15.8 Surface Integrals

Chapter 4 introduced integrals on intervals, Section 15.3 extended these ideas to integrals on paths in 2D and 3D, and Chapter 14 extended the ideas to integrals on 2D regions in the plane. This section goes one step further and considers integrals whose domains are parametric surfaces in 3D. In each previous situation the development was similar: partition, approximate on small pieces, sum, and take limits to achieve an integral. The approach here is the same. Surface integrals will be important in the coming sections on Stoke's Theorem and the Divergence Theorem and their applications.

Surface Integral for a Scalar Function

If S is a smooth surface in xyz-space parameterized by $\mathbf{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ with uv-domain R, then a partition of the uv-domain into small Δu by Δv rectangles ΔR (Fig. 1) is mapped by $\mathbf{r}$ to a partition of S into small patches in space with areas ΔS (Fig. 2), and the previous section showed that the area of each ΔS patch was $\Delta S \approx |\mathbf{r}_u \mathbf{x} \mathbf{r}_v| \Delta u \cdot \Delta v$.

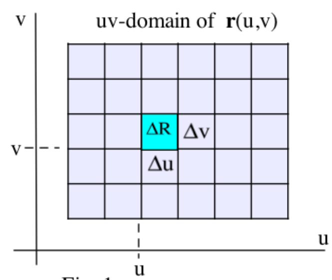

Fig. 1

Let (u*,v*) be a point in the uv-domain R. Then $\mathbf{r}(u^*,v^*) = \langle x(u^*,v^*),\ y(u^*,v^*),\ z(u^*,v^*)\rangle = \langle x^*,\ y^*,\ z^*\rangle$ is a point on a patch S*. If f(x,y,z) is a scalar-valued function on S, then the value of $f(\mathbf{r}(u^*,v^*)) \cdot \Delta S^*$ is approximately $f(\mathbf{r}(u^*,v^*)) \cdot \Delta S^* \approx f(\mathbf{r}(u^*,v^*)) \cdot |\mathbf{r}_{u^*} \mathbf{x} \mathbf{r}_{v^*}| \Delta u \cdot \Delta v$. Adding these values together for each of the uv-rectangles we have the Riemann sum

$$\sum_{u,v} f(x^*,y^*,z^*) \cdot \Delta S^* = \sum_{u,v} f(\mathbf{r}(u^*,v^*)) \cdot |\mathbf{r}_{u^*} \mathbf{x} \mathbf{r}_{v^*}| \Delta u \cdot \Delta v \cdot$$

Taking limits as $\Delta u,\ \Delta v \to 0$, we get

$$\iint_S f(x,y,z)\ dS = \iint_R f(\mathbf{r}(u,v)) \cdot |\mathbf{r}_u \mathbf{x} \mathbf{r}_v|\ dA\ .$$

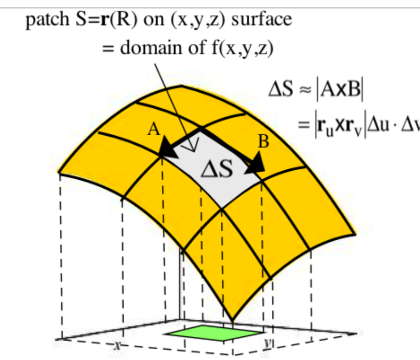

Fig. 2

> If S is a smooth surface parameterized by $\mathbf{r}(u,v)$ on domain R in the uv-domain, and f(x,y,z) is a scalar-valued function defined on S,
>
> then $\displaystyle\iint_S f(x,y,z)\ dS = \iint_R f(\mathbf{r}(u,v)) \cdot |\mathbf{r}_u \mathbf{x} \mathbf{r}_v|\ dA$.

This result enables us to evaluate many surface integrals in 3D as iterated integrals in u and v.

