15.3 Line Integrals

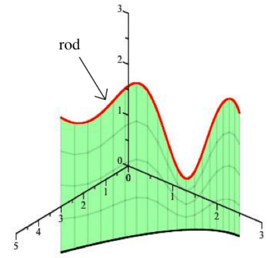


Fig. 1: Curtain on a bent rod

A curtain is hanging from a very bent rod (Fig. 1). If we have an equation z=f(x,y) or (x(t), y(t), z(t)) for the rod, how can we calculate the area of the curtain between the rod and the floor (the xy-plane). And if the rod has different densities, $\delta(x,y,z)$ at locations (x,y,z), along its length, how can we calculate the total mass of the rod?

These are two silly questions, but their solutions illustrate the main idea of this section: an integral along a curve, a **line integral**. One of the main applications of line integrals is to determine the work done to move an object in a force field in two or three dimensions, and there are others.

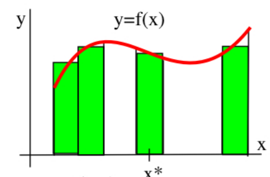


Fig. 2

Curtain area solution: In beginning calculus we created integral of f(x) on the interval a≤x≤b on the x-axis (Fig. 2) by partitioning the interval, picking a representative point x* in each subinterval, calculating the area of the little rectangle above the subinterval as $f(x^*)\cdot\Delta x$, forming the Riemann sum $\sum f(x^*)\cdot\Delta x$ of these little areas, and finally taking the limit as $\Delta x \to 0$ to get an integral: $\int_a^b f(x)\,dx$.

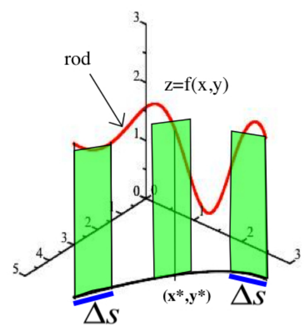


Fig. 3: Partitioned curtain

The same strategy works for the shower curtain, except here we partition the curve C in the xy-plane (Fig. 3), pick a representative point (x*, y*) in each subinterval, and find the area of each little rectangle as $f(x^*,y^*)\cdot\Delta s$ where Δs is the length of the subinterval. Then as $\Delta s \to 0$ we get that $\sum f(x^*,y^*)\cdot\Delta s \to \int f(x,y)\,ds$. If our curve in the xy-plane is parameterized by t, $\mathbf{r}(t)=\langle x(t),y(t)\rangle$ then z(t)=f(x(t), y(t)) and $\Delta s=\sqrt{(\Delta x)^2+(\Delta y)^2}$. The Riemann sum becomes

$$\sum f(x^*(t),y^*(t))\cdot\Delta s=\sum f(x^*(t),y^*(t))\cdot\frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{\Delta t}\cdot\Delta t=\sum z(t^*)\cdot\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2}\cdot\Delta t\ .$$

Taking the limit of this as $\Delta s \to 0$, we get that

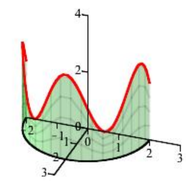
$$\textbf{\{area between f(x,y) and xy-plane\}}=\int_C f\,ds=\int_{t=a}^{t=b} z(t)\cdot\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt\ .$$

Example 1: Suppose the curve C is parameterized by $\mathbf{r}(t)=\langle x(t),y(t)\rangle$ with $x(t)=1+t^2$ and $y(t)=3-t$, and that $f(x,y)=2+\sin(3xy)=2+\sin(3(1+t^2)(3-t))$. This is the situation in Fig. 1. Write an integral in terms of t for the area between the curve C in the xy-plane and f(x,y) for t from 0 to 2. If the units of x and y are meters and the t units are seconds, then what are the units of the result.

Solution: area between f(x,y) and the xy - plane $= \int_0^2 \left(2+\sin(3(t^2+1)(3-t))\right)\sqrt{(2t)^2+(-1)^2}\ dt$

Unfortunately we can not evaluate this integral by hand, but a calculator can: area=10.754 m^2.

Fig. 4: <2cos(t), 2sin(t), 2+cos(5t)>



Practice 1: Represent the area between the curve C parameterized by $\mathbf{r}(t) = \langle 2\cos(t),\ 2\sin(t)\rangle$ f(r(t))=2+cos(5t) (Fig. 4) as a definite integral and evaluate the integral.

