15.3 The Fundamental Theorem of Line Integrals and Potential Functions

Section 15.2 introduced the line integral along a curve in a vector field F, $\int_C \mathbf{F} \bullet \mathbf{T}\ ds$, discussed some applications of these integrals and showed how to calculate them. But something curious is going on.

Let $\mathbf{F}(x,y) = \langle x,\ y+x \rangle$. If we take three different paths from (0,0) to (2,4) in this vector field we get three different values. Along the curve C_1 given by $\mathbf{r}_1(t) = \langle 2t,\ 4t \rangle$

$$\int_{C_1} \mathbf{F} \bullet \mathbf{T}\ ds = \int_{C_1} \mathbf{F}(\mathbf{r}_1(t)) \bullet \mathbf{r}'_1(t)\ dt = \int_{t=0}^{1} \langle 2t,\ 4t+2t \rangle \bullet \langle 2,\ 4 \rangle\ dt = \int_{t=0}^{1} 28t\ dt = 14 .$$

But along the curve C_2 given by $\mathbf{r}_2(t) = \left\langle 2t,\ 4t^2 \right\rangle$, $\int_{C_2} \mathbf{F} \bullet \mathbf{T}\ ds = 46/3$.

And along the curve C_3 given by $\mathbf{r}_3(t) = \left\langle 2t,\ 4t^5 \right\rangle$, $\int_{C_3} \mathbf{F} \bullet \mathbf{T}\ ds = 50/3$.

In this vector field the work to move an object from (0,0) to (2,4) depends on the path of the object.

However, if we change the vector field slightly, to $\mathbf{F}(x,y) = \langle x,\ y \rangle$, and calculate the work along the same three paths from (0,0) to (2, 4), the results are always the same:

$$\int_{C_1} \mathbf{F} \bullet \mathbf{T}\ ds = \int_{t=0}^{1} \langle 2t,\ 4t \rangle \bullet \langle 2,\ 4 \rangle\ dt = \int_{t=0}^{1} 20t\ dt = 10 ,$$

$$\int_{C_2} \mathbf{F} \bullet \mathbf{T}\ ds = \int_{t=0}^{1} \left\langle 2t,\ 4t^2 \right\rangle \bullet \langle 2,\ 8t \rangle\ dt = \int_{t=0}^{1} 32t^3 + 4t\ dt = 10 ,$$

and $\int_{C_3} \mathbf{F} \bullet \mathbf{T}\ ds = \int_{t=0}^{1} \left\langle 2t,\ 4t^5 \right\rangle \bullet \left\langle 2,\ 20t^4 \right\rangle\ dt = \int_{t=0}^{1} 80t^9 + 4t\ dt = 10 .$

And the result will be 10 no matter what other smooth paths we take from (0, 0) to (2, 4) in this field.

This field $\mathbf{F}(x,y) = \langle x,\ y \rangle$ is a special type of vector field called a **gradient field** or a **potential field** or a **conservative field**, and it has the wonderful property that the value of the line integral does not depend on the path. This property is called **path independence**.

Definition: $\int_C \mathbf{F} \bullet \mathbf{T}\ ds$ is **independent of path** if $\int_{C_1} \mathbf{F} \bullet \mathbf{T}\ ds = \int_{C_2} \mathbf{F} \bullet \mathbf{T}\ ds$ for any two paths C_1 and C_2 with the same initial and ending points.

Conservative Fields

But what are the conservative vector fields and how can we determine if a given field is conservative?

Definitions: Conservative Field and Potential Function

A vector field **F** is called a **conservative** (or gradient, or potential) **field** on a region R in 2D or 3D if there is a scalar function f on R so that $\nabla \mathbf{f} = \mathbf{F}$.

This scalar function f is called the **potential function** for the conservative field **F**.

Example 1: Each of these scalar-valued functions generates a conservative vector field. Determine the vector field

$\mathbf{F}$ for $f_1(x,y) = xy$, $f_2(x,y) = \sqrt{x^2+y^2}$ and $f_3(x,y,z) = x^2y + 2yz$.