Note: You should notice the similarity of this result with the result for a line integral of a scalar-valued function along a curve C: $\int_C \mathbf{f}\, ds = \int_{t=a}^{b} \mathbf{f}(\mathbf{r}(t)) \cdot |\mathbf{r}'(t)|\ dt$. In this new situation the curve C is replaced with the surface S, and $|\mathbf{r}'(t)|dt$ is replaced with $|\mathbf{r}_u \mathbf{x} \mathbf{r}_v| dA$.

Note: If f(x,y,z)=1 for all (x,y,z) on S, then $\iint_S 1\ dS = \iint_R |\mathbf{r}_u \mathbf{x} \mathbf{r}_v|\ dA$ is simply the surface area of S.

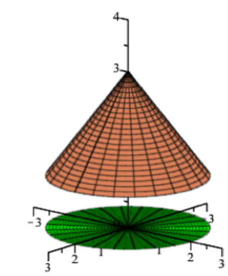

Fig. 3

Example 1: Let f(x,y,z)=1+z on the surface S parameterized by $\mathbf{r}(u,v) = \langle u \cdot \cos(v),\ u \cdot \sin(v),\ 3-u \rangle$ with $0 \le u \le 2$ and $0 \le v \le 2\pi$. (Fig. 3) (a) Evaluate $\iint_S f(x,y,z)\ dS$.

(b) If the units of x, y and z are meters (m) and f is the surface density (g/m^2) at location (a,y,z), what are the units of $\iint_S f(x,y,z)\ dS$?

Solution: (a) $\iint_S f(x,y,z)\ dS = \iint_R f(\mathbf{r}(u,v)) \cdot |\mathbf{r}_u \mathbf{x} \mathbf{r}_v|\ dA$ so we need $f(\mathbf{r}(u,v))$, $\mathbf{r}_u$ and $\mathbf{r}_v$.

$\mathbf{r}_u = \langle \cos(v),\ \sin(v),\ -1 \rangle$, $\mathbf{r}_v = \langle -u \cdot \sin(v),\ u \cdot \cos(v),\ 0 \rangle$ and

$$\mathbf{r}_u \mathbf{x} \mathbf{r}_v = \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & -1 \\ -u \cdot \sin(v) & u \cdot \cos(v) & 0 \end{vmatrix} = \langle -u \cdot \cos(v),\ u \cdot \sin(v),\ u \rangle \quad \text{so} \quad |\mathbf{r}_u \mathbf{x} \mathbf{r}_v| = u\sqrt{2} \ .$$

$$\iint_S f(x,y,z)\ dS = \iint_R f(\mathbf{r}(u,v)) \cdot |\mathbf{r}_u \mathbf{x} \mathbf{r}_v|\ dA = \int_{v=0}^{2\pi} \int_{u=0}^{2} [1+(3-u)](u\sqrt{2})\ du\ dv$$

$$= \int_{v=0}^{2\pi} \sqrt{2}\left(-\frac{1}{3}u^3 + 2u^2\right) \Big|_{u=0}^{2} dv = \int_{v=0}^{2\pi} \frac{16}{3}\sqrt{2}\ dv = \frac{32}{3}\sqrt{2}\pi \ .$$

(b) $\iint_S f(x,y,z)\ dS$ is the mass of the surface S, and the units are $(g/m^2)(m^2)= g$.

Practice 1: Evaluate $\iint_S (2+x)\ dS$ on the surface S in Example 1.

Example 2: Let f(x,y,z)=xy on the surface S that is the part of the plane $z = 3-x-y$ that is in the first octant. Evaluate $\iint_S f(x,y,z)\ dS$.

Solution: S can be parameterized by $\mathbf{r}(u,v) = \langle u,\ v,\ 3-u-v \rangle$ for $0 \le u \le 3$ and $0 \le v \le 3-u$. Then $r_u = \langle 1,\ 0,\ -1 \rangle$, $r_v = \langle 0,\ 1,\ -1 \rangle$, $\mathbf{r}_u \mathbf{x} \mathbf{r}_v = \langle 1,\ 1,\ 1 \rangle$ and $| r_u x r_v | = \sqrt{3}$. $f(\mathbf{r}(u,v)) = uv$ so

$$\iint_S f(x,y,z)\ dS = \int_0^3 \int_0^{u-3} uv\sqrt{3}\ dv\ du = \int_0^{u-3} \frac{\sqrt{3}}{2} u \cdot (3-u)^2\ dv = 3\sqrt{3} \ .$$

The units of the answer are (units of f)(units of S).

