15.6 Divergence and Curl in 3D

The divergence and curl were introduced in Section 15.2 for a 2D vector field F since it is easier in 2D to visualize what they measure and because we only needed the 2D versions for Green's Theorem in Section 15.5. Here we extend those definitions to a 3D vector field.

Divergence: div **F** in 3D

Assume that the vector field F describes the flow of a liquid in 3D. The **divergence** is the rate per unit volume that the water dissipates (**departs**, leaves) at the point P. A positive value for the divergence means that more water is leaving at P than is entering at P. But we can't really see a point so imagine a small sphere C centered at P, and consider whether more water s leaving or entering this sphere. Fig. 1(a) shows more water leaving than entering the sphere around P so div **F**(P) > 0, Fig. 1(b) has more entering than leaving so div **F**(P) < 0, and Fig, 1(c) shows the same amount leaving as entering so div **F** (P) = 0. This is much more difficult to visualize in 3D than in 2D, but the calculations are not any harder.

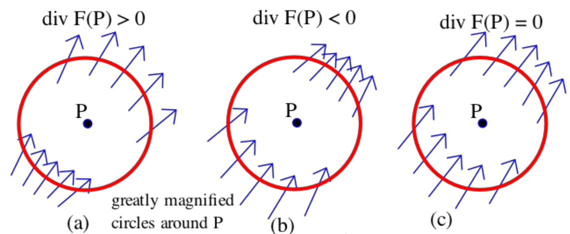
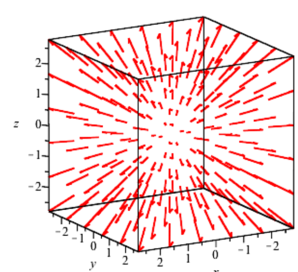


Fig. 1

Fig. 2: < x, y, z >



Example 1: Fig. 2 shows the radial field $\mathbf{F}(x,y,z) = \langle x,y,z \rangle$. Based on this figure, is div **F** positive, negative or zero at P=(1,2,0), Q=(1,0,2) and R=(0,0,0).

Solution: The vectors are increasing in magnitude as we move away from the origin so more are leaving little spheres at each of these points (in fact, at every point in 3D) so div **F** > 0 at every point in 3D .

> **Definition: Divergence F in 3D** div **F**
>
> For a vector field $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ with continuous partial derivatives,
>
> the divergence of F at Point P is $\text{div } \mathbf{F}(P) = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y} + \dfrac{\partial P}{\partial z}$. div $\mathbf{F} = \nabla \bullet \mathbf{F}$

Example 2: Calculate div **F** for $\mathbf{F}(x,y,z) = \langle x,y,z \rangle$ at the points in Example 1.

Solution: $\text{div } \mathbf{F} = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y} + \dfrac{\partial P}{\partial z} = 1 + 1 + 1 = 3$ at every point (x,y,z) so more water is leaving the is entering every tiny sphere in 3D.

Practice 1: Calculate div **F** for $\mathbf{F}(x,y,z) = \langle x^2, y^2, z \rangle$ at the points at (1,1,1), (2, –3,4) and (–0.5, 0,3).

Divergence of the inverse-square radial field $\mathbf{F} = \langle x, y, z\rangle / \sqrt{x^2+y^2+z^2}$

Radial inverse-square vector fields are very common in applications such as electricity and magnetism so it is worth calculating the divergence of such a field even if it is a bit tedious.

If $\mathbf{r} = \langle x, y, z\rangle$ then $|\mathbf{r}| = \sqrt{x^2+y^2+z^2}$ and the direction of r is $\dfrac{\mathbf{r}}{|\mathbf{r}|}$. If a field **F** has the same direction a **r** (a radial field) and follows an inverse-square law with magnitude $\dfrac{1}{|\mathbf{r}|^2}$ then $\mathbf{F} = \left(\dfrac{1}{|\mathbf{r}|^2}\right)\left(\dfrac{\mathbf{r}}{|\mathbf{r}|}\right) = \dfrac{\mathbf{r}}{|\mathbf{r}|^3}$.