Solution: All that is needed is the gradient of each function: $\mathbf{F}_1(x,y) = \langle y, x \rangle$,

$$\mathbf{F}_2(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle \text{ and } \mathbf{F}_3(x,y,z) = \left\langle 2xy, x^2+2z, 2y \right\rangle.$$

Each of these is a conservative vector field since each $\mathbf{F} = \nabla \mathbf{f}$ for a differentiable function f.

Practice 1: Determine the conservative vector fields generated by $f_1(x,y) = 3x - 2y$, $f_2(x,y) = \sin(xy)$ and

$$f_3(x,y,z) = \frac{k}{x^2+y^2+z^2}.$$

Fundamental Theorem of Line Integrals

If there is a scalar function f on an open, connected region R in 2D or 3D so that $\nabla \mathbf{f} = \mathbf{F}$ (**F** is a conservative field), and C is any piecewise smooth curve in R from point A (when t=a) to point B (when t=b)

then
$$\int_C \mathbf{F} \bullet \mathbf{T}\; ds = \int_{t=a}^{b} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\; dt = \int_{t=a}^{b} \mathbf{F} \bullet d\mathbf{r} = f(B) - f(A)$$

and the line integral does not depend on the path C (path independence).

This is considered one of the four fundamental theorems of vector calculus (with Green's and Stokes's and the Divergence theorems). The meanings of "open," and "connected" are given in the chapter Appendix.

Proof: Suppose $\mathbf{F}(x,y,z) = \nabla \mathbf{f}(x,y,z) = \mathbf{f_x}(x,y,z)\mathbf{i} + \mathbf{f_y}(x,y,z)\mathbf{j} + \mathbf{f_z}(x,y,z)k$ for some smooth scalar-valued function f. Then

$$\mathbf{F} \bullet \mathbf{r}' = \nabla \mathbf{f} \bullet \mathbf{r}' = \left\langle \frac{\partial \mathbf{f}}{\partial \mathbf{x}}, \frac{\partial \mathbf{f}}{\partial \mathbf{y}}, \frac{\partial \mathbf{f}}{\partial \mathbf{z}} \right\rangle \bullet \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \cdot \frac{dx}{dt} + \frac{\partial \mathbf{f}}{\partial \mathbf{y}} \cdot \frac{dy}{dt} + \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \cdot \frac{dz}{dt} = \frac{df}{dt} \quad \text{(by the Chain Rule)}$$

so $\displaystyle\int_{t=a}^{b} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\ dt = \int_{t=a}^{b} \frac{df(\mathbf{r}(t))}{dt}\ dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(B) - f(A)$.

Example 2: $\mathbf{F}(x,y) = \langle y, x \rangle$. Evaluate $\displaystyle\int_C \mathbf{F} \bullet \mathbf{T}\ ds$ for a curve C that starts at A=(1,2) and ends at B=(4,3).

Solution: If we recognize (from Example 1) that $\mathbf{F} = \nabla \mathbf{f}$ for f(x,y)=xy then by the Fundamental Theorem of Line Integrals, $\displaystyle\int_C \mathbf{F} \bullet \mathbf{T}\ ds = f(B) - f(A) = f(4,3) - f(1,2) = 12 - 2 = 10$ along any smooth path C from A to B. You might want to check this value by using the path $\mathbf{r}(t) = \langle 1+3t,\ 2+t \rangle$ for t from 0 to 1 and explicitly calculating $\displaystyle\int_{t=0}^{1} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\ dt$.

If we know a scalar function f whose gradient is the vector function $\mathbf{F}$, then a difficult calculus problem becomes an easy arithmetic problem. You should recognize the similarity of this calculation with the way integrals were evaluated in beginning calculus. In beginning calculus if f was an antiderivative of F (Df=F), then $\displaystyle\int_a^b F(x)\,dx = f(b) - f(a)$. Here if f is a potential function of $\mathbf{F}$ ($\nabla \mathbf{f} = \mathbf{F}$), then $\displaystyle\int_{t=a}^{b} \mathbf{F} \bullet d\mathbf{r} = f(B) - f(A)$.