Practice 2: Let f(x,y,z)=x on the surface $S = \{(x,y,z) : x^2 + y^2 = 3, 0 \le z \le 2\}$. Evaluate $\iint_S f(x,y,z)\ dS$.

> If the graph of f(x,y,z) with z=g(x,y) is a smooth surface S in xyz-space parameterized by $\mathbf{r}(u,v)$ on domain R in uv-space, ,
>
> then $\iint_S f(x,y,z)\ dS = \iint_R f(\mathbf{r}(u,v)) \cdot \sqrt{1 + (g_x)^2 + (g_y)^2}\ dA$.

Proof: We can parameterize this surface by setting u=x and v=y so $\mathbf{r}(u,v) = \langle u, v, g(u,v) \rangle$.

$\mathbf{r}_u = \langle 1, 0, g_x \rangle$ and $\mathbf{r}_v = \langle 0, 1, g_y \rangle$ so $\mathbf{r}_u \mathbf{x} \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{vmatrix} = \langle -g_x, -g_y, 1 \rangle$ and $|\mathbf{r}_u \mathbf{x} \mathbf{r}_v| = \sqrt{1 + (g_x)^2 + (g_x)^2}$

Then $\iint_S f(x,y,g(x,y))\ dS = \iint_R f(\mathbf{r}(u,v)) \cdot \sqrt{1 + (g_x)^2 + (g_x)^2}\ dA$.

Example 3: Let $f(x,y,z) = z\sqrt{x^2 + y^2}$ on the heliocoid surface S parameterized by

$\mathbf{r}(u,v) = \langle u \cdot \cos(v), u \cdot \sin(v), v \rangle$ with $0 \le u \le 1$ and $0 \le v \le 2\pi$.

Evaluate $\iint_S f(x,y,z)\ dS$. (Fig. 4 shows a heliocoid with $0 \le v \le 4\pi$.)

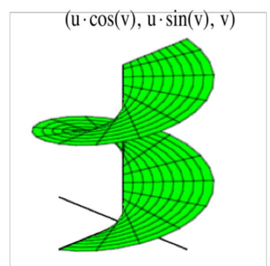

Fig. 4

Solution: $\iint_S f(x,y,z)\ dS = \iint_R f(\mathbf{r}(u,v)) \cdot \sqrt{1 + (g_x)^2 + (g_y)^2}\ dA$ so we need

$f(\mathbf{r}(u,v))$, $\mathbf{r}_u$ and $\mathbf{r}_v$. $\mathbf{r}_u = \langle \cos(v), \sin(v), 1 \rangle$, $\mathbf{r}_v = \langle -u \cdot \sin(v), u \cdot \cos(v), 0 \rangle$ and

$\mathbf{r}_u \mathbf{x} \mathbf{r}_v = \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & 1 \\ -u \cdot \sin(v) & u \cdot \cos(v) & 0 \end{vmatrix} = \langle -u \cdot \cos(v), -u \cdot \sin(v), u \rangle$ so $|\mathbf{r}_u \mathbf{x} \mathbf{r}_v| = u\sqrt{2}$.

$f(x,y,z) = z\sqrt{x^2 + y^2} = v \cdot u$ so

$$\iint_S f(x,y,z)\ dS = \iint_R f(\mathbf{r}(u,v)) \cdot |\mathbf{r}_u \mathbf{x} \mathbf{r}_v|\ dA = \int_{v=0}^{2\pi} \int_{u=0}^{1} [v \cdot u](u\sqrt{2})\ du\ dv = \left(2\pi^2\right)\left(\frac{1}{3}\sqrt{2}\right) .$$

Practice 3: Evaluate $\iint_S (1 + y)\ dS$ on the surface S in Example 3.

