15.5 Green's Theorem

Green's Theorem makes statements about the equivalence between what happens on the boundary of a 2D region with what is happening on the inside of the region. It says, in two ways, that a single integral around the boundary is equal to a double integral over the region enclosed by the boundary. In this way Green's Theorem is similar to the Fundamental Theorem of Calculus which relates the area above an interval to the values of the antiderivatives at the boundary (endpoints) of the interval.

As with some other theorems in mathematics. Green's Theorem allows us to trade one calculation for another one that may be easier. And it shows connections between ideas that do not seem to be related.

Green's Theorem:

If C is a simple, closed, and piecewise smooth curve (Fig. 1) and $\mathbf{F}(x,y) = \langle M(x,y), N(x,y) \rangle = M\mathbf{i} + N\mathbf{j}$ where M and N have continuous first partial derivatives in the region R enclosed by C,

then $$\oint_{\mathbf{C}} \mathbf{F} \bullet \mathbf{T}\, ds = \oint_{\mathbf{C}} M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$ Circulation-Curl Form

and $$\oint_{\mathbf{C}} \mathbf{F} \bullet \mathbf{n}\, ds = \oint_{\mathbf{C}} M\, dy - N\, dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA\ .$$ Flux-Divergence Form

Notation: The little circle on the integral sign indicates that C is a closed path parameterized in the counterclockwise direction..

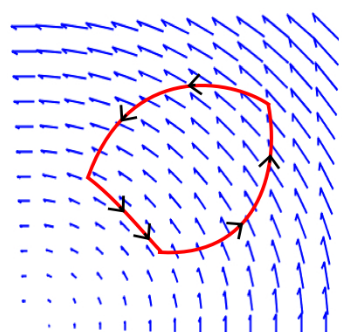

Fig. 1

The first conclusion says that the circulation is the accumulation of the interior curl. The second says that the flux is the accumulation of the interior divergence. It may help you make these connections by remembering that both circulation and curl deal with rotation while both flux and divergence deal with dissipation.

A proof for simple regions is given in the Appendix of this section.

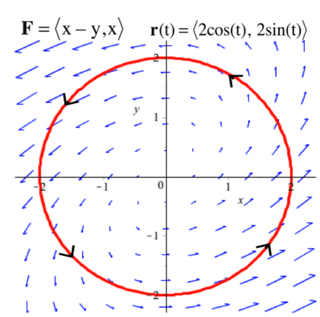

Fig. 2

Example 1: For $\mathbf{F} = \langle x - y,\ x \rangle$ and $\mathbf{r}(t) = \langle 2 \cdot \cos(t),\ 2 \cdot \sin(t) \rangle$ (Fig. 2),

evaluate $\int_C (Mdx + Ndy)$ and $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$.

Solution: First we need to write everything in terms of t:

$M = 2\cdot\cos(t) - 2\cdot\sin(t)$, $N = 2\cdot\cos(t)$, $dx = -2\sin(t)\,dt$, $dy = 2\cdot\cos(t)\,dt$. Then

$$\int_C (Mdx + Ndy) = \int_{t=0}^{2\pi} (2\cdot\cos(t) - 2\cdot\sin(t))(-2\sin(t)) + (2\cdot\cos(t))(2\cdot\cos(t))\ dt = \int_{t=0}^{2\pi} 4 - 4\cos(t)\cdot\sin(t)\,dt = 8\pi .$$

The circulation is 8π .

$\frac{\partial N}{\partial x} = 1$, $\frac{\partial M}{\partial y} = -1$ so $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2$ and $\iint_R (2)\ dA = 2(\text{area of circle of radius } 2) = 8\pi$.

As promised by the first conclusion of Green's Theorem, these two values are equal, and in this case the double integral was much easier to evaluate.

Practice 1: For $\mathbf{F} = \langle x - y,\ x\rangle$ and $\mathbf{r}(t) = \langle 2\cdot\cos(t),\ 2\cdot\sin(t)\rangle$, evaluate $\int_C (Mdy - Ndx)$ and $\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$.