Practice 2: $\mathbf{F}(x,y,z) = \langle 2xy,\ x^2 + 2z,\ 2y \rangle$. Evaluate $\displaystyle\int_C \mathbf{F} \bullet \mathbf{T}\ ds$ for the curve C that starts at A=(1,0,4) and ends at B=(4,3,2). (Suggestion: Look at the answers in Example 1.)

Example 3: Fig. 1 shows the level curves for a smooth function z=f(x,y) and the vector field $\mathbf{F} = \nabla \mathbf{f}$. Calculate the amount of work done moving an object from point A to point B along the two given paths.

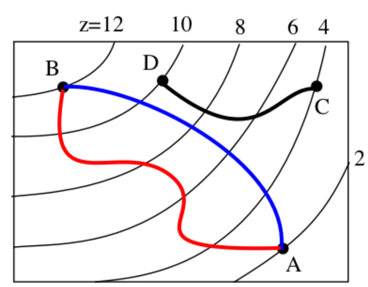


Fig. 1: Level curves of z=f(x,y)

Solution: work $= \displaystyle\int_C \mathbf{F} \bullet \mathbf{T}\ ds = f(B) - f(A) = 10$ along each path.

Practice 3: Using the same figure, calculate the amount of work done moving an object from point C to point D along the given curve. What is the total work to go from C to D and then back to C?

Theorem: $\displaystyle\int_C \mathbf{F} \bullet \mathbf{T}\ ds$ is **independent of path** on open, simply-connected region R if and only if $\displaystyle\int_C \mathbf{F} \bullet \mathbf{T}\ ds = 0$ for every closed path C in R.

Finding potential functions

Finding potential functions is a lot like finding antiderivatives but now we need to match two antiderivatives. If $\mathbf{F} = \langle M(x,y), N(x,y)\rangle$, then the potential function f must satisfy both $f_x = \frac{\partial f}{\partial x} = M$ and $f_y = \frac{\partial f}{\partial y} = N$. so typically two antiderivatives are needed, one with respect to x and one with respect to y.

Example 4: Find a potential function f for $\mathbf{F} = \langle 4y, 4x + 5\rangle$.

Solution: $\frac{\partial f}{\partial x} = 4y$ so (taking an antiderivative with respect to x) f(x,y)=4xy + g(y) (since $\frac{\partial g(y)}{\partial x} = 0$).

But $\frac{\partial}{\partial y}(4xy + g(y)) = 4x + g'(y)$ must equal N=4x+5, so $g'(y) = 5$ and $g(y) = 5y$ (typically the "+k" is omitted). Then $f(x,y) = 4xy + 5y$. A quick check shows that $f_x = 4y$ and $f_y = 4x + 5$ so $\nabla \mathbf{f} = \mathbf{F}$. .

Note: We could have first taken the antiderivative of N=4x+5 with respect to y.

Practice 4: Find a potential functions f for $\mathbf{F} = \left\langle y \cdot e^{xy} + 2, x \cdot e^{xy} + 3 \right\rangle$ and $\mathbf{F} = \left\langle 2xy + \cos(x), x^2 \right\rangle$.

But the vector field $\mathbf{F}(x,y) = \langle x, y + x\rangle$ at the beginning of this section was not path independent so it is not a conservative field and does not have a potential function. There is a very easy way to determine if a vector field has a potential function and is conservative.

Theorem: Test for a conservative field

If R is an open, connected, and simply-connected region then

2D: $\mathbf{F} = \langle M(x,y), N(x,y)\rangle$ is a conservative field on R if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

3D: $\mathbf{F} = \langle M(x,y), N(x,y), P(x,y)\rangle$ is a conservative field on R if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \quad \text{and} \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y} .$$

Proof: These follow from Clairaut's Theorem which says that mixed partial derivatives are equal: $f_{xy} = f_{yx}$.