General case for line integral of a scalar function f over a path C : $\int_C f\ ds$

Suppose the path C is parameterized by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ where x(t) and y(t) are smooth (differentiable) functions of t, and that t varies from t=a to t=b. Then

$$\int_C \mathbf{f}\ ds = \int_{t=a}^{b} \mathbf{f}(\mathbf{r}(t))\cdot\left|\frac{dr}{dt}\right|\ dt = \int_{t=a}^{b} \mathbf{f}(\mathbf{r}(t))\cdot|\mathbf{r}'(t)|\ dt$$

Note: $ds = |\mathbf{r}'(t)|\ dt$ simply says that {change in position}={speed} $\cdot$ {change in time}: distance =rate $\cdot$ time.

Example 2: C1 is the semicircular path from (2,0) to (-2,0) parameterized by $\mathbf{r}(t) = \langle 2\cos(t),\ 2\sin(t)\rangle$ for t= 0 to π, and f(x,y)= 5+2x+4y (Fig. 5). C2 is the straight line path from (2,0) to (-2,0) parameterized by the path $\mathbf{r}(t) = \langle 2 - 4t,\ 0\rangle$ for t=0 to t=1.

Evaluate $\int_{C1} f\ ds$ and $\int_{C2} f\ ds$.

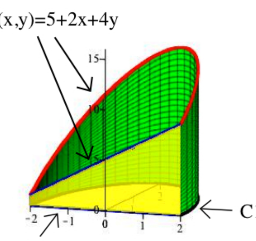


Fig. 5

Solution: Along C1 $|\mathbf{r}'(t)| = \sqrt{(-2\sin(t))^2+(2\cos(t))^2} = 2$ and

$f(\mathbf{r}(t)) = 5 + 2\cdot 2\cdot\cos(t) + 4\cdot 2\cdot\sin(t)$ so

$$\int_{C1} f\ ds = \int_0^{\pi} (5+4\cdot\cos(t)+8\cdot\sin(t))(2)\ dt = 32+10\pi \approx 63.4 .$$

Along C2 $|\mathbf{r}'(t)| = \sqrt{(-4)^2+(0)^2} = 4$ and $f(\mathbf{r}(t)) = 5+2\cdot(2-4t))+4\cdot(0) = 9-8t$ so

$$\int_{C2} f\ ds = \int_0^1 (9-8t)(4)\ dt = 20 .$$

Even though C1 and C2 begin and end at the same points, the values of the line integrals are different.

Note: We can always parameterize the straight line from A to B by $P(t) = (1-t)\cdot A + t\cdot B$ for t=0 to t=1. Circles are typically parameterized using variations of $x = r\cdot\cos(t)$ and $y = r\cdot\sin(t)$ that take directions and shifts into account.

Units: In the previous example, suppose f(x,y) is the density (in kg/m) of the curve **r** at the location (x,y), that the location (x,y) is in meters, m, and that time is in seconds, s. Then the units of the integral are

$$\int_C \mathrm{f\ ds} = \int_{t=a}^{b} \underset{\left(\frac{\mathrm{kg}}{\mathrm{m}}\right)}{\mathrm{f}(\mathbf{r}(\mathrm{t}))}\ \underset{\left(\frac{\mathrm{m}}{\mathrm{s}}\right)}{|\mathbf{r}'(\mathrm{t})|}\ \underset{(\mathrm{s})}{\mathrm{dt}} = \mathrm{kg}$$

Practice 2: C is the straight line path from (0,1) to (4,2) and f(x,y)=2x+y. Evaluate $\int_C \mathrm{f\ ds}$.

If the units of f are pounds, the x and y units are feet, and the t units are minutes, then what are the units of this line integral?

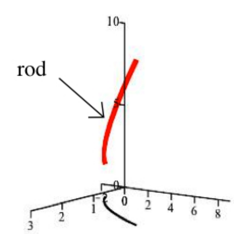
3D Mass of a Rod

Suppose a curve $\mathrm{C(t)} = \left(\mathrm{t,t^2,1+t^2}\right)$ meters (Fig. 6) has linear density $\delta(\mathrm{x,y,z}) = \mathrm{x - y + z - 1}$ g/m. We can represent the mass of the curve from t=1 to t=3 as an integral in t by proceeding as before and starting with a partition of the curve into segments of lengths $\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} = \sqrt{1 + (2t)^2 + (2t)^2} = \sqrt{1 + 8t^2}$.