Note: If z=g(x,y) and f(x,y,z)=1 for all (x,y), then $\iint_S f(x,y,z)\ dS$ is the surface area of S, and this area equals $\iint_R \sqrt{1 + (g_x)^2 + (g_x)^2}\ dA$, then same result we saw in Section 14.5 .

Oriented Surfaces and the Unit Normal Vector n

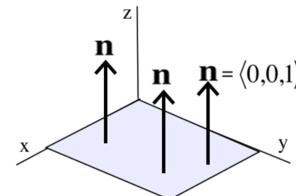

Fig. 5

Before investigating surface integrals for vector-valued functions, some vocabulary and technical issues need to be considered: oriented surfaces and an orientation for vector **n** that is normal (perpendicular) to the surface.

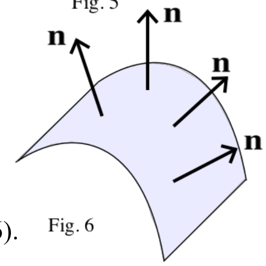

Fig. 6

A flat piece of paper in the xy-plane has a normal vector $\mathbf{n} = \langle 0,0,1 \rangle$ pointing upward at each point on the paper (Fig. 5). If we gently fold (but not crease) the paper, then the normal vector **n** will change continuously depending on its location on the paper (Fig. 6). If we follow a closed path on the paper that does not cross the paper's edge then the direction of the normal vector will change continuously and will return to the starting location pointing in the its original direction (Fig. 7). Such a surface is called oriented.

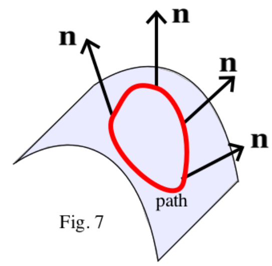

Fig. 7

A smooth surface S is **oriented** if

* S has a non–zero normal vector at each point,
* the direction of the normal vector varies continuously as we move along S (not crossing an edge),
* and, a normal vector returns to its original orientation when it returns to its original position after moving along any closed path on S (not crossing an edge).

Fortunately, most surfaces are oriented. The most famous example of a non-oriented surface is a Mobius strip (Fig. 8). If we start with a normal vector **n** at any point and travel along the middle of the strip (not crossing an edge), then we end up at the starting point again but with the normal vector now pointing in the direction **–n** (Fig. 9).

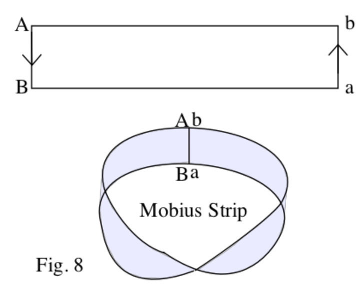

Fig. 8

The results that follow require that our surfaces be oriented.

However, at a point A on an oriented surface there are two normal vectors, and we need to select one of them for the orientation. If the surface encloses a region of space, the convention is to pick the normal vector which points outward from the enclosed region (Fig. 10).