$\dfrac{\partial |\mathbf{r}|^3}{\partial x} = \dfrac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{3/2} = 3x\cdot\left(x^2+y^2+z^2\right)^{1/2} = 3x\,|\mathbf{r}|$ and similarly for y and z, $\dfrac{\partial |\mathbf{r}|^3}{\partial y} = 3y\,|\mathbf{r}|$ and $\dfrac{\partial |\mathbf{r}|^3}{\partial z} = 3z\,|\mathbf{r}|$. But to calculate the divergence of $\dfrac{\mathbf{r}}{|\mathbf{r}|^3}$ we need the quotient rule for each partial derivative.

$$\frac{\partial}{\partial x}\frac{x}{|\mathbf{r}|^3} = \frac{|\mathbf{r}|^3\left(\dfrac{\partial x}{\partial x}\right) - x\cdot\left(\dfrac{\partial |\mathbf{r}|^3}{\partial x}\right)}{\left(|\mathbf{r}|^3\right)^2} = \frac{|\mathbf{r}|^3\left(\dfrac{\partial x}{\partial x}\right) - x\cdot 3x\,|\mathbf{r}|}{\left(|\mathbf{r}|^3\right)^2} = \frac{1}{|\mathbf{r}|^3} - \frac{3x^2}{|\mathbf{r}|^5}.$$

Similarly

$\dfrac{\partial}{\partial x}\dfrac{y}{|\mathbf{r}|^3} = \dfrac{1}{|\mathbf{r}|^3} - \dfrac{3y^2}{|\mathbf{r}|^5}$ and $\dfrac{\partial}{\partial x}\dfrac{z}{|\mathbf{r}|^3} = \dfrac{1}{|\mathbf{r}|^3} - \dfrac{3z^2}{|\mathbf{r}|^5}$ so $\operatorname{div}\mathbf{F} = \operatorname{div}\dfrac{\mathbf{r}}{|\mathbf{r}|^3} = \dfrac{3}{|\mathbf{r}|^3} - \dfrac{3(x^2+y^2+z^2)}{|\mathbf{r}|^5}$

$= \dfrac{3}{|\mathbf{r}|^3} - \dfrac{3|\mathbf{r}|^2}{|\mathbf{r}|^5} = 0$. The divergence of the inverse-square vector field **F** is 0 everywhere except at the origin where the field is not defined.

Curl: curl **F** in 3D

In section 15.2 the curl of a vector field F at a point P was introduced and described as a measure of the counterclockwise rotation of a small paddle wheel or circle at P caused by the vector field. (Fig. 3) We could also view this as vectors in the xy-plane causing rotation about an axis in the z direction. In 3D, instead of a small circle, imagine a small sphere that is fixed at its center and rotates about that center point (Fig. 4). The 3D curl is a vector with two important properties (these will proved in section 15.9 using Stoke's Theorem):

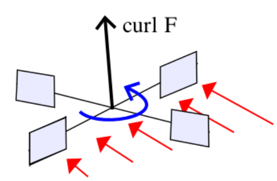


Fig. 3

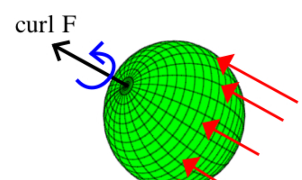


Fig. 4

* the magnitude of the curl gives the rate of the fluid's rotation, and
* the direction of the curl is normal to the plane of greatest circulation and points in the direction so that the circulation at the point has a right hand orientation (Fig. 5).

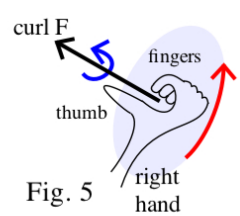


Fig. 5

If we have a small paddle wheel at point P and tilt it in different directions, then the wheel will spin fastest with a right-hand orientation when the axis points in the direction of the curl vector.