Example 2: Evaluate $\int_C Mdy - Ndx$ and $\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$ for $\mathbf{F} = \langle -y, x\rangle$ and the triangular region R bounded by the x-axis, the line x=2 and the line y=x (Fig. 3).

$\mathbf{F} = \langle -y,x\rangle$

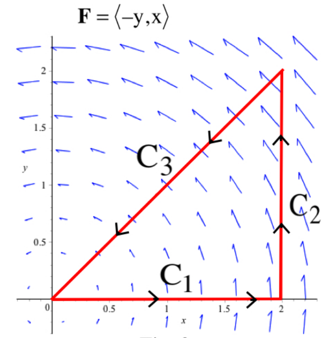

Fig. 3

Solution: $\frac{\partial M}{\partial x} = 0$ and $\frac{\partial N}{\partial y} = 0$ so we immediately have $\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = 0$.

The flux is 0.

The boundary of R consists of 3 line segments:

$C_1 = \langle 2t, 0\rangle$, $C_2 = \langle 2, 2t\rangle$, and $C_3 = \langle 2 - 2t, 2 - 2t\rangle$ with $0 \le t \le 1$. We need that choice for C_3 in order for the orientation to be counterclockwise.

On C_1, $\int_{t=0}^{1} Mdy - Ndx = \int_{t=0}^{1} (-0)(2) - (2t)(2)\,dt = \int_{t=0}^{1} -4t\,dt = -2$.

On C_2, $\int_{t=0}^{1} Mdy - Ndx = \int_{t=0}^{1} (-2t)(2) - (2)(0)\,dt = \int_{t=0}^{1} -4t\,dt = -2$.

On C_3, $\int_{t=0}^{1} Mdy - Ndx = \int_{t=0}^{1} (2t - 2)(-2) - (2 - 2t)(-2)\,dt = \int_{t=0}^{1} 8 - 8t\,dt = 4$.

So $\int_{C_1} + \int_{C_2} + \int_{C_3} = (-2) + (-2) + (4) = 0$. Certainly the double integral was easier.

Practice 2: Evaluate $\int_C (Mdx + Ndy)$ and $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$ for $\mathbf{F} = \langle -y, x\rangle$ and the triangular region R bounded by the x-axis, the line x=2 and the line y=x .

Green's Theorem can also be used to evaluate line integrals.

Example 3: Evaluate $\oint_C x^2y\,dy - y^2\,dx$ where C is the boundary of the rectangle

$R = \{(x,y):\ 0 \le x \le 2,\ 0 \le y \le 1\}$ oriented counterclockwise.

Solution: If we can match the form of the line integral with one of the forms of Green's Theorem then we can evaluate one double integral instead of the four line integrals around R.

If we use the Flux-Divergence form $\oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$ then we need

$M = x^2y$ and $N = y^2$ so $\oint_C x^2y\,dy - y^2\,dx = \iint_R (2xy + 2y)\ dA = \int_0^1 \int_0^2 (2xy + 2y)\ dx\,dy = \int_0^1 8y\,dy = 4$.

Practice 3: Use the Circulation-Curl form of Green's Theorem to evaluate the same line integral on the same region R.

In the previous examples we always traded a line integral for a double integral, but sometimes the opposite trade is useful.

Using Green's Theorem to Find Area

If the boundary of R is a simple closed curve C then area of $R = \iint_R 1\ dA$. If we can find M and N so that $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 1$ then we can use $\iint_R 1\ dA = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \oint_C M dy - N dx$. Putting $M = \frac{x}{2}$ and $N = \frac{y}{2}$ works so $\iint_R 1\ dA = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \oint_C M dy - N dx$.

> If the boundary of R is a simple closed curve C, then area of $R = \frac{1}{2}\oint_C x\,dy - y\,dx$.

Example 4: Use this result to determine the area of the elliptical region $\frac{x^2}{4} + \frac{y^2}{25} \le 1$.