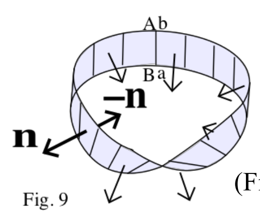

Fig. 9

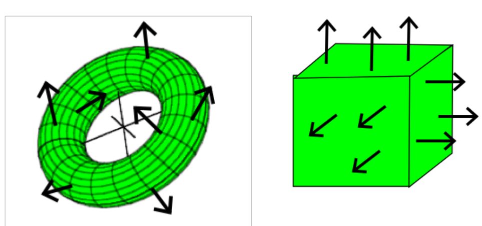
Fig. 10

Surface Integral for a Vector–Valued Function

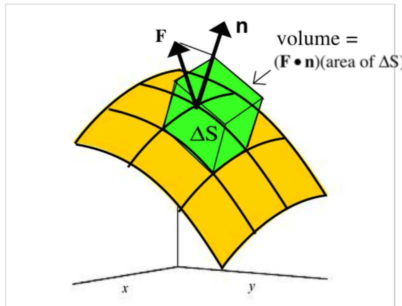

Fig. 11

Let ΔS be a small patch on the smooth, oriented surface S with oriented normal vector **n** (at some point of S). Then the magnitude of the vector F crossing the patch is the projection of F onto **n**. If we think of the vector field as water moving **F** at each point, then the amount of water passing through the patch ΔS in the direction of **n** is $(\mathbf{F} \bullet \mathbf{n})(\text{area of } \Delta S)$. Visually, that volume of that water (per unit of time) is the volume of the prism in Fig. 11. As before, adding the values of $(\mathbf{F} \bullet \mathbf{n})(\text{area of } \Delta S)$ for all of the patches we have the Riemann sum $\displaystyle\sum_{\Delta u, \Delta v} (\mathbf{F} \bullet \mathbf{n})(\text{area of } \Delta S)$.

Taking the limit as all of the Δu and Δv approach zero, we have the surface integral $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, d\mathbf{S}$. If the surface S is parameterized by the $\mathbf{r}(u,v)=(\mathbf{x}(u,v), \mathbf{y}(u,v), \mathbf{z}(u,v))$ for (u,v) in a region R, then $\Delta S = |\mathbf{r}_u \mathsf{x} \mathbf{r}_v| \cdot \Delta u \cdot \Delta v$. The vector $\mathbf{r}_u \mathsf{x} \mathbf{r}_v$ is normal to the surface S and the unit normal vector is $\mathbf{n} = \dfrac{\mathbf{r}_u \mathsf{x} \mathbf{r}_v}{|\mathbf{r}_u \mathsf{x} \mathbf{r}_v|}$ so $\mathbf{F} \bullet \mathbf{n}\, d\mathbf{S} = \mathbf{F} \bullet \dfrac{\mathbf{r}_u \mathsf{x} \mathbf{r}_v}{|\mathbf{r}_u \mathsf{x} \mathbf{r}_v|} \cdot |\mathbf{r}_u \mathsf{x} \mathbf{r}_v| \cdot \Delta u \cdot \Delta v$ and $\mathbf{F} \bullet \mathbf{n}\, d\mathbf{S} = \mathbf{F} \bullet (\mathbf{r}_u \mathsf{x} \mathbf{r}_v)\, dA$.

> Definition: Surface Integral of **F** over S
>
> If **F** is a continuous vector field over the oriented surface S parameterized by **r**(u,v) and having unit normal vector **n**,
>
> then the surface integral of **F** over S is $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, d\mathbf{S} = \iint_{\mathbf{R}} \mathbf{F} \bullet (\mathbf{r}_u \mathsf{x} \mathbf{r}_v)\, dA$.
>
> This integral is also called the **flux** of **F** across S.

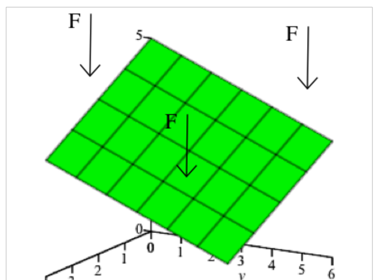

Fig. 12

Example 4: Suppose S is the part of the plane 3x+2y+6z=30 with domain $0 \le x \le 4$ and $0 \le y \le 6$ (Fig. 12). This surface can be parameterized by $\mathbf{r}(u,v) = \left\langle u, v, 5 - \frac{u}{2} - \frac{v}{3} \right\rangle$ so $\mathbf{r}_u = \left\langle 1, 0, -\frac{1}{2} \right\rangle$, $\mathbf{r}_v = \left\langle 0, 1, -\frac{1}{3} \right\rangle$ and. If $\mathbf{F}(x,y,z) = \langle 0, 0, -2 \rangle$ then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{\mathbf{R}} \mathbf{F} \bullet (\mathbf{r}_u \mathsf{x} \mathbf{r}_v)\, dA = \iint_{\mathbf{R}} \langle 0, 0, -2 \rangle \bullet \left\langle \frac{1}{2}, \frac{1}{3}, 1 \right\rangle dA = \int_{u=0}^{4} \int_{v=0}^{6} -2\, dv\, du$$