If has a potential function f, then $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial M}{\partial y}$ and $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial N}{\partial x}$ so $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

The proof for the 3D case follows from Clairaut's Theorem that $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$ and $f_{yz} = f_{zy}$.

The proof in 2D that $M_y = N_x$ implies that $\mathbf{F} = \langle M, N\rangle$ is conservative requires Green's Theorem which appears later.

For the vector field $\mathbf{F}(x,y) = \langle x,\ y+x \rangle$, $\frac{\partial M}{\partial y} = 0$ and $\frac{\partial N}{\partial x} = 1$ so the field does not have a potential function.

Practice 5: Which of these fields are conservative: $\mathbf{F_1} = \langle 2xy^2+3,\ 2x^2y \rangle$, $\mathbf{F_2} = \langle -y,\ x \rangle$, $\mathbf{F_3} = \langle yz,\ xz,\ xy+2x \rangle$

and $\mathbf{F_4} = \langle yz,\ xz,\ xy+2 \rangle$?

Potential or Gradient fields are also called Conservative fields because the total energy, kinetic plus potential, is constant at each point in the field, energy is conserved. The derivation of this result is shown in an Appendix after the Practice Answers.

Summary of results

If R is an open, connected, simply-connected region, then the following are equivalent:

(1) $\mathbf{F}$ is a conservative field.

(2) There is a potential function f so that $\mathbf{F} = \nabla f$ . .

(3) $\int_C \mathbf{F} \bullet \mathbf{T}\ ds = \int_{t=a}^{b} \mathbf{F} \bullet \mathbf{dr} = f(B) - f(A)$ for all points A and B in R and for all smooth curves C.

(4) $\int_C \mathbf{F} \bullet \mathbf{T}\ ds = \int_{t=a}^{b} \mathbf{F} \bullet \mathbf{dr} = 0$ for all simple, closed, smooth curves in R.

Problems

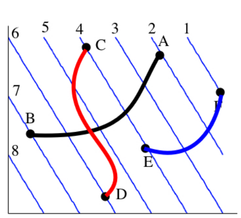
In problems 1 to 6, determine the conservative field generated by the given potential function.

1. $f(x,y) = x^2 + 3y^2$
2. $f(x,y) = 3x^2 - 4y^2$
3. $f(x,y) = \sin(3x+2y)$
4. $f(x,y) = x^2y^3 + xy^2$
5. $f(x,y) = \ln(2x+5y) + e^y$
6. $f(x,y) = \tan(x) - \sec(y)$

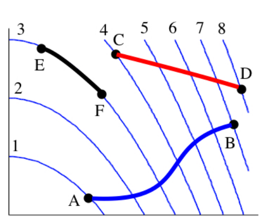
In problems 7 to 12, determine the work to move an object in the given field from point A to point B.

7. $\mathbf{F} = \langle 2x,\ 2y \rangle$, $A = (1,2)$, $B = (5,1)$
8. $\mathbf{F} = \langle y,\ x \rangle$, $A = (1,3)$, $B = (3,5)$
9. $\mathbf{F} = \langle x,\ x \rangle$, $A = (0,2)$, $B = (3,6)$
10. $\mathbf{F} = \langle 3x^2y,\ 3x^2 \rangle$, $A = (1,0)$, $B = (3,1)$
11. $\mathbf{F} = \langle yz,\ xz,\ xy \rangle$, $A = (1,0,0)$, $B = (4,2,1)$
12. $\mathbf{F} = \langle -x,-y,-z \rangle$, $A = (0,0,0)$, $B = (2,4,6)$

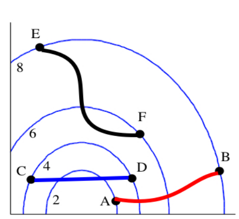
In problems 13 to 16, determine the work to move the object along the given paths from A to B and from C to D. Each figure shows the level curves of z=f(x,y) for some smooth function f.