Solution: The boundary of this region can be parameterized by $\mathbf{r}(t) = \langle 2\cos(t),\ 5\sin(t)\rangle$.

Then $\text{area} = \frac{1}{2}\oint_C (2\cos(t))(5\cos(t)) - (5\sin(t))(-2\sin(t))\ dt = \frac{1}{2}\int_0^{2\pi} 10\ dt = 10\pi$.

Practice 4: Use the same method to determine the area of the general elliptical region $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$.

Green's Theorem in More General Regions

The proof of Green's Theorem in the Appendix is valid for simple regions R (Fig. 4) in the plane for which any line parallel to an axis cuts the region R in at most 2 points or along an edge of R.

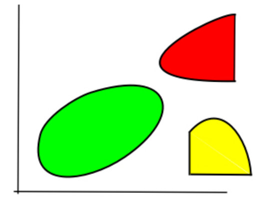

Fig. 4: Some simple regions

But Green's Theorem is true in much more complex regions if they can be decomposed into a union of simple regions.

If the region R is "bent," (Fig. 5) there are usually a finite number of cuts parallel to an axis so that R is the union of the some simple regions. Since the counterclockwise orientation on each simple region moves along each cut once in each direction (Fig. 6) the sum of those integral pieces is 0 and we are left with the integral around the original boundary of R.

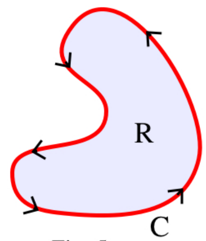

Fig. 5

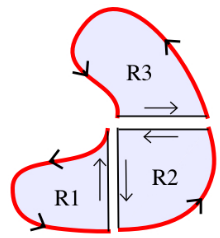

Fig. 6

Practice 5: Decompose the region in Fig. 7 into several simple regions. Indicate the direction(s) of the paths along each cut.

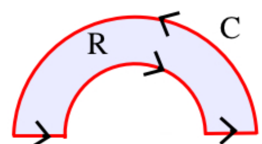

Fig. 7

Similarly, if R contains a finite number of holes (Fig. 8), then we can again create a single boundary for R by adding paths that connect to the holes. Then the integral along this new path will be the sum of the counterclockwise integrals around the outer boundary of R minus the sum of the counterclockwise integrals around the holes. The integrals along the added paths sum to 0 since they are traveled once in each direction. Remember, for a counterclockwise orientation of the curve C the region is always on our left hand side as we walk along C.

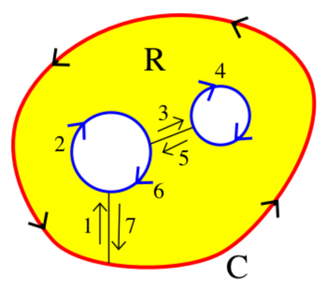

Fig. 8

Practice 6: Decompose the region in Fig. 9 into several simple regions. Indicate the directions and order of the paths along each cut.

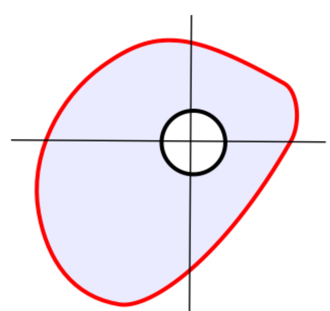

Fig. 9

Example 5: An interesting situation. $\mathbf{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$.

$\left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$

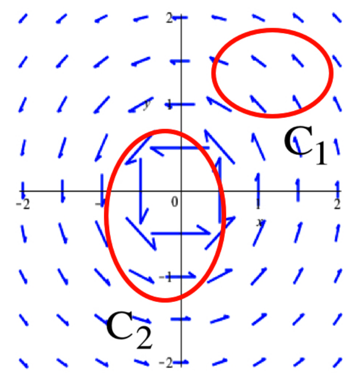

Fig. 10

Let C_1 be any simple closed curve that does not enclose the origin and let C_2 be a simple closed curve that does enclose the origin (Fig. 10). Use the Circulation-Curl form of Green's Theorem to calculate the circulations around C_1 and C_2.

