15.7 Parametric Surfaces

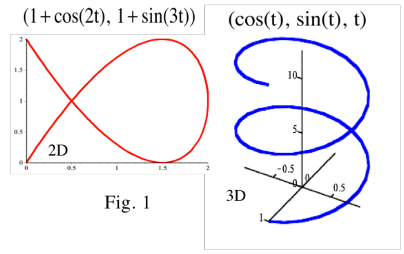

Fig. 1

In earlier work we extended the basic idea of a function of one variable, y = f(x), to parametric equations in which both x and y were functions of a single parameter t: x=x(t) and y=y(t) (in 3D, z=(t), also). Basically this mapped a 1-dimensional object (think of a piece of wire) into 2 or 3 dimensions (Fig. 1). And by treating this mapping as a vector-valued function, $\mathbf{r}(t) = \langle x(t),\, y(t),\, z(t) \rangle$, we could use the ideas and tools of vectors. In this section we will do something similar, except now we will map a 2-dimension object (think of a sheet of paper) into 3 dimensions by treating x, y, and z as functions of two parameters, u and v: x=x(u,v), y=y(u,v) and z=z(u,v). These graphs will be surfaces in 3D. By using parametric surfaces we can work with more complicated and more general shapes (Fig. 2). And by treating these surfaces as vector-valued functions, $\mathbf{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v) \rangle$, we can again use the ideas and tools of vectors.

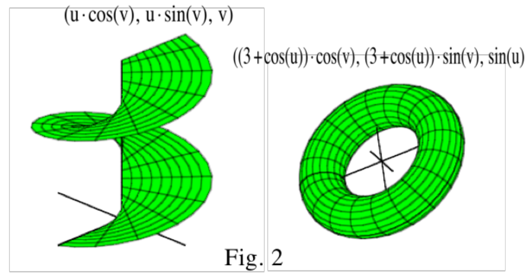

Fig. 2

Definitions: Parametric Function and Parametric Surface

Let x, y and z be functions of the parameters u and v for all (u, v) in a region D.
The vector-valued function $\mathbf{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v) \rangle$ is called a **parametric function** with domain D.
The set of points $S = (\, x(u,v),\, y(u,v),\, z(u,v)\,)$ is called the **parametric surface** of the function **r** on domain D.

Note: As with the parametric functions of 1 variable, $\mathbf{r}(t) = \langle x(t),\, y(t),\, z(t) \rangle$, we will work with **r** as a vector $\langle x(u,v),\, y(u,v),\, z(u,v) \rangle$ but only plot the points $S = (\, x(u,v),\, y(u,v),\, z(u,v)\,)$.

All of the rectangular, cylindrical and spherical coordinate functions we have used so far can be easily converted into parametric functions, and parametric functions give us even more freedom.

Example 1: (a) Convert $f(x,y) = x^2 + y^2$ with $-2 \le x \le 2$ and $-2 \le y \le 2$ into parametric form with u and v.

(b) Convert the spherical coordinate function $(\, 3,\, \theta,\, \varphi)$, $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi/2$ (the top half of a sphere) into parametric form with u and v.

Solution: (a) Simply replace x with u and y with v and rewrite the z coordinate in terms of u and v: $(u, v, u^2 + v^2)$ with $-2 \le u \le 2$ and $-2 \le v \le 2$ (Fig. 3).

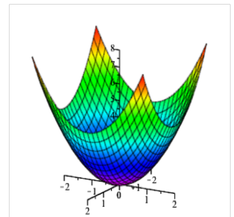

Fig. 3

(b) $x = \rho \cdot \sin(\varphi) \cdot \cos(\theta)$, $y = \rho \cdot \sin(\varphi) \cdot \sin(\theta)$, $z = \rho \cdot \cos(\varphi)$ so we can replace θ and φ with u and v and rewrite x, y and z as

$x = 3 \cdot \sin(v) \cdot \cos(u)$, $y = 3 \cdot \sin(v) \cdot \sin(u)$, $z = 3 \cdot \cos(v)$

with $0 \le u \le 2\pi$, $0 \le v \le \pi/2$. (Fig. 4)

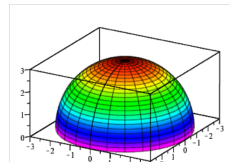

Fig. 4

Practice 1: Convert the cylindrical coordinate function $(r, \theta, 1 + \sin(3\theta))$, $1 \le r \le 2$, $0 \le \theta \le 2\pi$ (Fig. 5) into parametric form with u and v.