The density at each location (x*, y*, z*) on the curve is

$\delta(\mathrm{x,y,z}) = \mathrm{x - y + z - 1} = (\mathrm{t}) - (\mathrm{t}^2) + (1 + \mathrm{t}^2) - 1 = \mathrm{t}$ so the mass along each segment is $\{\text{segment mass}\} = \text{density} \cdot \text{length} = \mathrm{t} \cdot \sqrt{1 + 8t^2}$. Finally, the total mass of the rod is $\displaystyle\int_{t=1}^{3} \mathrm{t} \cdot \sqrt{1 + 8t^2}\ \mathrm{dt} = \frac{1}{24}\left(1 + 8t^2\right)^{3/2} \Big|_1^3 = \frac{73}{24}\sqrt{73} - \frac{9}{8} \approx 24.86\ g$.

Fig. 6: <t, t^2, 1+t^2>



Generalizing this approach to any continuous function f(x,y,z) along a smooth curve C parameterized by $\mathbf{r}(\mathrm{t}) = \langle \mathrm{x(t),y(t),z(t)} \rangle$ from t=a to t=b, we again have

$$\int_{\mathrm{C}} \mathrm{f\ ds} = \int_{t=a}^{t=b} \mathrm{f}(\mathbf{r}(\mathrm{t})) \cdot |\mathbf{r}'(\mathrm{t})| \cdot dt\ .$$

Practice 3: Represent the total mass of the curve $\mathrm{C(t)} = \left(\mathrm{2t,t^2,t}\right)$ that has density $\delta(\mathrm{x,y,z}) = \mathrm{x + z}$ from t=1 to t=4 as an integral and evaluate the integral.

Note: The various application formulas for first and second moments and centers of mass in section 14.6 also apply here, but we replace the triple integrals $\iiint_R$ with $\int_{\mathrm{C}}$ and the dV with ds.

Work moving along a curve

Previously in this section the function f was a scalar-valued function, but very interesting situations arise when we move along a curve in a vector field.

In section 11.4 we saw that the elementary idea that "work=force times distance" could be extended to "work=(force in the direction of movement) $\cdot$ (displacement) = $\mathbf{F} \bullet \mathbf{D}$," the dot product of **F** and **D**, where **F** is a force vector and **D** is the displacement vector. This is exactly the idea we need to calculate the work moving an object along a curve in two or three dimensions.

Line integral of a vector-valued function F over a path C

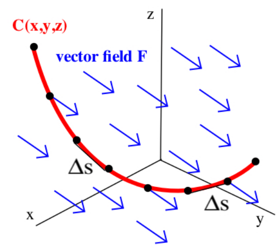


Fig. 7: Curve C and vector fiekd F

Work: Suppose C is a smooth curve in 3D parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \le t \le b$ and that $\mathbf{F}(x,y,z)$ is a 3D force vector field. If we partition C (Fig. 7) into small time increments, then at location P*=(x(t*), y(t*), z(t*) the displacement is tangent to the curve and has length Δs so the displacement is $\Delta s \cdot \mathbf{T}(t^*)$ where $\mathbf{T}(t^*)$ is the unit tangent vector at P* (Fig. 8). The work for **F** to move the object along that small Δs segment of the curve is $\mathbf{F} \bullet \mathbf{T} \cdot \Delta s$. Then the total work is approximately

$$\sum \mathbf{F}(x(t^*), y(t^*), z(t^*)) \bullet \mathbf{T}(x(t^*), y(t^*), z(t^*)) \cdot \Delta s$$

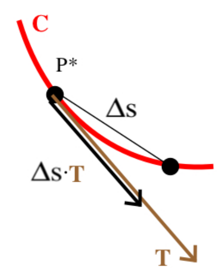


Fig. 8: work on one segment

As $\Delta s \to 0$ the approximations become better and better, and the work to move an object along C is defined to be

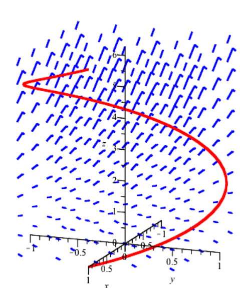
$$\text{work} = \int_C \mathbf{F} \bullet \mathbf{T}\ ds = \int_C \mathbf{F}(x(t), y(t), z(t)) \bullet \mathbf{T}(x(t), y(t), z(t))\ ds$$

But $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ and $ds = |\mathbf{r}'(t)|\ dt$ so

$$\boxed{\text{work} = \int_C \mathbf{F} \bullet \mathbf{T}\ ds = \int_{t=a}^{b} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\ dt}$$

Note: The units for work are the units of **F** times the units of length, ds.