Solution: On C_1 circulation

$$\oint_{\mathbf{C}} \mathbf{F} \bullet \mathbf{T}\, ds = \oint_{\mathbf{C}} M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA .$$

$$\frac{\partial M}{\partial y} = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} . \quad \frac{\partial N}{\partial x} = \frac{(x^2+y^2)(1) - (x)(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \text{so} \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

so $\text{circulation} = \oint_{\mathbf{C}} M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \iint_R 0\; dA = 0$ on C_1.

Lets begin the "encloses the origin" by looking at the particular circle C that encloses the origin: $\mathbf{r}(t) = \langle h \cdot \cos(t), h \cdot \sin(t) \rangle$ for a positive value of h. In this case we can work with the line integral for circulation directly by putting everything in terms of t:

$$\text{circulation} = \oint_{\mathbf{C}} M\,dx + N\,dy = \int_0^{2\pi} \left(\frac{-h \cdot \sin(t)}{h^2} \right)(-h \cdot \sin(t)) + \left(\frac{h \cdot \cos(t)}{h^2} \right)(h \cdot \cos(t))\; dt = \int_0^{2\pi} 1\; dt = 2\pi .$$

If C_2 is any simple closed curve that encloses the origin, we can take h small enough that the circle C is inside C_2. Then the region R bounded by D= the union of C_2 counterclockwise and C clockwise (Fig 10) does not contain the origin so $0 = \int_D = \int_{C_2} + \int_C = \int_{C_2} + (-2\pi)$

and $\{\text{circulation around } C_2\} = \oint_{C_2} M\,dx + N\,dy = 2\pi$.

For this vector field the circulation is 0 for any simple closed curve that does not surround the origin, and the circulation is always 2π for any simple. closed, positively-oriented curve that does surround the origin.

Problems

In problems 1 to 6 evaluate the line integral directly and by using Green's Theorem where C is positively oriented.

1. $\int_C x^2y\,dx + 3y\,dy$ when C is the rectangle $0 \le x \le 2$, $0 \le y \le 1$.
2. $\int_C xy^2\,dx + 5xy\,dy$ when C is the square $0 \le x \le 2$, $0 \le y \le 2$.
3. $\int_C 3xy\,dx + 2x^2\,dy$ when C is the triangle with vertices (0,0), (1,0) and (1,2).
4. $\int_C x^2\,dx + xy\,dy$ when C is the triangle with vertices (0,0), (0,2) and (2,2).
5. $\int_C ax\,dx + by\,dy$ when C is the circle $x^2 + y^2 = r^2$.
6. $\int_C ay\,dx + bx\,dy$ when C is the circle $x^2 + y^2 = r^2$.

In problems 7 to 10 use Green's Theorem to find the counterclockwise circulation and the outward flux for the field **F** and the curve C.

7. $\mathbf{F} = \langle x + 2y, y - x \rangle$, C is the square $0 \le x \le 2$, $0 \le y \le 2$.
8. $\mathbf{F} = \langle 3x + 2y, 4y - 5x \rangle$, C is the rectangle $0 \le x \le 3$, $0 \le y \le 1$.
9. $\mathbf{F} = \langle x^2 + y^2, x^2 - y^2 \rangle$, C is the triangle with vertices (0,0), (0,2) and (2,2).
10. $\mathbf{F} = \langle x^2y, 3x + y^2 \rangle$, C is the triangle bounded by the lines x=0, y=1 and x=2y.

In problems 11 and 12 use Green's Theorem to find the area enclosed by the curve C.

11. C is given by $\mathbf{r}(t) = \langle t, t^2 \rangle$ as t goes from –2 to 2 and by $\mathbf{r}(t) = \langle t, 8 - t^2 \rangle$ as t goes from 2 to –2.
12. C is the curve bounded by the x-axis and the cycloid $\mathbf{r}(t) = \langle A(t - \sin(t)), A(1 - \cos(t)) \rangle$.