Definition: Curl F in 3D

For a vector field $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ with continuous partial derivatives, then

the curl of F at Point P is $\text{curl F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$. curl $\mathbf{F} = \nabla \mathbf{x} \mathbf{F}$

Note: The k component of curl, $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$, is just the 2D curl **F** from section 15.2.

Example 3: Calculate curl **F** for $\mathbf{F}(x,y,z) = \langle y - z,\ z - 2x,\ x + 3z \rangle$.

Solution:

$$\text{curl } \mathbf{F} = \nabla \mathbf{x} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z & z - 2x & x + 3z \end{vmatrix} = \langle 0 - 1,\ -(1 + 1),\ -2 - 1 \rangle = \langle -1,\ -2,\ -3 \rangle \ .$$

The curl of this field is the same at every point in 3D, and the little paddle wheel will spin fastest if the axis of the wheel is oriented in the direction of $\langle -1, -2, -3 \rangle$.

Practice 2: Calculate curl **F** for $\mathbf{F}(x,y,z) = \left\langle 2y + z,\ x^2,\ 3z \right\rangle$.

Example 4: (a) Calculate the gradient vector field **F** of $f(x,y,z) = x^3z + 3xy^2 + 4z$.

(b) Calculate curl **F** .

Solution: (a) $\mathbf{F} = \nabla f = \left\langle 3x^2z + 3y^2,\ 6xy.\ x^3 + 4 \right\rangle$.

(b)

$$\text{curl } \mathbf{F} = \nabla \mathbf{x} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2z + 3y^2 & 6xy & x^3 + 4 \end{vmatrix} = \left\langle 0 - 0,\ -(3x^2 - 3x^2),\ 6y - 6y \right\rangle = \langle 0,\ 0,\ 0 \rangle \ .$$

The result in the previous example was not a lucky accident – the curl of every gradient field is 0 everywhere.

Theorem: If **F** is a conservative field ($\mathbf{F} = \nabla f$),

then curl $\mathbf{F} = \langle 0,\ 0,\ 0 \rangle$ at every point in 3D. curl $(\nabla f) = \mathbf{0}$

If curl $\mathbf{F} \neq \mathbf{0}$ at any point, then **F** is a not a conservative field.

Proof: If **F** is a gradient field then there is a potential function f so that $\mathbf{F} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$.

Then $\text{curl } \mathbf{F} = \nabla \mathbf{x} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$.

The $\mathbf{i}$ component is $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = 0$ by Clairaut's Theorem of Mixed Partial Derivatives.

You can easily verify that the $\mathbf{j}$ and $\mathbf{k}$ components of this curl are also 0.

Practice 3: Show that $\mathbf{F} = \langle -y, x, z \rangle$ is not a conservative vector field.

Unfortunately, knowing that curl F = 0 is not sufficient to guarantee that F is a conservative field. However there is a partial converse to the previous theorem.

> Theorem: If **F** is defined and has continuous partial derivatives at every point in 3D
> and curl **F** = 0
> then **F** is a conservative field.

The proof requires Stoke's Theorem and is not given here.

Curls of the radial and inverse-square radial fields

We could use the definition of curl to calculate the curls for $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{F} = \left(\frac{1}{|\mathbf{r}|^2}\right)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\right) = \frac{\mathbf{r}}{|\mathbf{r}|^3}$ but it is much easier to recognize that both **r** and **F** are gradient fields and then invoke the theorem. It is easy to check that $\mathbf{r} = \nabla\left[\frac{1}{2}(x^2 + y^2 + z^2)\right]$ and $\mathbf{F} = \nabla\sqrt{x^2 + y^2 + z^2}$ so curl **r** = 0 and curl **F** = 0 everywhere where each of them is defined. **r** is defined everywhere and is a conservative field. **F** is not defined at the origin and is not a conservative field in a domain that includes the origin.

> Theorem: If **F** has continuous partial second derivatives, then div (curl **F**) = 0.

Proof: The proof is straightforward and the last step depends on Clairaut's Theorem.

$$\text{div (curl } \mathbf{F}) = \text{div}\left\{\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}\right\}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} + \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 0$$

Problems

In problems 1 to 6, calculate the divergence and curl of each vector field.

