Theorem**

If E is a solid, closed, simple 3D region with a piecewise-smooth boundary surface S, and **n** is the outward unit normal vector to S

then for the vector field $\mathbf{F} = \langle M, N, P \rangle$ whose components have continuous partial derivatives in a open region containing E

$$\iiint_{\mathbf{E}} \text{div } \mathbf{F} \bullet dV = \iint_S \mathbf{F} \bullet d\mathbf{S} = \iint_S \mathbf{F} \bullet \mathbf{n} \, dS = \text{flux across S}$$

A proof for a common special case is given in the Appendix.

The theorem is rather obvious in the case where the 3D region E contains a finite number if sources and sinks. If the region E inside of the boundary surface S contains a number of springs (pipes adding water) and sinks (pipes removing water), then the flux of water across the boundary surface S is the (springs input)–(sinks outputs). If several pipes inside E are adding water at a total rate of 5 $m^3/s$ and several pipes inside E are removing water at a rate of 2 $m^3/s$ then the (outward) flux across the boundary S is 3 $m^3/s$. The sum of the pipes adding water (positive divergence) plus the sum of the pipes removing water (negative divergence) equals the outward flux across the boundary of the region. The Gauss/Divergence Theorem extends this finite case idea to the case where potentially every point in the region E is a source or sink.

**Example 1:** Suppose pipes P1 at (1,0,0) and P2 at (2,2,1) are adding water at the rates of 5 $m^3/s$ and 3 $m^3/s$ , respectively, and P3 at (0,2,0) and P4 at (0,0,4) are removing water at the rates of 2 $m^3/s$ and 1 $m^3/s$, respectively. What is the outward flux of water across (a) the sphere S with center at (0,0,0) and radius 3.5 m, and (b) a tiny cube with center at (2,2,1)?

**Solution:** (a) Only P1, P2 and P3 are inside S so the flux is (5)+(3)–(2)= 6 $m^3/s$.

(b) Only P2 is inside this tiny cube so flux = 3 $m^3/s$ .

**Practice 1:** If the same pipes as in Example 1 are adding and removing grams of water per second, what is the flux across (a) the sphere S with center at (1,1,1) and radius 5 m and (b) a tiny cube with center at (1,2,3)?

**Example 2:** Calculate the flux across the sphere $x^2 + y^2 + z^2 = R^2$ for the radial vector field $\mathbf{F} = \langle x, y, z \rangle$ .

**Solution:** E = the 3D sphere with radius R so the volume of E is $\frac{4}{3}\pi R^3$ .

$$\text{div } \mathbf{F} = \nabla \bullet \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3 \text{ so}$$

$$\text{flux across S} = \iiint_{\mathbf{E}} \text{div } \mathbf{F} \bullet dV = 3 \cdot (\text{volume of the sphere}) = 4\pi R^3 .$$

**Practice 2:** Calculate the flux across the sphere $x^2 + y^2 + z^2 = R^2$ for the more general radial vector field $\mathbf{F} = \langle ax, by, cz \rangle$ .

**Example 3:** Calculate the outward flux across the boundary D of the solid unit cube $E = \{(x,y,z): 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$ for the field $\mathbf{F} = \langle xy, yz, xz \rangle$.

**Solution:** E = the solid cube, and $\text{div } \mathbf{F} = \nabla \bullet \mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xz) = y + z + x$ so

$$\text{flux across D} = \iiint_{\mathbf{E}} \text{div } \mathbf{F} \bullet dV = \int_0^1 \int_0^1 \int_0^1 x + y + z \, dz \, dy \, dz$$

$$= \int_0^1 \int_0^1 \left( xz + yz + \frac{1}{2}z^2 \right) \Big|_{z=0}^{1} dy \, dz = \int_0^1 \int_0^1 \left( x + y + \frac{1}{2} \right) dy \, dz = \ldots = \frac{3}{2} .$$

**Practice 3:** Calculate the outward flux across the boundary D of the solid unit cube $E = \{(x,y,z): 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$ for the field $\mathbf{F} = \langle xyz, xyz, xyz \rangle$.