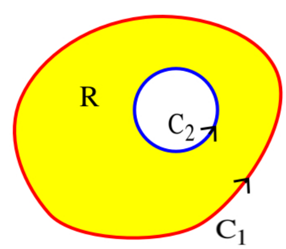

Fig. P13

13. $\mathbf{F} = \langle M, N \rangle$ and $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 5$ on region R in Fig. P13. The area of R is 100, and $\int_{C_2} \mathbf{F} \bullet d\mathbf{r} = 20$. Use Green's Theorem to determine $\int_{C_1} \mathbf{F} \bullet d\mathbf{r}$.

14. $\mathbf{F} = \langle M,N \rangle$ and $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 7$ on region R (inside C_1, outside C_2) in Fig. P14. If $\displaystyle\int_{C_2} \mathbf{F} \bullet d\mathbf{r} = 3\pi$, use Green's Theorem to determine $\displaystyle\int_{C_1} \mathbf{F} \bullet d\mathbf{r}$.

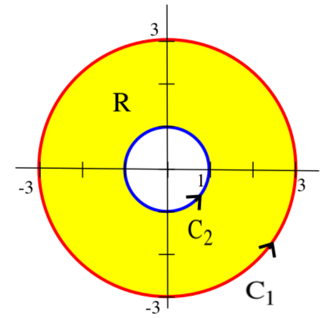

Fig. P14

15. $\mathbf{F} = \langle M,N \rangle$ and $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 9$ on region R in Fig. P15, $\displaystyle\int_{C_2} \mathbf{F} \bullet d\mathbf{r} = 3\pi$ and $\displaystyle\int_{C_3} \mathbf{F} \bullet d\mathbf{r} = 4\pi$. Use Green's Theorem to determine $\displaystyle\int_{C_1} \mathbf{F} \bullet d\mathbf{r}$.

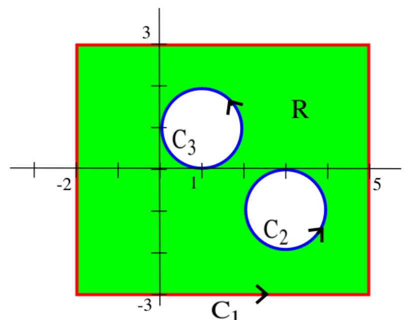

Fig. P15

16. $\mathbf{F} = \langle M,N \rangle$ and $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 5$ on region R in Fig. P16, and $\displaystyle\int_{C_2} \mathbf{F} \bullet d\mathbf{r} = 2\pi$. . Use Green's Theorem to determine $\displaystyle\int_{C_1} \mathbf{F} \bullet d\mathbf{r}$.

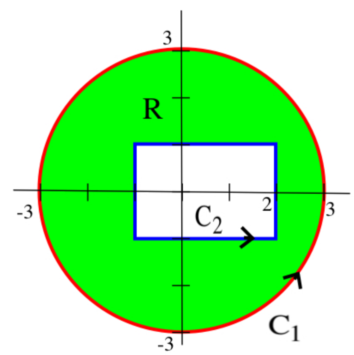

Fig. P16

17. Show that the circulation and flux of a constant field $\mathbf{F} = \langle a,b \rangle$ are 0 over every simply connected region R.

18. Show that the flux across any counterclockwise oriented simple closed curve C of a linear vector field $\mathbf{F} = \langle ax + by,\ cx + dy \rangle$ is always a constant multiple of the area of the region enclosed by C. Find the constant.

19. Show that the circulation around any counterclockwise oriented simple closed curve C of a linear vector field $\mathbf{F} = \langle ax + by,\ cx + dy \rangle$ is always a constant multiple of the area of the region enclosed by C. Find the constant.