Fig. 9: C=(cos(t),sin(t),t) F=<1,1,z>



Example 2: Represent the work done in the field $\mathbf{F}(x,y,z) = \langle 1, 1, z \rangle$ to move an object along the helix $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ for t=0 to t=2π (Fig. 9) as an integral and then evaluate the integral.

Solution: $\mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) = \langle 1, 1, z \rangle \bullet \langle -\sin(t), \cos(t), 1 \rangle = -\sin(t) + \cos(t) + t$ so $\text{work} = \int_{t=0}^{2\pi} (-\sin(t) + \cos(t) + t)\ dt = 2\pi^2$

Practice 4: Represent the work done in the constant field $\mathbf{F}(x,y,z) = \langle 3,2,1 \rangle$ to move an object along the curve $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ for t=0 to t=2 as an integral and then evaluate the integral.

Definition

If C is a smooth curve given by the vector function $\mathbf{r}(t)$ for a≤t≤b, and $\mathbf{F}$ is a continuous vector field defined on C, then the **line integral of F along C** is

$$\int_C \mathbf{F}(\mathbf{r}(t)) \bullet d\mathbf{r} = \int_{t=a}^{b} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \bullet \mathbf{T}\, ds$$

Note: This pattern will occur often in future sections, and you should be familiar with all three notations. The middle integral is usually the easiest to use for computations.

Note: If $\mathbf{F} = \langle M,N,P \rangle$ and $\mathbf{T} = \left\langle \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \right\rangle$ then $\mathbf{F} \bullet \mathbf{T} ds = M dx + N dy + P dz$. Units = (F units)(ds units).

You should also recognize what is happening geometrically (Fig. 10). If the angle θ between **F** and **r'** is 90^o then $\mathbf{F} \bullet \mathbf{T} = |\mathbf{F}|\,|\mathbf{T}| \cos(\theta) = 0$, if the **F** and **r'** angle is acute $(-90^o < \theta < 90^o)$ then $\mathbf{F} \bullet \mathbf{T} = |\mathbf{F}|\,|\mathbf{T}| \cos(\theta) > 0$, and if the **F** and **r'** angle is obtuse $(90^o < \theta < 270^o)$ then $\mathbf{F} \bullet \mathbf{T} = |\mathbf{F}|\,|\mathbf{T}| \cos(\theta) < 0$.

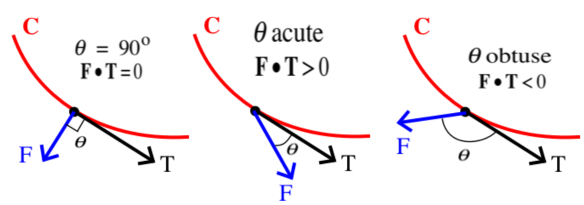


Fig. 10: Geometry of $\mathbf{F} \bullet \mathbf{T}$

Practice 5: C is the semicircle given by $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $0 \le t \le \pi$, and four vector fields are $\mathbf{F1} = \langle 1,0 \rangle$, $\mathbf{F2} = \langle -1,0 \rangle$, $\mathbf{F3} = \langle 0,1 \rangle$ and $\mathbf{F4} = \langle x,y \rangle$. Use sketches of the path C and a few vectors from the field $\mathbf{F}$ to estimate whether each line integral $\int_C \mathbf{F}(\mathbf{r}(t)) \bullet d\mathbf{r}$ is positive, negative or zero.

Work in Gravitational, Electrical and Magnetic Fields: In all three of these situations the force between points is inversely proportional to the square of the distance between the points and acts along the straight line connecting the points. If one point is at the origin and the other is at (x,y,z) then the magnitude of $\mathbf{F}$ is

$|\mathbf{F}| = \frac{k}{|\mathbf{r}|^2} = \frac{k}{(x^2+y^2+z^2)}$ and the direction of $\mathbf{F}$ is $\frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\langle x,y,z \rangle}{(x^2+y^2+z^2)^{1/2}}$ so $\mathbf{F}(\mathbf{r}) = \frac{k\langle x,y,z \rangle}{(x^2+y^2+z^2)^{3/2}} = \frac{k\mathbf{r}}{|\mathbf{r}|^3}$.

Example 3: Find the work to moving an object along the path from (1,1,1) in a straight line to (c,c,c) where c>1.