**Flux for the inverse–square vector field** $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{\sqrt{x^2+y^2+z^2}} = \dfrac{\mathbf{r}}{|\mathbf{r}|^3}$

We cannot apply the Divergence Theorem to this field on a solid that includes the origin since the field is not defined at the origin, but we can apply it to a region between two spheres (or other smooth surfaces) so that the region doe not include the origin. Let $D = \{(x,y,z):\ 0 < A^2 < x^2+y^2+z^2 < B^2\}$ be the region outside of a sphere centered at the origin with radius A but inside a sphere centered at the origin with radius B. The boundary of D consists of the two spheres $S_1 = \{(x,y,z):\ A^2 = x^2+y^2+z^2\}$ and $S_2 = \{(x,y,z):\ x^2+y^2+z^2 = B^2\}$ but the unit normal vectors, pointing outward from the region D, have opposite directions (Fig. 1). Then

$\iiint_D \text{div}\,\mathbf{F}\,dV = \iint_S \mathbf{F}\bullet\mathbf{n}\,dS = \iint_{S_1} \mathbf{F}\bullet\mathbf{n}\,dS - \iint_{S_2} \mathbf{F}\bullet\mathbf{n}\,dS$ . But in Section 15.6 we determined that $\text{div}\,\mathbf{F} = 0$ for this field so $\iint_{S_1} \mathbf{F}\bullet\mathbf{n}\,dS = \iint_{S_2} \mathbf{F}\bullet\mathbf{n}\,dS$ meaning that the outward flux (away from the origin) across $S_2$ equals the inward flux (toward the origin) across $S_1$.

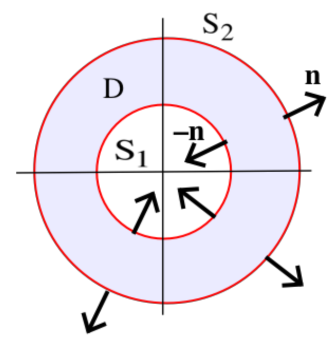


Fig. 1

We can evaluate $\iint_{S_1} \mathbf{F}\bullet\mathbf{n}\,dS$ by noting that on $S_1$ we have $|\mathbf{r}| = A$ and

$$\text{flux} = \iint_{S_1} \mathbf{F}\bullet\mathbf{n}\,dS = \iint_{S_1}\left(\frac{\mathbf{r}}{|\mathbf{r}|^3}\right)\bullet\left(\frac{\mathbf{r}}{|\mathbf{r}|}\right)dS = \iint_{S_1}\left(\frac{\mathbf{r}\bullet\mathbf{r}}{|\mathbf{r}|^4}\right)dS = \iint_{S_1}\frac{1}{|\mathbf{r}|^2}\,dS = \iint_{S_1}\frac{1}{A^2}\,dS$$

$$= \frac{1}{A^2}(\text{surface area of } S_1) = \frac{1}{A^2}(4\pi A^2) = 4\pi \ .$$

The resulting flux is the same $4\pi$ for a any region between two smooth surfaces that surround the origin.

**Example 4:** Determine the flux for the solid region D outside the sphere $S = \{(x,y,z):\ x^2+y^2+z^2 = 1\}$ and inside the ellipsoid $E = \left\{(x,y,z):\ \dfrac{x^2}{4}+\dfrac{y^2}{9}+\dfrac{z^2}{16} = 1\right\}$ for $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{\sqrt{x^2+y^2+z^2}} = \dfrac{\mathbf{r}}{|\mathbf{r}|^3}$ .

**Solution:** Based on the previous discussion we can immediately conclude that flux across D is $4\pi$ .

**Practice 4:** Determine the flux for the solid region D between a sphere centered at (2,3,4) with radius 4 and another sphere centered at (2,3,4) with radius 1 for $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{\sqrt{x^2+y^2+z^2}} = \dfrac{\mathbf{r}}{|\mathbf{r}|^3}$ .