Practice Answers

Practice 1: $M = 2\cdot\cos(t) - 2\cdot\sin(t)$, $N = 2\cdot\cos(t)$, $dx = -2\sin(t)\,dt$, $dy = 2\cdot\cos(t)\,dt$, $\frac{\partial M}{\partial x} = 1$, $\frac{\partial N}{\partial y} = 0$

$$\int_C (Mdy - Ndx) = \int_{t=0}^{2\pi} ((2\cdot\cos(t) - 2\cdot\sin(t))(2\cdot\cos(t)) - (2\cdot\cos(t)(-2\sin(t))\,dt = \int_{t=0}^{2\pi} 4\cdot\cos^2(t)\,dt = 4\pi .$$

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \iint_R (1+0)\ dA = \iint_R (1)\ dA = 1(\text{area of circle of radius 2}) = 4\pi.$$

As promised by the second conclusion of Green's Theorem, these two values are equal.

Practice 2: $\frac{\partial N}{\partial x} = 1$ and $\frac{\partial M}{\partial y} = -1$ so $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R (2)\ dA = 2(\text{triangle area}) = 2(2) = 4$.

The circulation is 4.

$\mathbf{F} = \langle -y, x\rangle$. $C_1 = \langle 2t, 0\rangle$, $C_2 = \langle 2, 2t\rangle$, and $C_3 = \langle 2-2t, 2-2t\rangle$ with $0 \le t \le 1$.

On C_1, $\int_{t=0}^{1} Mdx + Ndy = \int_{t=0}^{1} (-0)(2) + (2t)(0)\,dt = \int_{t=0}^{1} 0\,dt = 0$.

On C_2, $\int_{t=0}^{1} Mdx + Ndy = \int_{t=0}^{1} (-2t)(0) + (2)(2)\,dt = \int_{t=0}^{1} 4\,dt = 4$.

On C_3, $\int_{t=0}^{1} Mdx + Ndy = \int_{t=0}^{1} (2t-2)(-2) + (2-2t)(-2)\,dt = \int_{t=0}^{1} 0\,dt = 0$.

$\int_{C_1} + \int_{C_2} + \int_{C_3} = (0) + (4) + (0) = 4$. Again the double integral was easier.

The flows along C_1 and C_3 make sense in terms of Fig. 3. Along C_1 and C_3 the vector field is perpendicular to the boundary so there is no flow.

Practice 3: $\oint_C x^2y\,dy - y^2\,dx = \oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$ so $M = -y^2$ and $N = x^2y$.

$$\oint_C x^2y\,dy - y^2\,dx = \iint_R (2xy + 2y)\ dA = \int_0^1\int_0^2 (2xy + 2y)\ dx\,dy = \int_0^1 8y\,dy = 4 .$$

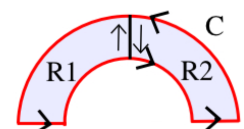

Fig. 7

Practice 4: Take $\mathbf{r}(t) = \langle a\cdot\cos(t),\ b\cdot\sin(t)\rangle$. Then

$$\text{area} = \frac{1}{2}\oint_C (a\cdot\cos(t))(b\cdot\cos(t)) - (b\cdot\sin(t))(-a\cdot\sin(t))\,dt = \frac{1}{2}\int_0^{2\pi} ab\,dt = ab\pi .$$

Practice 5: See Fig. 7

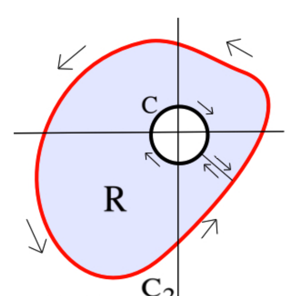

Fig. 9

Practice 6: Fig. 9 shows one solution.

Appendix A: Proof of Green's Theorem for Simple Regions

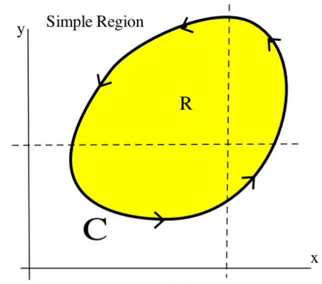

Fig. A1

A simple region R is one in which lines parallel to an axis intersect the boundary of R in at most two places (Fig. A1).