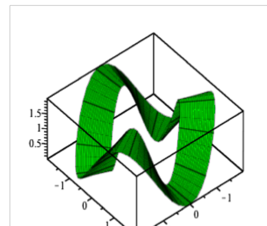

Fig. 5

One common parametric surface is the torus which can be thought of as a small circular tube around a larger circle (Fig. 6). The parametric equation for a torus with large radius R and small radius r is

$$((R - r \cdot \cos(u)) \cdot \cos(v),\ (R - r \cdot \cos(u)) \cdot \sin(v),\ r \cdot \sin(u))$$

The Appendix discusses the derivation of this surface as well as how to create a parametric representations of a small tubes around other curves in space.

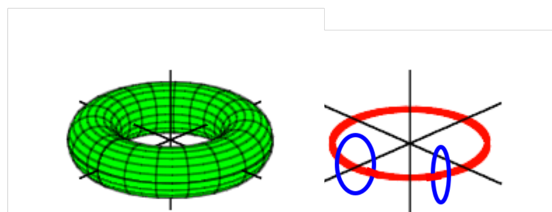

Fig. 6

Example 2: Write parametric equations for the surface generated by rotating the curve $z = \sqrt{y}$ around the y-axis for $0 \le y \le 4$. (Fig. 7)

Solution: Put y=u for $0 \le u \le 4$. Then the radius of the circle of revolution is $\sqrt{u}$ so $x = \sqrt{u} \cdot \cos(v)$ and $z = \sqrt{u} \cdot \sin(v)$ works.

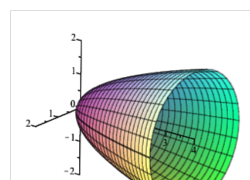

Fig. 7

Practice 2: Write parametric equations for the surface generated by rotating the curve $x=2+\sin(z)$ for $0 \le z \le 5$ around the z-axis.

Surface Area of a Parametric Surface

The derivation and result for the surface area of a parametric surface **r**(u,v) are similar to the method for a surface defined by z = f(x,y) in Section 14.5. First partition the uv-domain D into small Δu by Δv rectangles (Fig. 8) with (u,v) at the lower left point of the rectangle. Call this rectangle R. Then **r**(u,v) maps this rectangle R onto a patch S on the parametric surface (Fig. 9). The bottom

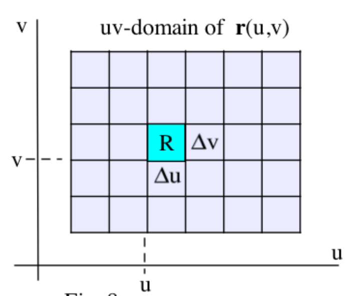

Fig. 8

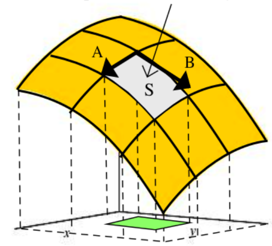

Fig. 9

corners of the R rectangle, (u,v) and (u+ Δu ,v), are mapped to **r**(u,v) and **r**(u+ Δu,v). The left edge corners of R, (u,v) and (u,v+ Δv), are mapped to **r**(u,v) and **r**(u, v+ Δv). The area of the patch S on the parametric surface is approximated by the area of the rectangle whose corners are the images under r of the corners of the R rectangle.