Solution: The line can be parameterized by $\mathbf{r}(t) = \langle t,t,t \rangle$ as t goes from 1 to c, and

$$\mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) = \frac{k\langle t,t,t \rangle}{|\langle t,t,t \rangle|^3} \bullet \langle 1,1,1 \rangle = \frac{k3t}{(3t^2)^{3/2}} = \frac{k}{\sqrt{3}}\frac{1}{t^2}$$ so the work done is

$$\text{work} = \int_{t=1}^{c} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\,dt = \frac{k}{\sqrt{3}} \int_{t=1}^{c} \frac{1}{t^2}\,dt = \frac{k}{\sqrt{3}}\left(-\frac{1}{t}\right)\Big|_{t=1}^{c} = \frac{k}{\sqrt{3}}\left(1-\frac{1}{c}\right)$$

. As the object is moved farther and farther away, as $c \to \infty$, the work approaches the finite value $k/\sqrt{3}$.

Flow along a curve C in a vector field F

Flow: If a smooth curve C is parameterized by r(t) in a continuous vector field F, then the **flow along C** from t=a to t=b is $\text{flow} = \int_C \mathbf{F} \bullet \mathbf{T}\; ds$.

F might represent the velocity field of a fluid in a region of space (water in a river channel, air in a wind tunnel), then the integral of $\mathbf{F} \bullet \mathbf{T}$ along a curve C in that space is the "flow" along that curve. In this case, if the units of F are m/sec, then the units for flow are m^2/sec.

If C is a closed loop, then $\int_C \mathbf{F} \bullet \mathbf{T}\; ds$ is called the **circulation** around C.

Flow is calculated in the same way as work.

Flux across a closed curve C in 2D

Just as flow measured the accumulation of a vector field along a curve C, the flux measures the accumulation as the vector field crosses perpendicular to the curve so we need the velocity of the field in the direction of the normal vector to the curve at each point. And then we want to accumulate all of those little values, an integral.

$\{\text{Flux of F across closed curve C}\} = \int_C \mathbf{F} \bullet \mathbf{n}\, ds$ where $\mathbf{n}$ is the (outward) unit normal vector to C and the curve C is traversed exactly once in the counterclockwise direction.

Flow is the line integral of the scalar component $\mathbf{F} \bullet \mathbf{T}$ of $\mathbf{F}$ in the direction of the unit tangent vector to C.

Flux is the line integral of the scalar component $\mathbf{F} \bullet \mathbf{n}$ of $\mathbf{F}$ in the direction of the unit normal vector to C.

If $\mathbf{F}(x,y,z) = \langle \mathbf{M(x,y), N(x,y)} \rangle = M\mathbf{i} + N\mathbf{j}$ is a vector field in 2D and C is parameterized in the xy-plane in the counterclockwise direction by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ then the unit normal vector is $\mathbf{n} = \mathbf{T} x \mathbf{k}$ where $\mathbf{T}$ is the unit tangent vector and $\mathbf{k} = \langle 0,0,1 \rangle$ (Fig. 11). Then

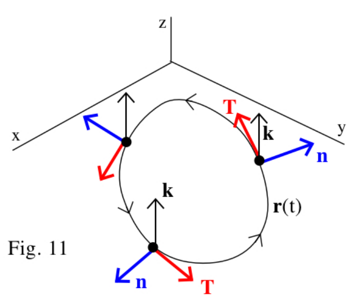


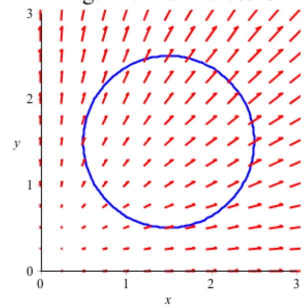
Fig. 11

$$\mathbf{n} = \mathbf{T}\mathbf{x}\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left\langle \frac{dy}{ds}, -\frac{dx}{ds}, 0 \right\rangle \text{ so}$$

$$\mathbf{F} \bullet \mathbf{n} = \langle M, N \rangle \bullet \left\langle \frac{dy}{ds}, -\frac{dx}{ds} \right\rangle = M\frac{dy}{ds} - N\frac{dx}{ds} \quad \text{and}$$

$$\text{flux} = \int_C \mathbf{F} \bullet \mathbf{n}\, ds = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds} \right) ds = \int_C M dy - N dx .$$