Label Fig. A2 so $f_1(x) \le y \le f_2(x)$ for $a \le x \le b$. C_1 is the curve $y = f_1(x)$ oriented counterclockwise as x goes from a to b , and C_2 is $y = f_2(x)$ which has a counterclockwise orientation as x goes from b to a. The closed curve C is the union of C_1 and C_2. With labeling we will compute $\oint_C M\,dx$ and $\iint_R \frac{\partial M}{\partial y}\,dA$.

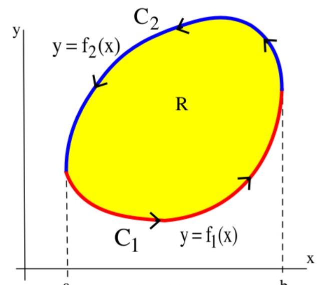

Fig. A2

$$\oint_C M\,dx = \oint_{C_1} M\,dx + \oint_{C_2} M\,dx = \int_a^b M(x,f_1)\,dx + \int_{\mathbf{b}}^{\mathbf{a}} M(x,f_2)\,dx$$

$$= \int_a^b M(x,f_1)\,dx - \int_{\mathbf{a}}^{\mathbf{b}} M(x,f_2)\,dx = \int_a^b M(x,f_1)\,dx - M(x,f_2)\,dx.$$

$$\iint_R \frac{\partial M}{\partial y}\,dA = \int_a^b \int_{f_1}^{f_2} \frac{\partial M}{\partial y}\,dy\,dx = \int_a^b M(x,f_2) - M(x,f_1)\,dx \quad \text{so} \quad \boxed{\oint_C M\,dx = -\iint_R \frac{\partial M}{\partial y}\,dA} \qquad (1)$$

To get the other part of the result we need, re-label the region R as in Fig. A3 so $x = g_1(y)$ for $c \le y \le d$. C_1 is the curve $x = g_1(x)$ oriented counterclockwise as y goes from c to d, and C_2 is $x = g_2(y)$ which has counterclockwise orientation as y goes from d to c. With this labeling we will compute $\oint_C N\,dy$ and $\iint_R \frac{\partial N}{\partial x}\,dA$.

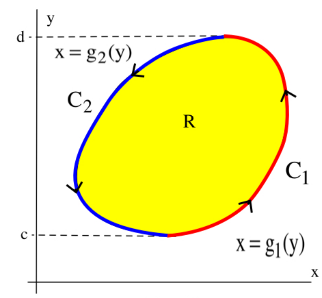

Fig. A3

$$\oint_C N\,dy = \oint_{C_1} N\,dy + \oint_{C_2} N\,dy = \int_c^d N(g_2,y)\,dy + \int_{\mathbf{d}}^{\mathbf{c}} N(g_1,y)\,dy$$

$$= \int_c^d N(g_2,y)\,dy - \int_{\mathbf{c}}^{\mathbf{d}} N(g_1,y)\,dy = \int_c^d N(g_2,y) - N(g_1,y)\,dy$$

$$\iint_R \frac{\partial N}{\partial x}\,dA = \int_c^d \int_{g_1}^{g_2} \frac{\partial N}{\partial x}\,dx\,dy = \int_c^d N(g_2,y) - N(g_1,y)\,dy \quad \text{so} \quad \boxed{\oint_C N\,dy = \iint_R \frac{\partial N}{\partial x}\,dA} \qquad (2)$$

Adding result (1) and result (2)), we have $\boxed{\oint_C M\,dx + N\,dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA}$, the Circulation-Curl form of Green's Theorem.

Using a similar approach, you can show that $-\oint_C N\,dx = \iint_R \frac{\partial N}{\partial y}\,dA$ and $\oint_C M\,dy = \iint_R \frac{\partial M}{\partial x}\,dA$ so

$$\oint_C M\,dy - N\,dx = \iint_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,dA,$$ the Flux-Divergence form of Green's Theorem.