Let A be the vector from **r**(u,v) and **r**(u+ Δu,v), and B be the vector from **r** (u,v) and **r**(u, v+ Δv). Then (Fig. 10)

A= **r**(u+ Δu,v)– **r**(u,v) and **B**= **r**(u, v+ Δv)– **r** (u,v) so $\mathbf{A} = \frac{\mathbf{r}(u+\Delta u,v)-\mathbf{r}(u,v)}{\Delta u}\cdot \Delta u$ and

$\mathbf{B} = \frac{\mathbf{r}(u,v+\Delta v)-\mathbf{r}(u,v)}{\Delta v}\cdot \Delta v$. If Δu and Δv are small, then $\frac{\mathbf{r}(u+\Delta u,v)-\mathbf{r}(u,v)}{\Delta u} \approx \frac{\partial \mathbf{r}(u,v)}{\partial u} = \mathbf{r}_u(u,v)$

and $\frac{\mathbf{r}(u,v+\Delta v)-\mathbf{r}(u,v)}{\Delta v} \approx \frac{\partial \mathbf{r}(u,v)}{\partial v} = \mathbf{r}_v(u,v)$ so $\mathbf{A} \approx \mathbf{r}_u(u,v)\cdot \Delta u$ and $\mathbf{B} \approx \mathbf{r}_v(u,v)\cdot \Delta v$.

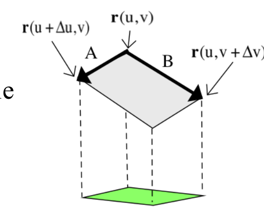

Fig. 10

Finally, the area of the patch S is approximately $|\mathbf{r}_u \mathrm{x} \mathbf{r}_v|\cdot \Delta u\cdot \Delta v$. Summing over all of the rectangles R in the uv-domain and taking limits as Δu and Δv both approach 0,

$$\sum_v \sum_u |\mathbf{r}_u \mathrm{x} \mathbf{r}_v|\cdot \Delta u\cdot \Delta v \to \iint_D |\mathbf{r}_u \mathrm{x} \mathbf{r}_v|\cdot dA .$$

Surface Area of a Parametric Surface

If S is a smooth surface given by **r** (u,v) $= \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ with uv-domain D,

then {surface area of S} = $\iint_D |\mathbf{r}_u \mathrm{x} \mathbf{r}_v|\cdot dA$.

(Note: The surface S may fold over on itself, but that will not happen if the R rectangle is very small.)

If z=f(x,y), then we could parameterize the surface by x=u, y=v and z=f(u,v), and it is straightforward to derive that {Surface Area} = $\iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\cdot dA$ as we did in Section 14.5.

Proof: Parameterize this surface by x=u, y=v and z=f(u,v). Then $\mathbf{r}(u,v) = \langle u,v,f\rangle$ so $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \langle 1,0,f_u\rangle$ and

$\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \langle 0,1,f_v\rangle$. $\mathbf{r}_u \mathrm{x} \mathbf{r}_v = \langle -f_u,-f_v,1\rangle$ so $|\mathbf{r}_u \mathrm{x} \mathbf{r}_v| = \sqrt{(f_u)^2+(f_v)^2+1}$ and the surface area

is $\iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\cdot dA$.

Example 3: Use the parametric surface form to find the surface area of the curve $z=\sqrt{y}$ around the y-axis for $0 \le y \le 4$.

Solution: This surface was parameterized in Example 2 by y=u for $0 \le u \le 4$, $x=\sqrt{u}\cdot\cos(v)$ and $z=\sqrt{u}\cdot\sin(v)$ for $0\le v\le 2\pi$. $\mathbf{r}_u=\frac{\partial \mathbf{r}}{\partial u}=\left\langle \frac{1}{2\sqrt{u}}\cdot\cos(v),\ 1,\ \frac{1}{2\sqrt{u}}\cdot\sin(v)\right\rangle$.