Fig. 12: Flux across C



Example 4: Calculate the flux across the circle C parameterized by

$\mathbf{r}(t) = \langle 1.5 + \cos(t), 1.5 + \sin(t) \rangle$ for the vector field $\mathbf{F}(x,y) = \langle x,y \rangle = x\mathbf{i} + y\mathbf{j}$. (Fig. 12)

Solution: M=1.5+cos(t), N=1.5+sin(t), dx=-sin(t) dt, dy=cos(t) dt, $0 \le t \le 2\pi$. This path traverses the circle in the counterclockwise direction. Then

$$\text{flux} = \int_C M dy - N dx = \int_{t=0}^{2\pi} (1.5 + \cos(t))(\cos(t)\, dt) - (1.5 + \sin(t))(-\sin(t)\, dt)$$

$$= \int_{t=0}^{2\pi} \left(1.5\cos(t) + \cos^2(t) + 1.5\sin(t) + \sin^2(t) \right) dt = 2\pi$$

The net outward flow across the circle is positive – more is leaving than is entering the circular region.

Practice 6: What flux do you expect for this curve C if the field is reversed to become $\mathbf{F}(x,y) = \langle -x,-y \rangle$? Calculate the flux over this C for the field.

If the simple closed curve C encloses a source of water, then the flux across C will be positive and equal to the rate of water input from the source. If C contains a sink (a drain), then the flux across C will be negative. If C contains both sources and sinks, then the flux across C will be the signed sum of the sources (counted as positive) and the sinks (counted as negative).

Wrap up

This section has examined line integrals and a variety of their applications, but there are really only two mathematical situations: when the function is scalar-valued and when the function is vector-valued.

If f is a **scalar-valued** function then:

$$\int_C \mathbf{f}\, ds = \int_{t=a}^{b} \mathbf{f}(\mathbf{r}(t)) \cdot \left| \frac{dr}{dt} \right| dt = \int_{t=a}^{b} \mathbf{f}(\mathbf{r}(t)) \cdot |\mathbf{r}'(t)|\ dt$$

$$\text{area} = \int_C f(\mathbf{r}(t)) ds = \int_{t=a}^{b} f(\mathbf{r}(t)) \cdot |\mathbf{r}'(t)|\ dt$$

$$\text{mass} = \int_C \delta(\mathbf{r}(t)) ds = \int_{t=a}^{b} \delta(\mathbf{r}(t)) \cdot |\mathbf{r}'(t)|\ dt$$

If $\mathbf{F}$ is a **vector-valued** function then:

$$\int_C \mathbf{F}\bullet\mathbf{T}\,ds = \int_C \mathbf{F}(\mathbf{r}(t))\bullet d\mathbf{r} = \int_{t=a}^{b} \mathbf{F}(\mathbf{r}(t))\bullet\mathbf{r}'(t)\,dt = \int_C (M dx + N dy + P dz)$$

$$\text{work} = \int_C \mathbf{F}\bullet\mathbf{T}\,ds$$

$$\text{flow} = \int_C \mathbf{F}\bullet\mathbf{T}\,ds$$

$$\text{flux} = \int_C \mathbf{F}\bullet\mathbf{n}\,ds = \int_C (M dy - N dx)$$ (C a simple, closed 2D curve oriented counterclockwise)

Later in this chapter we will consider 3D electric and magnetic vector fields and their flows and fluxes.

Problems

1. Determine the area between the curve $\mathbf{r}(t) = \langle 2t+1, 3+t^2\rangle$ for t from 0 to 3 in the xy-plane and $f(x,y) = x^2 - 4y + 11$.
2. Determine the area between the curve $\mathbf{r}(t) = \langle \sin(t), 1+\cos(t)\rangle$ for t from 0 to π in the xy-plane and $f(x,y) = 2 + xy$
3. Create an integral to calculate the area between $r(t) = \langle x(t), y(t), z(y)\rangle$ $(x,y,z \geq 0)$ and the xz-plane.
4. Create an integral to calculate the area between $r(t) = \langle x(t), y(t), z(y)\rangle$ $(x,y,z \geq 0)$ and the yz-plane.
5. A pipe is parameterized by $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t)\rangle$ for $0 \leq t \leq \pi/2$ and the density of the pipe is $\delta(x,y) = 1 + x + 2y$ at location (x,y). Find the total mass of the pipe.
6. Find the mass of the pipe in Problem 5 if $\delta(x,y) = 1 + 3x$.
7. A pipe is parameterized by $\mathbf{r}(t) = \langle 1+t, 3t, 2+t\rangle$ for $0 \leq t \leq 2$ and has density $\delta(x,y,z) = y + z$ at (x,y,z). Find the mass of the pipe.
8. Find the mass of the pipe in Problem 7 if $\delta(x,y,z) = x + 2y + 3z$.