**Units of flux**

Suppose **v** is a velocity vector field in meters per second (m/s) of a material that has constant density $\delta$ given in grams per cubic meter ( $g/m^3$). Then the field $\mathbf{F} = \delta \cdot \mathbf{v}$ has units $\left(\frac{g}{m^3}\right)\left(\frac{m}{s}\right) = \left(\frac{g}{m^2 \cdot s}\right)$ which measures the amount of material per a square meter flowing past a point each second. This field **F** is sometimes called the **flux density**. But in mathematics flux is defined as a surface integral, and then the units of flux become

$$\text{flux} = \iint_S \mathbf{F} \bullet \mathbf{n}\, dS = \{\text{amount per square meter per second}\}\{\text{area of S in square meters}\}$$

$$= \left\{\frac{g}{m^2 \cdot s}\right\}\left(m^2\right) = \frac{\mathbf{g}}{\mathbf{s}} = \text{the amount (mass) per second flowing across the surface S.}$$

> "In the case of fluxes, we have to take the integral, over a surface, of the flux through every element of the surface. The result of this operation is called the surface integral of the flux. It represents the quantity which passes through the surface."
>
> —James Clerk Maxwell

In other fields the flux density **F** has other units as does the surface integral:

**Heat** flux density has units $\frac{J}{m^2 \cdot s}$ so the units of the surface integral (our flux) are $\frac{J}{s}$ (Joules per second).

**Magnetic** flux density has units $\mathbf{B} = \frac{Wb}{m^2}$ (Weber per square meter =Tesla) so the units of magnetic flux are $\Phi_{\mathbf{B}} = \iint_S \mathbf{B} \bullet d\mathbf{S}$ = Weber.

**Gauss's Law for magnetism** states that the total magnetic flux through a closed surface is 0: $\Phi_{\mathbf{B}} = \iint_S \mathbf{B} \bullet d\mathbf{S} = \mathbf{0}$ for any closed surface S (since every magnetic north pole is attached to a magnetic south pole).

**Electric** flux density has units $\mathbf{E} = \frac{\mathbf{F}}{q}$ (force/charge= Newtons/coulomb=volts/meter) the magnetic flux is $\Phi_{\mathbf{E}} = \iint_S \mathbf{E} \bullet d\mathbf{S} = \frac{Q}{4\pi\varepsilon_0}(4\pi) = \frac{Q}{\varepsilon_0}$.

**Gauss' Law for electric fields** states that the total electric flux through a closed surface S is $\Phi_{\mathbf{E}} = \iint_S \mathbf{E} \bullet d\mathbf{S} = \frac{Q}{4\pi\varepsilon_0}(4\pi) = \frac{Q}{\varepsilon_0}$ where Q is the total electric charge inside S and $\varepsilon_0$ is a (very small) constant called the electric constant or permittivity of free space. The total electric flux need not equal 0 for a closed surface S since the divergence at a point charge inside S may not equal 0 and the flux is a constant times the total of the charges inside S.

## Problems

For problems 1 to 6 pipe P1=(1,2,3) is inputting 7 $m^3/s$ of water, P2=(–2,1,1) is removing 3 $m^3/s$ of water, P3=(2,2,0) is inputting 2 $m^3/s$ of water, P4=(1,1,0) is inputting 6 $m^3/s$ of water, P5=(2,2,2) is removing 4 $m^3/s$ of water, and P6=(3,1,3) is inputting 6 $m^3/s$ of water,

1. What is the net flux across the sphere with center at the origin and (a) radius 2 and (b) radius 4?
2. What is the net flux across the sphere with center at the origin and (a) radius 3 and (b) radius 5?
3. What is the net flux across the sphere with center at (2,2,0) and (a) radius 1 and (b) radius 3?
4 What is the net flux across the sphere with center at (2,2,0) and (a) radius 4 and (b) radius 5?
5. What is the net flux across the boundary of the region outside the sphere with center at the origin and radius 3 and inside the sphere with center at the origin and radius 5?
6. What is the net flux across the boundary of the region outside the sphere with center at the origin and radius 1 and inside the sphere with center at the origin and radius 3?

In problems 7 to 14 find the outward flux of the field F across the boundary surface of the given solid.