$\mathbf{r}_v=\frac{\partial \mathbf{r}}{\partial v}=\left\langle -\sqrt{u}\cdot\sin(v),\ 0,\ \sqrt{u}\cdot\cos(v)\right\rangle$, $\mathbf{r}_u \mathbf{x} \mathbf{r}_v=\left\langle \sqrt{u}\cdot\cos(v),\ \frac{1}{2},\ \sqrt{u}\cdot\sin(v)\right\rangle$ so

$|\mathbf{r}_u \mathbf{x} \mathbf{r}_v|=\sqrt{u\cdot\sin^2(v)+\frac{1}{4}+u\cdot\cos^2(v)}=\sqrt{\frac{1}{4}+u}$. Finally,

$$\text{Surface area}=\iint_D |\mathbf{r}_u \mathbf{x} \mathbf{r}_v|\cdot dA=\int_{v=0}^{2\pi}\int_{u=0}^{4}\sqrt{\frac{1}{4}+u}\ du\,dv$$

$$=\int_{v=0}^{2\pi}\frac{2}{3}\left(\frac{1}{4}+u\right)^{3/2}\Big|_{u=0}^{4} dv=\left(\frac{2}{3}\left(\frac{17}{4}\right)^{3/2}-\frac{2}{3}\left(\frac{1}{4}\right)^{3/2}\right)\cdot 2\pi=\left(\frac{1}{12}(17)^{3/2}-\frac{1}{12}\right)\cdot 2\pi$$

the same result as Practice 3 in Section 14.5.

And now we can extend surface area calculations to other systems such as cylindrical coordinates (r,θ,z).

Surface Area in Cylindrical Coordinates

If S is a smooth surface in cylindrical coordinates $(r,\theta,f(r,\theta))$ on domain D,

then $$\{\text{surface area of S}\}=\iint_D \sqrt{r^2(f_r)^2+(f_\theta)^2+r^2}\cdot dr\cdot d\theta .$$

Proof: Parameterize this surface by $x=u\cdot\cos(v)$, $y=u\cdot\sin(v)$ (so u=r and v= θ) and $z=f(u\cdot\cos(v),\ u\cdot\sin(v))$. Then

$\mathbf{r}_u=\frac{\partial \mathbf{r}}{\partial u}=\langle \cos(v),\ \sin(v),\ f_u\rangle$ and $\mathbf{r}_v=\frac{\partial \mathbf{r}}{\partial v}=\langle -u\cdot\sin(v),\ u\cdot\cos(v),\ f_v\rangle$ so

$$\mathbf{r}_u \mathbf{x} \mathbf{r}_v=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(v) & \sin(v) & f_u \\ -u\cdot\sin(v) & u\cdot\cos(v) & f_v \end{vmatrix}=\left\langle f_v\cdot\sin(v)-f_u\cdot u\cdot\cos(v), -(f_v\cdot\cos(v)+f_u\cdot u\cdot\sin(v)),\ u\cdot\cos^2(v)+u\cdot\sin^2(v)\right\rangle.$$

Finally, after some simplifying, surface area $=\iint_D |\mathbf{r}_u \mathbf{x} \mathbf{r}_v|=\iint_D \sqrt{u^2(f_u)^2+(f_v)^2+u^2}\cdot du\cdot dv$.

Example 4: Use the cylindrical coordinate integral form to calculate the surface area of the hemisphere $f(r,\theta)=\sqrt{R^2-r^2}$ for $0\le r\le R$ and $0\le\theta\le 2\pi$.

Solution: $f_r = \dfrac{-r}{\sqrt{R^2 - r^2}}$ and $f_\theta = 0$ so

$$\iint_D \sqrt{r^2(f_r)^2 + (f_\theta)^2 + r^2} \cdot dr \cdot d\theta = \int_0^{2\pi}\int_0^R \frac{Rr}{\sqrt{R^2 - r^2}}\, dr\, d\theta = \int_0^{2\pi} -R \cdot \sqrt{R^2 - r^2}\ \Big|_0^R\ d\theta = \int_0^{2\pi} R^2\, d\theta = 2\pi R^2 \ .$$

The surface area of the entire sphere is $4\pi R^2$.

Practice 3: Use the parametric surface form to find the surface area of $z = x^2 - y^2$ for $0 \le x^2 + y^2 \le 4$.