$$= (-2)(\text{area of R}) = (-2)(24) .$$

If $\mathbf{F} = \langle 0, 0, -2 \rangle$ is the velocity of water in m/s, and x and y are given in meters (m), then the units of $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\mathbf{S}$ are m^3/s : 48 m^3/s pass through the surface S. The negative sign in the answer results because the angle between $\mathbf{F}$ and $\mathbf{n}$ is greater than 90^o. If we had picked the opposite normal vector, then the answer would have been +48.

Practice 4: Use the surface S from Example 3 and calculate $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\mathbf{S}$ for $\mathbf{F} = \langle 0, -3, 0 \rangle$ and for $\mathbf{F} = \langle 1, 2, 3 \rangle$.

If F is not a constant vector field as in the previous Example and Practice problems, then we need to rewrite **F**(x,y,z) as **F**(x(u,v), y(u,v), z(u,v)).

Practice 5: Suppose S is the part of the surface z=f(x,y) above the region R in the xy-plane and that $\mathbf{F} = \langle M, N, P \rangle$.

Show that $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\mathbf{S} = \iint_R \mathbf{F} \bullet (\mathbf{r}_u \mathbf{x} \mathbf{r}_v)\, dA = \iint_R \{-M \cdot f_x - N \cdot f_y + P\}\, dA$ ·

When S is a sphere

Spheres occur often in applications so it is worthwhile to see the calculations for a sphere. A sphere of radius R centered at the origin is easily described in spherical coordinates as (R, θ, φ) with $0 \le \theta \le 2\pi$ and $0 \le \varphi \le \pi$. Setting $u = \theta$ and $v = \varphi$ and then converting to rectangular coordinates we have (as in 15.7 Example 1) $x(u,v) = R \cdot \sin(v) \cdot \cos(u)$, $y(u,v) = R \cdot \sin(v) \cdot \sin(u)$ and $z(u,v) = R \cdot \cos(v)$. Since $\mathbf{r}(u,v) = \langle x, y, z \rangle$, $\mathbf{r}_u = \langle -R \cdot \sin(v) \cdot \sin(u), R \cdot \sin(v) \cdot \cos(u), 0 \rangle$ and $\mathbf{r}_v = \langle R \cdot \cos(v) \cdot \cos(u), R \cdot \cos(v) \cdot \sin(u), -R \cdot \sin(v) \rangle$.
Finally, $\mathbf{r}_u \mathbf{x} \mathbf{r}_v = -R^2 \left\langle \sin^2(v) \cdot \cos(u), \sin^2(v) \cdot \sin(u), \sin(v) \cdot \cos(v) \right\rangle$. For an outward facing normal vector **n**, take $\mathbf{r}_u \mathbf{x} \mathbf{r}_v = R^2 \left\langle \sin^2(v) \cdot \cos(u), \sin^2(v) \cdot \sin(u), \sin(v) \cdot \cos(v) \right\rangle$.

Example 5: Suppose S is a sphere of radius 1 centered at the origin and $F(x,y,z) = \langle x,y,z \rangle$ is a radial vector field.

(a) Determine the flux of **F** across S. (b) Determine the flux of **F** across S when S has radius R.