On a Modified Simple Region

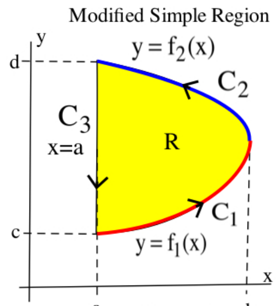

Fig. A4

Suppose the region R has one edge that if parallel to the y-axis (Fig. A4) labeled so $f_1(x) \le y \le f_2(x)$ for a≤x≤b. C_1 is the curve $y = f_1(x)$ oriented counterclockwise as x goes from a to b , and C_2 is $y = f_2(x)$ which has a counterclockwise orientation as x goes from b to a. C_3 is oriented counterclockwise with x=a as y goes from d to .The closed curve C is the union of C_1, C_2 and C_3.

$\oint_C M\,dx$ and $\iint_R \frac{\partial M}{\partial y}\,dA$.

$$\oint_C M\,dx = \oint_{C_1} M\,dx + \oint_{C_2} M\,dx + \oint_{C_3} M\,dx = \int_a^b M(x,f_1)\,dx + \int_{\mathbf{b}}^{\mathbf{a}} M(x,f_2)\,dx + \int_{\mathbf{a}}^{\mathbf{a}} M(x,f_2)\,dx$$

$$= \int_a^b M(x,f_1)\,dx + \int_{\mathbf{b}}^{\mathbf{a}} M(x,f_2)\,dx = \int_a^b M(x,f_1)\,dx - \int_{\mathbf{a}}^{\mathbf{b}} M(x,f_2)\,dx = \int_a^b M(x,f_1)\,dx - M(x,f_2)\,dx\,.$$

$$\iint_R \frac{\partial M}{\partial y}\,dA = \int_a^b \int_{f_1}^{f_2} \frac{\partial M}{\partial y}\,dy\,dx = \int_a^b M(x,f_2) - M(x,f_1)\,dx \quad \text{so} \quad \oint_C M\,dx = -\iint_R \frac{\partial M}{\partial y}\,dA\,.$$

To get the other part of the result we need, label the region R as in Fig. A5 so C_1 is the curve $x = g(y)$ oriented counterclockwise as y goes from c to d, and C_2 is $x = a$ which has counterclockwise orientation as y goes from d to c. With this labeling we will compute $\oint_C N\,dy$ and $\iint_R \frac{\partial N}{\partial x}\,dA$.

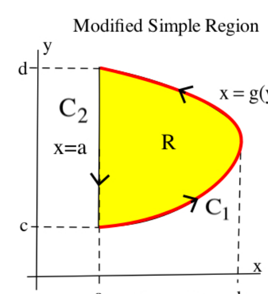

Fig. A5

$$\oint_C N\,dy = \oint_{C_1} N\,dy + \oint_{C_2} N\,dy = \int_c^d N(g,y)\,dy + \int_{\mathbf{d}}^{\mathbf{c}} N(a,y)\,dy$$

$$= \int_c^d N(g,y)\,dy - \int_{\mathbf{c}}^{\mathbf{d}} N(a,y)\,dy = \int_c^d N(g,y) - N(a,y)\,dy$$

$$\iint_R \frac{\partial N}{\partial x}\,dA = \int_c^d \int_a^g \frac{\partial N}{\partial x}\,dx\,dy = \int_c^d N(g,y) - N(a,y)\,dy \quad \text{so}$$

$$\oint_C N\,dy = \iint_R \frac{\partial N}{\partial x}\,dA \quad \text{and, adding these results,} \quad \oint_C M\,dx + N\,dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA$$

Other "simple" regions can be handled in similar ways.

The proof for general regions is difficult and is not included here.

If the region R lies on a plane in 3D (R is flat), then after a rotation of axes R can be made to lie in a new x'y'-plane and Green's Theorem applies.