Problems

For Problems 1 to 10, sketch the parametric surface.

1. $x = u,\ y = v,\ z = \sqrt{v},\ 0 \le u \le 2,\ 0 \le v \le 4$.
2. $x = u,\ y = v,\ z = u^2,\ 0 \le u \le 2,\ 0 \le v \le 1$.
3. $x = 2 \cdot \cos(u),\ y = 2 \cdot \sin(u),\ z = v,\ 0 \le u \le 2\pi,\ 0 \le v \le 3$.
4. $x = \cos(u),\ y = v,\ z = \sin(u),\ 0 \le u \le 2\pi,\ 0 \le v \le 4$.
5. $x = u \cdot \cos(v),\ y = u \cdot \sin(v),\ z = u,\ 0 \le u \le 2,\ 0 \le v \le 2\pi$.
6. $x = u \cdot \cos(v),\ y = u^2,\ z = u \cdot \sin(v),\ 0 \le u \le 2,\ 0 \le v \le 2\pi$.
7. $x = u,\ y = \sqrt{u} \cdot \cos(v),\ z = \sqrt{u} \cdot \sin(v),\ 0 \le u \le 4,\ 0 \le v \le 2\pi$.
8. $x = \sqrt{u} \cdot \cos(v),\ y = u,\ z = 2 + \sqrt{u} \cdot \sin(v),\ 0 \le u \le 4,\ 0 \le v \le 2\pi$.
9. $x = u,\ y = v,\ z = \sin(v),\ 0 \le u \le 2,\ 0 \le v \le 2\pi$.
10. $x = \sin(u),\ y = v,\ z = u,\ 0 \le u \le \pi,\ 0 \le v \le 4$.

For problems 11 to 20, write the integral that represents the surface area of each surface in problems 1 to 10.

11. Write the surface area integral for the surface in problem 1.

12. Write the surface area integral for the surface in problem 2.

and so on for problems 13 to 20.

Practice Answers

Practice 1: $x = r \cdot \cos(\theta)$, $y = r \cdot \sin(\theta)$, $z = z$ Replace r and θ with u and v so we have $(\ u \cdot \cos(v),\ u \cdot \sin(v),\ 1 + \sin(3v)\ )$ with $1 \le u \le 2,\ 0 \le v \le 2\pi$.

Practice 2: $x = (2 + \sin(u)) \cdot \cos(v)$, $y = (2 + \sin(u)) \cdot \sin(v)$, $z = u$ works.

Practice 3: Parameterize this surface by $x = r \cdot \cos(\theta)$, $y = r \cdot \sin(\theta)$ and

$f = r^2 \cdot \cos^2(\theta) - r^2 \cdot \sin^2(\theta) = r^2 \cdot \cos(2\theta)$ for $0 \le r \le 2,\ 0 \le \theta \le 2\pi$. Then

$f_r = 2r \cdot \cos(2\theta)$, and $f_\theta = -2r^2 \cdot \sin(2\theta)$ so $r^2(2r \cdot \cos(2\theta))^2 + \left(-4r^2 \cdot \sin(2\theta)\right)^2 + r^2 = 4r^4 + r^2$

Surface area $= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} r \cdot \sqrt{4r^2 + 1}\, dr\, d\theta = \left[\frac{1}{12}(17)^{3/2} - \frac{1}{12}\right] \cdot 2\pi$.

Strangely, the paraboloid of Example 3 and the hyperboloid of Practice 3 have the same surface area.

Appendix: Building a Torus and Other Tubes

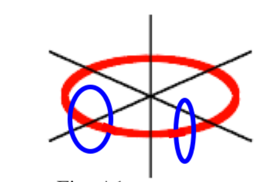
Fig. A1

A torus is the collection of little circles centered on a large circle where the plane of each small circle is perpendicular to the large circle (Fig. A1). We can extend this idea to the collection of small circles centered on any curve in 2D or 3D so that the plane of each small circle is perpendicular to the curve (Fig. A2).