In problems 9 to 14, evaluate $\int_C f\,ds$ for the given function f on the curve C.

9. $f(x,y) = 2x + y$ on C given by $\mathbf{r}(t) = \langle 3t+2, 5-4t\rangle$ for $0 \leq t \leq 2$.
10. $f(x,y) = x - 2y$ on C given by $\mathbf{r}(t) = \langle 12t+1, 5t-4\rangle$ for $1 \leq t \leq 4$.
11. $f(x,y) = x^2 y + y$ on C given by $\mathbf{r}(t) = \langle 3, t^2+1\rangle$ for $0 \leq t \leq 3$.
12. $f(x,y) = xy$ on C given by $\mathbf{r}(t) = \langle 4t^2, 3t^2+3\rangle$ for $0 \leq t \leq 1$.
13. $f(x,y) = x + y$ on C given by $\mathbf{r}(t) = \langle 2-\sin(t), 1+\cos(t)\rangle$ for $0 \leq t \leq \pi$.
14. $f(x,y) = x - 2y$ on C given by $\mathbf{r}(t) = \langle 2+5\cos(t), 1+2\cos(t)\rangle$ for $0 \leq t \leq 2\pi$.

15. In Fig. 13 is the work along each path A and B positive, negative or zero?
16. In Fig. 13 is the work along each path C and D positive, negative or zero?
17. In Fig. 14 is the flow along each path A and B positive, negative or zero?
18. In Fig. 14 is the flow along each path C and D positive, negative or zero?
19. Calculate the work to move an object along the path $\mathbf{r}(t) = \langle 2+3t,\ 4t\rangle$ for $0 \le t \le 3$ in the field $\mathbf{F} = \langle x, x+y\rangle$.
20. Calculate the work to move an object along the path $\mathbf{r}(t) = \left\langle t^2,\ t\right\rangle$ for $0 \le t \le 2$ in the field $\mathbf{F} = \langle -y, x\rangle$
21. Calculate the work to move an object along the path $\mathbf{r}(t) = \langle \cos(t),\ t,\ \sin(t)\rangle$ for $0 \le t \le \pi$ in the field $\mathbf{F} = \langle 1,2,3\rangle$.
22. Calculate the work to move an object along the path $\mathbf{r}(t) = \langle \cos(t),\ t,\ \sin(t)\rangle$ for $0 \le t \le \pi$ in the field $\mathbf{F} = \langle x,y,z\rangle$.

Fig. 13

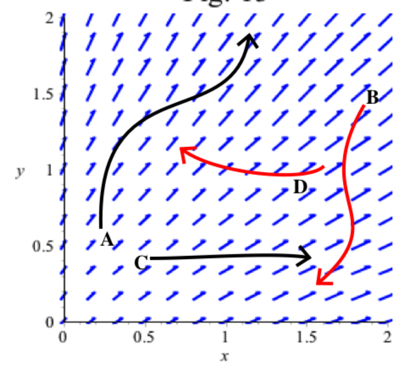
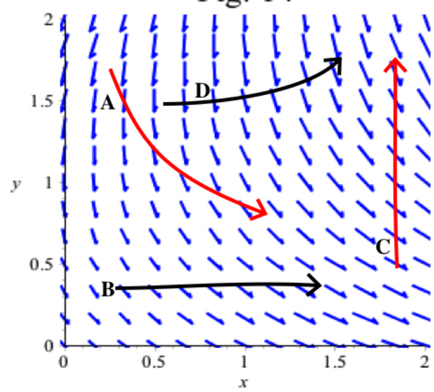


Fig. 14



23. Calculate the flow along the path $\mathbf{r}(t) = \left\langle t,\ 4t,\ t^2\right\rangle$ for $1 \le t \le 2$ in the field $\mathbf{F} = \langle z,\ 2y,\ x\rangle$.
24. Calculate the flow along the path $\mathbf{r}(t) = \left\langle t^2,\ 3+t\right\rangle$ for $0 \le t \le 5$ in the field $\mathbf{F} = \langle -y,\ 2x\rangle$.
25. Calculate the flux around the closed path $\mathbf{r}(t) = \langle \cos(t),\ 2\cdot\sin(t)\rangle$ for $0 \le t \le 2\pi$ in the field $\mathbf{F} = \langle 2x,\ 1+2y\rangle$.
26. Calculate the flux around the closed path $\mathbf{r}(t) = \langle \cos(t),\ 2\cdot\sin(t)\rangle$ for $0 \le t \le 2\pi$ in the field $\mathbf{F} = \langle 3x,\ 1+2y\rangle$.
27. If the circulation is 0 around a closed path in a vector field **F**, can the flux be positive, negative, zero? (Think about the unit circle path in a radial vector field.)
28. If the work along a path C in a vector field is positive, what can be said about the flow along that path?