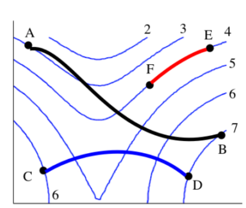
Problem 13



Problem 14



Problem 15



Problem 16

In problems 17 to 24, find a potential function for the given vector field F or show that F does not have a potential function.

17. $\left\langle 3x^2+4,\ 6\right\rangle$

18. $\left\langle \frac{2}{2x+3y},\ \frac{3}{2x+3y}+3y^2\right\rangle$

19. $\left\langle y\cdot\cos(xy)+3x^2y,\ x\cdot\cos(xy)+x^3\right\rangle$

20. $\left\langle xy^3+3x^2y,\ 3xy^2+x^2\right\rangle$

21. $\langle \sin(y),\ x\cdot\cos(y)\rangle$

22. $\langle 2y+3z,\ 2x,\ 3x\rangle$

23. $\langle yz\cdot\cos(xyz),\ xz\cdot\cos(xyz)+z,\ xy\cdot\cos(xyz)\rangle$

24. $\langle y\cdot\cos(xy),\ x\cdot\cos(xy)+z,\ y\rangle$

25. (a) For f(x,y)=arctan(y/x) show that $\nabla f=\left\langle \frac{-y}{x^2+y^2},\ \frac{x}{x^2+y^2}\right\rangle=\langle M, N\rangle$.

(b) Show that $M_y=N_x$.

(c) Show that for the closed circle $\mathbf{r}(t)=(\cos(t), \sin(t))$ for $0\le t\le 2\pi$, $\int_C \mathbf{F}\bullet\mathbf{T}\ ds=\int_{t=a}^{b}\mathbf{F}\bullet d\mathbf{r}=2\pi$.

(d) Why do these three results not violate the equivalent statements in the Summary?

Practice Answers

Practice 1: $\mathbf{F}_1(x,y)=\langle 3,-2\rangle$, $\mathbf{F}_2(x,y)=\langle y\cdot\cos(xy),\ x\cdot\cos(xy)\rangle$ and

$$\mathbf{F}_3(x,y,z)=\left\langle \frac{-2kx}{\left(x^2+y^2+z^2\right)^2},\frac{-2ky}{\left(x^2+y^2+z^2\right)^2},\frac{-2kz}{\left(x^2+y^2+z^2\right)^2}\right\rangle=\frac{2k}{\left(x^2+y^2+z^2\right)^2}\langle -x,\ -y,\ -z\rangle$$

Note: $\mathbf{F}_3$ is a radial field with $|\mathbf{F}_3|=|2k|$ and it always points toward the origin.

Practice 2: From Example 1, $\mathbf{F}=\nabla\mathbf{f}$ for $f(x,y)=x^2y+2yz$ so $\int_C \mathbf{F}\bullet\mathbf{T}\ ds=f(4,3,2)-f(1,0,4)=72$.

Practice 3: $\text{work}=\int_C \mathbf{F}\bullet\mathbf{T}\ ds=f(D)-f(C)=6$ along the given path, and, in fact, along every piecewise smooth path from C to D.

The work to go from C to D and then back to C is 0: the work from C to D is f(D)-f(C), and the work from D to C is f(C)-f(D) so the total work is {f(D)-f(C)}+{f(C)-f(D)}=0.

Practice 4: $\nabla(e^{xy}+22+3y)=\left\langle y\cdot e^{xy}+2,\ x\cdot e^{xy}+3\right\rangle$ and $\nabla(x^2y+\sin(x))=\left\langle 2xy+\cos(x),\ x^2\right\rangle$

Practice 5: $\mathbf{F_1}$ and $\mathbf{F_4}$ are conservative. $\mathbf{F_2}$ and $\mathbf{F_3}$ are not conservative.

Appendix: A bit of vocabulary about Curves and Regions

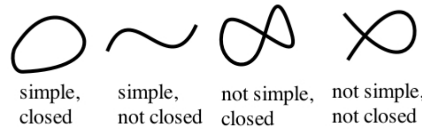


Fig. A1: Curves

A curve C parameterized by r(t) for $a \le t \le b$ is **simple** if

$\mathbf{r}(t_1) \ne \mathbf{r}(t_2)$ for $a < t_1 < t_2 < b$. (Fig. A1)

A curve C parameterized by r(t) for $a \le t \le b$ is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$.