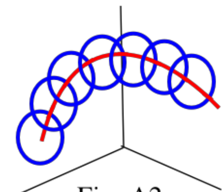
Fig. A2

In order to build these tubular surfaces (collections of small circles), we first need to know how to describe a circle in 3D at a given point, with a given radius and whose plane has a given normal vector (Fig. A3).

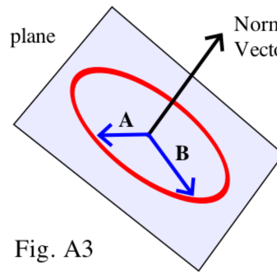

Fig. A3

Circle Algorithm: Center point at = (cx,cy,cz), radius R, and

normal vector V= <vx, vy, vz>.

Then the plane of this circle is vx(x–cz)+vy(y–cy)+vz(z–cz)=0.

Pick two non-colinear unit vectors A and B perpendicular to V.

Then the equation of the circle we want is

$$\mathbf{C}(t) = \langle \text{center point} \rangle + \mathbf{A} \cdot R \cdot \cos(t) + \mathbf{B} \cdot R \cdot \sin(t).$$

Sometimes it is convenient for A and B to be perpendicular, but the only firm requirement on these unit vectors is that they are not co-linear.

Example A1: Find an equation for a circle with radius 2, center at P=(3,1,2) and normal vector $\mathbf{N} = \langle 4,2,1 \rangle$.

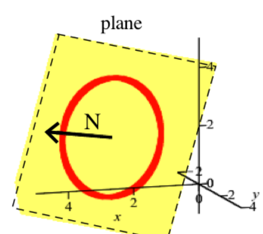

Fig. A4

Solution: First we can find the equation of the plane that contains the point C and has normal vector N:

4(x–3)+2(y–1)+1(z–2)=0 so z= 2–4(x–3)–2(y-1). (Fig A4)

Next we need two unit vectors perpendicular to N:

$\mathbf{A} = \langle 1,-1,-2 \rangle / \sqrt{6}$ works as a first one.

For the second vector calculate $\mathbf{AxN} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \\ 4 & 2 & 1 \end{vmatrix} = \frac{3}{\sqrt{6}} \langle 1,-3,2 \rangle$

and divide by its magnitude to create the second unit vector $\mathbf{B} = \langle 1,-3,2 \rangle / \sqrt{14}$ that is perpendicular to both N and A. Then the parametric equation of the circle:

$$\mathbf{C}(u) = P + \mathbf{A} \cdot 2 \cdot \cos(u) + \mathbf{B} \cdot 2 \cdot \sin(u) = \langle 3,1,2 \rangle + \langle 1,-1,-2 \rangle \cdot \frac{2}{\sqrt{6}} \cdot \cos(u) + \langle 1,-3,2 \rangle \cdot \frac{2}{\sqrt{14}} \cdot \sin(u)$$

with $0 \le u \le 2\pi$. The component functions are: $x(t) = 3 + \frac{2}{\sqrt{6}} \cdot \cos(u) + \frac{2}{\sqrt{14}} \cdot \sin(u)$,

$$y(t) = 1 - \frac{2}{\sqrt{6}} \cdot \cos(u) - \frac{6}{\sqrt{14}} \cdot \sin(u) \text{ , and } z(t) = 2 - \frac{4}{\sqrt{6}} \cdot \cos(u) + \frac{4}{\sqrt{14}} \cdot \sin(u) \text{ .}$$

Fig. A4 shows the vector **N**, the plane normal to **N** at P and the parametric circle.

Practice A1: Find an equation for a circle with radius 2, center at P=(2,4,3) and normal vector $\mathbf{N} = \langle -1,3,-2 \rangle$.

Once we can create a circle at a point P with those properties, then we can create a tube around a curve simply by moving the point P along the curve using the tangent vector to the curve as the vector **N**.

Tube Algorithm: To create a tube of radius r along the curve $\mathbf{F}(v) = \langle x(v), y(v), z(v) \rangle$ for a≤v≤b, apply the Circle Algorithm at each point $\mathbf{F}(v)$ using $\mathbf{N} = \mathbf{F}\,'(v)$.