In problems 29 to 34, assume that the orientation of the path is reversed. Is the original value changed or not?

29. What happens to the area? Why?
30. What happens to the mass? Why?
31. What happens to the work? Why?
32. What happens to the flow? Why?
33. What happens to the circulation? Why?
34. What happens to the flux? Why?

Practice Answers

Practice 1: **area** $= \int_{t=-\pi/2}^{\pi/2} (2+\cos(5t))\cdot\sqrt{(-2\sin(t))^2+(2\cos(t))^2}\cdot dt = \int_{t=-\pi/2}^{\pi/2} ((2+\cos(5t)))\cdot 2\cdot dt$

$$= 2\left(2t+\frac{1}{5}\sin(5t)\right)\Big|_{-\pi/2}^{\pi/2} = 4\pi+\frac{4}{5}\approx 13.37$$

Practice 2: C is parameterized by $\mathbf{r}(t) = \langle (1-t)\cdot 0 + t\cdot 4, (1-t)\cdot 1 + t\cdot 2\rangle = \langle 4t, 1+t\rangle$ for t=0 to t=1. Then

$|\mathbf{r}'(t)| = \sqrt{17}$ and $f(\mathbf{r}(t)) = 2(4t)+(1+t) = 9t+1$ so

$\int_C f\,ds = \int_{t=0}^{1} f(\mathbf{r}(t))\left|\frac{dr}{dt}\right|\,dt = \int_{t=0}^{1}(9t+1)(\sqrt{17})\,dt = \frac{11}{2}\sqrt{17}$. The units are $(\$)\left(\frac{\text{feet}}{\text{min}}\right)(\text{min}) = \$\cdot\text{feet}$

Practice 3: total mass $= \int_{t=1}^{4} 3t\cdot\sqrt{5+4t^2}\,dt = \frac{1}{4}\left(5+4t^2\right)^{3/2}\Big|_1^4 = \frac{69}{4}\sqrt{73}-\frac{27}{4}\approx 136.54$

Practice 4: $\mathbf{F}(\mathbf{r}(t))\bullet\mathbf{r}'(t) = \langle 3,2,1\rangle\bullet\left\langle 1, 2t, 3t^2\right\rangle = 3+4t+3t^2$ and work $= \int_{t=0}^{2}(3+4t+3t^2)\,dt = 22$

Practice 5: C and the vector fields are shown in Fig. P5. The line integral is negative for F1, positive for F2, and 0 for F3 (by symmetry) and F4.

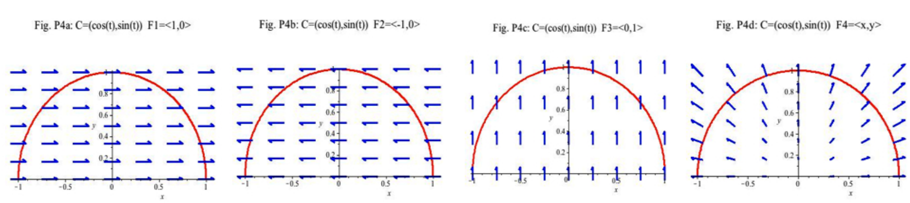


Fig. P5

Practice 6: M=-(1.5+cos(t)), N=-(1.5+sin(t)), dx=-sin(t) dt, dy=cos(t) dt, $0\le t\le 2\pi$.

$$\text{flux} = \int_C M\,dy - N\,dx = \int_{t=0}^{2\pi} -(1.5+\cos(t))(\cos(t)\,dt) + (1.5+\sin(t))(-\sin(t)\,dt)$$

$= -\int_{t=0}^{2\pi}\left(1.5\cos(t)+\cos^2(t)+1.5\sin(t)+\sin^2(t)\right)dt = -2\pi$, the opposite of the Example 4 value.