7. $\mathbf{F} = \langle x,y,z\rangle$ across the solid sphere $E = \{(x,y,z): x^2 + y^2 + z^2 \le 4\}$ .
8. $\mathbf{F} = \langle -y,x,z\rangle$ across the solid sphere $E = \{(x,y,z): x^2 + y^2 + z^2 \le 4\}$ .
9. $\mathbf{F} = \langle x,\ 2y,\ -z\rangle$ across the solid sphere $E = \{(x,y,z): x^2 + y^2 + z^2 \le 9\}$ .
10. $\mathbf{F} = \langle -x,\ 2y,\ -z\rangle$ across the solid sphere $E = \{(x,y,z): x^2 + y^2 + z^2 \le 9\}$ .
11. $\mathbf{F} = \langle x,\ 2y,\ z\rangle$ across the solid box $E = \{(x,y,z): 0 \le x \le 2,\ 0 \le y \le 3,\ 0 \le z \le 4\}$ .
12. $\mathbf{F} = \langle 2x,\ 3y,\ 4z\rangle$ across the solid box $E = \{(x,y,z): 1 \le x \le 2,\ 1 \le y \le 3,\ 1 \le z \le 4\}$ .
13. $\mathbf{F} = \langle x^2,\ y^2,\ z^2\rangle$ across the solid box $E = \{(x,y,z): 0 \le x \le 2,\ 0 \le y \le 3,\ 0 \le z \le 4\}$ .
14. $\mathbf{F} = \langle x^2,\ -y^2,\ z^2\rangle$ across the solid box $E = \{(x,y,z): 1 \le x \le 2,\ 1 \le y \le 3,\ 1 \le z \le 4\}$ .

15. If div $\mathbf{F} = 0$ for the region between two concentric spheres, what is the relationship between the fluxes across the surfaces of the spheres?
16. E is a convex solid with volume V. Find the flux of F across the boundary of E (a) if div $\mathbf{F} = 0$ at every point, and (b) if div $\mathbf{F} = 3$ at every point.
17. $\mathbf{F} = \langle 1,\ -2,\ 3\rangle$. Without doing any calculations determine whether the flux across each sphere is positive, negative or zero. (a) Sphere with radius 2 and center at the origin. (b) Sphere with radius 1 and center at (1, 2, 3) (c) Sphere with radius 1 and center at (3, –2, 1).

18. $\mathbf{F} = \langle a, b, c \rangle$. Without doing any calculations determine whether the flux across each sphere is positive, negative or zero. (a) Sphere with radius 2 and center at the origin. (b) Sphere with radius 1 and center at $(1, 2, 3)$ (c) Sphere with radius 1 and center at $(3, -2, 1)$.
19. $\mathbf{F} = \langle 0, y, 0 \rangle$. Without doing any calculations determine whether the flux across each sphere is positive, negative or zero. (a) Sphere with radius 2 and center at the origin. (b) Sphere with radius 1 and center at $(1, 2, 3)$ (c) Sphere with radius 1 and center at $(3, -2, 1)$.
20. $\mathbf{F} = \langle 0, 1, z \rangle$. Without doing any calculations determine whether the flux across each sphere is positive, negative or zero. (a) Sphere with radius 2 and center at the origin. (b) Sphere with radius 1 and center at $(1, 2, 3)$ (c) Sphere with radius 1 and center at $(3, -2, 1)$.
21. $\mathbf{F} = \langle x^2, y^2, z^2 \rangle$. Without doing any calculations determine whether the flux across each sphere is positive, negative or zero. (a) Sphere with radius 2 and center at the origin. (b) Sphere with radius 1 and center at $(1, 2, 3)$ (c) Sphere with radius 1 and center at $(3, -2, 1)$.
22. Suppose $\mathbf{v}$ is a velocity vector field in meters per second (m/s) and $\delta$ is a density given in cows per cubic meter ( $c/m^3$), and F is the vector field $\mathbf{F} = \delta \cdot \mathbf{v}$. What are the units of flux for this field across the boundary of a solid region?

## Practice Answers

**Practice 1:** (a) All of the pipes are inside this sphere so flux = (5)+(3)–(2)–(1) = 5 g/s.

(b) None of the pipes are inside this tiny cube so flux = 0 g/s.

**Practice 2:** $\text{div } \mathbf{F} = \nabla \bullet \mathbf{F} = \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz) = a + b + c$ so

$$\text{flux across S} = \iiint_{\mathbf{E}} \text{div } \mathbf{F} \bullet dV = (a + b + c) \cdot (\text{volume of the sphere}) = (a + b + c) \cdot \frac{4}{3}\pi R^3 \ .$$

**Practice 3:** E = the solid cube, and $\text{div } \mathbf{F} = \nabla \bullet \mathbf{F} = yz + xz + xy$ so

$$\text{flux across D} = \iiint_{\mathbf{E}} \text{div } \mathbf{F} \bullet dV = \int_0^1 \int_0^1 \int_0^1 (yz + xz + xy)\, dz\, dy\, dz = \ \dots \ = \frac{3}{4} \ .$$

**Practice 4:** The smaller sphere is inside the larger one and the larger one does not contain the origin so, as was shown in section 15.5, div $\mathbf{F}$ = 0 inside the larger sphere. Then by the Divergence Theorem, the flux across each sphere is 0 so the flux across the boundary of D is 0.