1. $\mathbf{F} = \langle x^2 y,\ xyz,\ xz^3 \rangle$
2. $\mathbf{F} = \langle yz,\ xz,\ xy + 2 \rangle$
3. $\mathbf{F} = \langle x \cdot e^z,\ z \cdot e^y,\ y \cdot e^x \rangle$
4. $\mathbf{F} = \langle y,\ z,\ x^3 \rangle$
5. $\mathbf{F} = \langle x + y,\ y + z,\ z + x \rangle$
6. $\mathbf{F} = \langle x^2 + y^2,\ y^2 + z^2,\ z^2 + x^2 \rangle$

In problems 7 to 10, F is a vector field and f is a scalar function in 3D. Determine whether the given calculation is meaningful. If it is not, explain why. If it is, determine whether the result is a scalar or a vector.

7. div f , div F , div (curl F) , gradient (curl F), curl F
8. curl f , curl (div F) , gradient (div F) , div (curl (gradient f))
9. gradient f , curl (curl F) , curl (div (gradient f))
10. gradient F , curl (gradient F) , div (div F)

In problems 11 to 16, determine if the vector field F is conservative. If F is conservative, find a function f so the F=gradient f.

11. $\mathbf{F} = \langle x^2 y,\ xyz,\ xz^3 \rangle$
12. $\mathbf{F} = \langle yz,\ xz,\ xy + 2 \rangle$
13. $\mathbf{F} = \langle y,\ z,\ x^3 \rangle$
14. $\mathbf{F} = \langle y^2,\ 2xy,\ 3z^2 \rangle$
15. $\mathbf{F} = \langle \sin(y \cdot z),\ x \cdot z \cdot \cos(y \cdot z),\ x \cdot y \cdot \cos(y \cdot z) + 2 \rangle$
16. $\mathbf{F} = \langle y^3,\ 3xy^2 + 5z,\ 5y \rangle$
17. Suppose $\text{curl}\ \mathbf{F}(1,2,3) = \langle 2,\ 4,\ 1 \rangle$. If you are at the location (5, 10, 5) and look towards (1,2,3) will you see the rotation at (1,2,3) to be clockwise or counterclockwise?
18. Suppose $\text{curl}\ \mathbf{F}(5,1,4) = \langle -1,\ 3,\ 1 \rangle$. If you are at the location (2, 10, 7) and look towards (5,1,4) will you see the rotation at (5,1,4) to be clockwise or counterclockwise?
19. Suppose $\text{curl}\ \mathbf{F}(6,3,2) = \langle 2,\ -1,\ 1 \rangle$. If you are at the location (4, 4, 3) and look towards (6,3,2) will you see the rotation at (6,3,2) to be clockwise or counterclockwise?
20. Suppose $\text{curl}\ \mathbf{F}(7,-4,5) = \langle -1,\ -2,\ 3 \rangle$. If you are at the location (9, 0, –1) and look towards (7,–4,5) will you see the rotation at (7,–4,5) to be clockwise or counterclockwise?

Practice Answers

Practice 1: div $\mathbf{F} = 2x + 2y + 1$ so div $\mathbf{F}(1,1,1) = 5$, div $\mathbf{F}(2,-3,4) = -1$ and div $\mathbf{F}(-0.5,0,3) = 0$.

Practice 2:

$$\text{curl } \mathbf{F} = \nabla \mathbf{x} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y+z & x^2 & 3z \end{vmatrix} = \langle 0-0,\ -(0-1),\ 2x-2 \rangle = \langle 0,\ 1,\ 2x-2 \rangle\ .$$

Practice 3:

$$\mathbf{F} = \langle -y,x,z \rangle.\ \text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} = (0-0)\mathbf{i} + (0-0)\mathbf{j} + (1-(-1))\mathbf{k} = \langle 0,0,2 \rangle \neq \mathbf{0}$$

so $\mathbf{F}$ is not a conservative field.