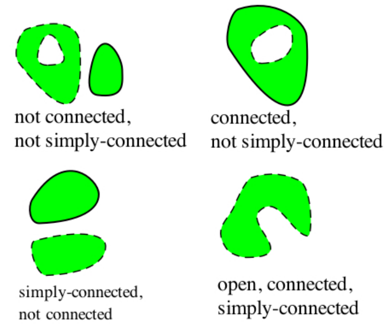


Fig. A3: Regions

A region R is **open** if every point in the R is inside a circle that lies completely in R.

R= $\{(x,y):\ x^2 + y^2 < 1\}$ is **open**.

R= $\{(x,y):\ x^2 + y^2 \le 1\}$ is **not open** since every circle around a boundary point contains points not in R.

A region R is **connected** if any two points in R can be joined by a continuous curve that lies in R.

A region R is **simply-connected** if every closed path in R can be contracted to a path without leaving R. A region with any holes (even just missing a single point) is not simply-connected.

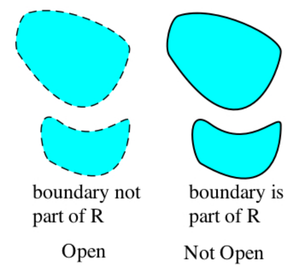


Fig. A2: Open and not open

Appendix Problems

For problems A1-A4, give examples of curves with the specified properties.

A1. closed, not simple

A2. not closed, not simple

A3. closed, simple

A4. not closed, simple

For problems A5-A10, give examples of regions with the specified properties. Give an example of a region that is

A5. open, connected and not simply-connected.

A6. open, simply-connected and not connected.

A7. connected, simply-connected, and not open.

A8. open, not connected and not simply-connected.

A9. connected, not open and not simply-connected.

A10. simply-connected, not open and not connected.

Appendix: Conservation of Energy

We need Newton's Second Law, $\mathbf{F} = m\mathbf{a}$ where $\mathbf{a}$ is the acceleration, so $\mathbf{F}(\mathbf{r}(t)) = m \cdot \mathbf{a}(t) = m \cdot \mathbf{r}''(t)$.

We also need that for any vector $\mathbf{w}$,

$$\frac{d}{dt}(\mathbf{w} \bullet \mathbf{w}) = w \bullet \frac{d\mathbf{w}}{dt} + \frac{d\mathbf{w}}{dt} \bullet \mathbf{w} = \mathbf{2}\frac{d\mathbf{w}}{dt} \bullet \mathbf{w} \quad \text{so} \quad \frac{d\mathbf{w}}{dt} \bullet \mathbf{w} = \frac{1}{2}\frac{d}{dt}(\mathbf{w} \bullet \mathbf{w}) \ ,$$

so if $\mathbf{w} = \mathbf{r'}$ then $\mathbf{r''} \bullet \mathbf{r'} = \frac{d\mathbf{r'}}{dt} \bullet \mathbf{r'} = \frac{1}{2}\frac{d}{dt}(\mathbf{r'} \bullet \mathbf{r'})$.

Kinetic energy: work to move from A to B

$$\text{work} = \int_C \mathbf{F} \bullet dr = \int_{t=a}^{b} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r'}(t)\, dt = \int_{t=a}^{b} m \cdot \mathbf{r''}(t) \bullet \mathbf{r'}(t)\, dt$$

$$= \int_{t=a}^{b} m \cdot \frac{1}{2}\frac{d}{dt}(\mathbf{r'} \bullet \mathbf{r'})\, dt = \frac{m}{2}\mathbf{r'} \bullet \mathbf{r'} \Big|_{\mathbf{t=a}}^{\mathbf{b}} = \frac{m}{2} |\mathbf{r'(t)}|^2 \Big|_{\mathbf{t=a}}^{\mathbf{b}} = \frac{m}{2} |\mathbf{v(t)}|^2 \Big|_{\mathbf{t=a}}^{\mathbf{b}} = \frac{m}{2} |\mathbf{v}(b)|^2 - \frac{m}{2} |\mathbf{v}(a)|^2$$

But kinetic energy $K = \frac{1}{2} m\mathbf{v}^2$ so work =K(B)–K(A).