**Appendix: Proof of the Gauss/Divergence Theorem**

There are two intuitive ways to think of the Gauss/Divergence Theorem that make the result seem "obvious." These are not proofs (see later in this Appendix), but they contain the essence of the theorem.

**First intuitive approach:** Imagine that the interior of the solid region E is partitioned into lots of cells and that the divergence of each cell is the total outward flow from that cell. If two cells share a boundary surface then the outward flow from one (a positive divergence) is the inward flow into the other (a negative divergence). So adding the divergences for all of the cells (a triple sum in x, y and z), all of the inside divergences sum to 0, and the final result is just the sum of those divergences on the surface S of the solid region E: $\sum_{x,y,z}\sum\sum \text{div}\,\mathbf{F} = \sum_{\text{surface}}\sum \mathbf{F}\bullet\mathbf{n}$

**Second intuitive approach:** Partition the solid region E into thin slices parallel to the xy-plane (Fig. A1). Then each slice will contain a 2D region R with boundary S. Applying Green's Theorem to this slice we have $\iint_{\mathbf{R}} \text{div}\,\mathbf{F}\ dA = \oint_C \mathbf{F}\bullet\mathbf{n}\ ds$. And by adding these results together

$$\sum_{\Delta z}\left(\iint_{\mathbf{R}} \text{div}\,\mathbf{F}\ dA\right)dz = \sum_{\Delta z}\left(\oint_C \mathbf{F}\bullet\mathbf{n}\ ds\right)dz$$ we expect the Divergence Theorem

$$\iiint_E \text{div}\,\mathbf{F}\ dV = \iint_S \mathbf{F}\bullet\mathbf{n}\ dS\ .$$

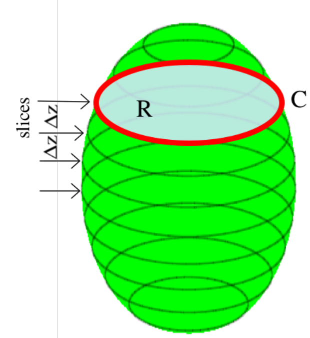

Fig. A1

Neither of these is a proof, but they might give you a better understanding of the "why" of the Divergence Theorem.

**Proof for a common special case: E is a convex region**

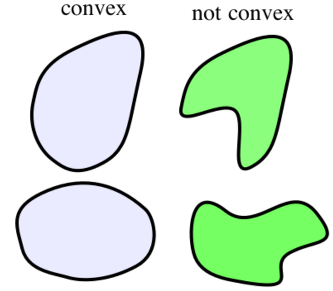

Fig. A2

A region E is called **convex** if a straight line connecting any two points in E lies in E. Fig. A2 shows some convex and non-convex 2D regions.

Assume that E is a convex region with a piecewise-smooth boundary S, that $\mathbf{F} = \langle M,N,P\rangle$ has continuous partial derivatives, and that $\mathbf{n} = \langle \mathbf{n_1}, \mathbf{n_2}, \mathbf{n_3}\rangle$ is the unit, outward pointing normal vector to S.

If E is convex, then the projection of E onto the xy, xz and yz-planes is a 2D convex region. Let D be the projection of E onto the xy-plane (Fig. A3). Then for each (x,y)

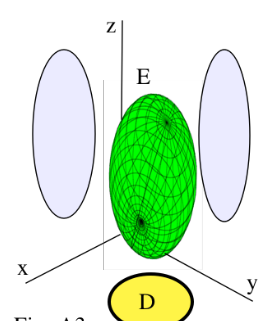

Fig. A3

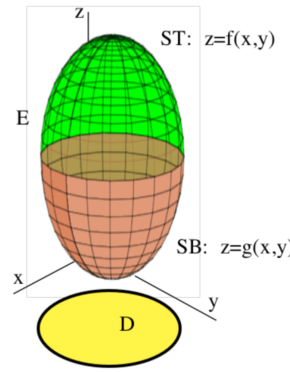

Fig. A4

in D the solid E is bounded by a top surface

$ST = \{(x,y,z): (x,y) \text{ is in D and } z = f(x,y)\}$ and a bottom surface

$SB = \{(x,y,z): (x,y) \text{ is in D and } z = g(x,y)\}$ (Fig. A4). If we can show that