Example A2: Create parametric equations for a torus centered at the origin with large radius R and tube radius r . (Fig. A5)

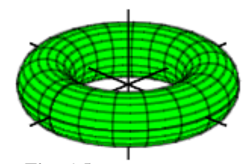
Fig. A5

Solution: For the large circle in the xy-plane, take $\mathbf{F}(v) = \langle R \cdot \cos(v), R \cdot \sin(v), 0 \rangle$ with $0 \le v \le 2\pi$. Then the unit tangent vector is $\mathbf{N} = \mathbf{T} = \langle -\sin(v), \cos(v), 0 \rangle$ and the unit vectors $\mathbf{A} = \mathbf{T}\,' = \langle -\cos(v), -\sin(v), 0 \rangle$ and $\mathbf{B} = \mathbf{N}\mathbf{x}\mathbf{A} = \langle 0,0,1 \rangle$ are each perpendicular to **N** and are not co-linear. Putting this together, the parametric equation of the torus is

$$C(u,v) = \mathbf{F}(v) + \mathbf{A} \cdot r \cdot \cos(u) + \mathbf{B} \cdot r \cdot \sin(u)$$

$$= \langle R \cdot \cos(v), R \cdot \sin(v), 0 \rangle + \langle -\cos(v), -\sin(v), 0 \rangle \cdot r \cdot \cos(u) + \langle 0,0,1 \rangle \cdot r \cdot \sin(u) \quad \text{for } 0 \le u,v \le 2\pi.$$

$$x(u,v) = R \cdot \cos(v) - \cos(v) \cdot r \cdot \cos(u) + 0 = (R - r \cdot \cos(u)) \cdot \cos(v)$$

$$y(u,v) = R \cdot \sin(v) - \sin(v) \cdot r \cdot \cos(u) + 0 = (R - r \cdot \cos(u)) \cdot \sin(v)$$

$$z(u,v) = 0 + 0 + r \cdot \sin(u) = r \cdot \sin(u)$$

Practice A2: Create parametric equations for a tube of radius r=1/2 centered on the curve $\mathbf{F}(v) = \langle 0, v, v^2 \rangle$ for $-1 \le v \le 2$.

Appendix Practice Answers

Solution A1: Thee are many correct answers depending on the unit vectors **A** and **B**, both perpendicular to **N** and not co-linear, that are chosen: $\mathbf{A} = \langle 1,1,1 \rangle / \sqrt{3}$ and $\mathbf{B} = \langle 5,-1,-4 \rangle / \sqrt{35}$ work. Then the parametric equation of the circle is

$$C(u) = \mathbf{P} + \mathbf{A} \cdot 2 \cdot \cos(u) + \mathbf{B} \cdot 2 \cdot \sin(u)$$

$$= \langle 2,4,3 \rangle + \langle 1,1,3 \rangle \cdot \frac{2}{\sqrt{3}} \cdot \cos(u) + \langle 5,-1,-3 \rangle \cdot \frac{2}{\sqrt{35}} \cdot \sin(u) \text{ with } 0 \le u \le 2\pi.$$

Solution A2: For this curve $\mathbf{N} = \mathbf{T}(v)/\,|\,\mathbf{T}(v)\,| = \langle 0,1,2v\rangle/\sqrt{1+4v^2}$. Then $\mathbf{A} = \langle 1,0,0\rangle$ and $\mathbf{B} = \langle 0,2v,-1\rangle$ are both perpendicular to $\mathbf{N}$ and to each other. Putting all of this together (Fig. A6), $x(u,v) = \frac{1}{2}\cdot\cos(u)$, $y(u,v) = v + v\cdot\sin(u)$, and $z(u,v) = v^2 - \frac{1}{2}\cdot\sin(u)$ for $-1 \le v \le 2$, $0 \le u \le 2\pi$.

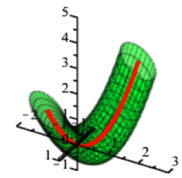

Fig. A6