Potential energy: work to move from A to B

If f is a conservative field so $\mathbf{F} = -\nabla f$ for a potential function f, then

$$\text{work} = \int_C \mathbf{F} \bullet dr = \int_C -\nabla f \bullet dr = (-f(\mathbf{r}(t))) \Big|_{\mathbf{t=a}}^{\mathbf{b}} = -f(\mathbf{r}(b)) + f(\mathbf{r}(a)) = f(A) - f(B)$$

Putting both type of energy together, K(B)–K(A) = work = $f(A) - f(B)$ so K(B)+f(B)=K)A)+f(A).

The total energy, kinetic plus potential, at A is the same as at B for any two points A and B in R. In a conservative field the total energy is conserved.

Gravitational, electrical and magnetic fields are conservative fields.

Problem Answers

1. $\mathbf{F} = \langle 2x, 2y \rangle$

2. $\mathbf{F} = \langle 6x, -8y \rangle$

3. $\mathbf{F} = \langle 3\cos(3x+2y), 2\cos(3x+2y) \rangle$

4. $\mathbf{F} = \langle 2xy^3 + y^2, 3x^2y^2 + 2xy \rangle$

5. $\mathbf{F} = \left\langle \frac{2}{2x+5y}, \frac{5}{2x+5y} + e^y \right\rangle$

6. $\mathbf{F} = \langle \sec^2(x), \sec(y)\tan(y) \rangle$

7. $f(x,y) = x^2 + y^2 \quad f(5,1) - f(1,2) = 21$

8. $f(x,y) = xy \quad f(3,5) - f(1,3) = 12$

9. $M_y = 0$, $N_x = 1$ so F is not conservative. $\mathbf{r}(t) = \langle 3t, 2+4t \rangle$ (t from 0 to 1).

$F(r(t)) \bullet r'(t) = \langle 3t, 3t \rangle \bullet \langle 3, 4 \rangle = 21t$ so $\text{work} = \int_{t=0}^{1} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt = \int_{t=0}^{1} 21t\, dt = \frac{21}{2}$

10. $M_y = 3x^2$, $N_x = 6x$ so F is not conservative. $\mathbf{r}(t) = \langle 1+2t, t \rangle$ (t from 0 to 1).

$F(r(t)) \bullet r'(t) = \left\langle 3(1+2t)^2(t), 3(1+2t)^2 \right\rangle \bullet \langle 2.1 \rangle = 3 + 15t + 24t^2 + 12t^3$

$\text{work} = \int_{t=0}^{1} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt = \int_{t=0}^{1} \{3 + 15t + 24t^2 + 12t^3\}\, dt = \frac{43}{2}$

11. $f(x,y,z) = xyz \quad f(4,2,1) - f(1,0,0) = 8$

12. $f(x,y,z) = -\frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} \quad f(2,4,6) - f(0,0,0) = -28$

13. A to B: 5 C to D: 2 E to F: 3

14. A to B: 6 C to D: 4 E to F: 0

15. A to B: 4 C to D: 0 E to F: -2

16. A to B: 4 C to D: 1 E to F: 0

17. $f(x,y) = x^3 + 4x + 6y$

18. $f(x,y) = \sin(xy) + x^3 y$

19. $f(x,y) = \sin(xy) + x^3 y$

20. $M_y = 3xy^2 + 3x^2 \neq N_x = 3y^2 + 3x^2$

21. $f(x,y) = x \cdot \sin(y)$

22. $f(x,y,z) = 2xy + 3xz$

23. $N_z \neq P_y$

24. $f(x,y,z) = \sin(xy) + yz$