$$\iiint_E \frac{\partial M}{\partial x}\, dV = \iint_S M\, dS = \iint_S \mathbf{F} \bullet \mathbf{n_1}\, dS,$$

$$\iiint_E \frac{\partial N}{\partial y}\, dV = \iint_S N\, dS = \iint_S \mathbf{F} \bullet \mathbf{n_2}\, dS, \text{ and}$$

$$\iiint_E \frac{\partial P}{\partial z}\, dV = \iint_S P\, dS = \iint_S \mathbf{F} \bullet \mathbf{n_3}\, dS \text{ then}$$

$$\iiint_E \text{div } \mathbf{F}\, dV = \iiint_E \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\, dV = \iint_S (M + N + P)\, dS = \iint_S \mathbf{F} \bullet \mathbf{n_1} + \mathbf{F} \bullet \mathbf{n_2} + \mathbf{F} \bullet \mathbf{n_3}\, dS$$
$$= \iint_S \mathbf{F} \bullet \mathbf{n}\, dS$$

which is the Divergence Theorem.

Working to show $\displaystyle\iiint_E \frac{\partial P}{\partial z}\, dV = \iint_S P\, dS$:

$\displaystyle\iiint_E \frac{\partial P}{\partial z}\, dV$: By the Fundamental Theorem of Calculus,

$$\iiint_E \frac{\partial P}{\partial z}\, dV = \iint_D \left( \int_{g(x,y)}^{f(x,y)} \frac{\partial P}{\partial z}\, dz \right) dx\, dy = \iint_D P(x,y,f(x,y)) - P(x,y,g(x,y))\, dx\, dy.$$

$\displaystyle\iint_S P\, dS$: The surface S consists of the two surfaces ST where z=f(x,y) and SB where z=g(x,y). On the top surface ST, the normal $\mathbf{n}$ points up and $\mathbf{n} = \langle -f_x, -f_y, 1 \rangle$ . On the bottom surface SB, the normal vector $\mathbf{n}$ points down and $\mathbf{n} = \langle g_x, g_y, -1 \rangle$.

$$\iint_{ST} P\, dS = \iint_{ST} P(x,y,z)\, dS = \iint_D \langle 0,0,P(x,y,f(x,y)) \rangle \bullet \langle -f_x, -f_y, 1 \rangle\, dx\, dy = \iint_D P(x,y,f(x,y))\, dx\, dy .$$

$$\iint_{SB} P\, dS = \iint_{SB} P(x,y,z)\, dS = \iint_D \langle 0,0,P(x,y,g(x,y)) \rangle \bullet \langle g_x, g_y, -1 \rangle\, dx\, dy$$
$$= \iint_D -P(x,y,g(x,y))\, dx\, dy .$$

Putting these last two results together,

$$\iint_S P\, dS = \iint_{ST} P\, dS + \iint_{SB} P\, dS = \iint_D P(x,y,f(x,y))\, dx\, dy - \iint_D P(x,y,g(x,y))\, dx\, dy$$

$$= \iint_D P(x,y,f(x,y)) - P(x,y,g(x,y))\, dx\, dy \text{ which is the same result we got for } \iiint_E \frac{\partial P}{\partial z}\, dV.$$

If the ST and SB surfaces are connected by vertical walls SV, the result is still true since **n** has the form $\mathbf{n} = \langle a, b, 0 \rangle$ so $\iint\limits_{SV} P \; dS = \iint\limits_{D} \langle 0,0,P(x,y,g(x,y)) \rangle \bullet \langle a, b, 0 \rangle \; dx\, dy = 0$ .

Similarly $\iiint\limits_{E} \frac{\partial M}{\partial x} \; dV = \iint\limits_{S} M \; dS$ and $\iiint\limits_{E} \frac{\partial N}{\partial y} \; dV = \iint\limits_{S} N \; dS$ , and then the Divergence Theorem is proven for convex solid regions.

Te proof for more general solid regions is more complicated and is not given here.