Solution: (a) $F(x,y,z) = \langle x,y,z \rangle = \langle \sin(v) \cdot \cos(u), \sin(v) \cdot \sin(u), \cos(v) \rangle$. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\mathbf{S} = \iint_{\mathbf{R}} \mathbf{F} \bullet (\mathbf{r}_u \mathbf{x} \mathbf{r}_v)\, dA$$

$$= \iint_{\mathbf{R}} \langle \sin(v) \cdot \cos(u), \sin(v) \cdot \sin(u), \cos(v) \rangle \bullet \left\langle \sin^2(v) \cdot \cos(u), \sin^2(v) \cdot \sin(u), \sin(v) \cdot \cos(v) \right\rangle dA$$

$$= \iint_{\mathbf{R}} \sin^3(u) \cdot \cos^2(v) + \sin^3(v) \cdot \sin^2(u) + \sin(v) \cdot \cos^2(v)\, dA = \int_{u=0}^{2\pi} \int_{v=0}^{\pi} \sin(v)\, dv\, du = 4\pi \;.$$

(b) The only change in the calculation from part (a) is that now $\mathbf{r}_u \mathsf{x} \mathbf{r}_v$ has the factor R^2 so the result from part (a) needs to be multiplied by R^2 : flux = $4\pi R^2$.

Practice 6: Suppose S is the hemisphere $S = \{(x,y,z):\ x^2 + y^2 + z^2 = 1 \text{ and } 0 \le z\}$ and $F(x,y,z) = \langle z,x,y \rangle$. Determine the flux of F across S.

Connections with line integrals

There are nice parallels between the integrals of scalar and vector-valued functions along a curves C in 2D (section 15.3) and those on surfaces S in 3D.

scalar f on a curve C parameterized by $\mathbf{r}(t)$: $\int_C \mathbf{f}\, ds = \int_{t=a}^{b} \mathbf{f}(\mathbf{r}(t)) \cdot |\mathbf{r}'(t)|\ dt$

scalar f on a surface S parameterized by $\mathbf{r}(u,v)$: $\iint_S f(x,y,z)\ dS = \iint_R f(\mathbf{r}(u,v)) \cdot |\mathbf{r}_u \mathsf{x} \mathbf{r}_v|\ dA$

vector–valued **F** on a curve C parameterized by $\mathbf{r}(t)$: $\int_C \mathbf{F} \bullet \mathbf{T}\ ds = \int_{t=a}^{b} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\ dt$

vector–valued **F** on a surface S parameterized by $\mathbf{r}(u,v)$: $\iint_S \mathbf{F} \cdot \mathbf{n}\ d\mathbf{S} = \iint_{\mathbf{R}} \mathbf{F} \bullet (\mathbf{r}_u \mathsf{x} \mathbf{r}_v)\ dA$

Problems

1. $f(x,y,z)=x+y+z$ and $\mathbf{r}(u,v) = \langle u + 3v,\ 2u - v,\ 3u + v \rangle$. What is the value of $f(\mathbf{r}(u,v)) \cdot |\mathbf{r}_u \mathsf{x} \mathbf{r}_v|\ \Delta A$ when u=1, v=2, $\Delta u = 0.3$ and $\Delta v = 0.1$?

2. $f(x,y,z)=x+y+z$ and $\mathbf{r}(u,v) = \langle 2u + v,\ 3u - v,\ u + 2v \rangle$. What is the value of $f(\mathbf{r}(u,v)) \cdot |\mathbf{r}_u \mathsf{x} \mathbf{r}_v|\ \Delta A$ when u=2, v=3, $\Delta u = 0.1$ and $\Delta v = 0.2$?

3. $f(x,y,z) = 2x + y^2 - z$ and $\mathbf{r}(u,v) = \langle u^2,\ 3u + v,\ v^2 \rangle$. What is the value of $f(\mathbf{r}(u,v)) \cdot |\mathbf{r}_u \mathsf{x} \mathbf{r}_v|\ \Delta A$ when u=2, v=1, $\Delta u = 0.3$ and $\Delta v = 0.2$?

4. $f(x,y,z) = 2x + y^2 - z$ and $\mathbf{r}(u,v) = \langle u^2,\ 3u + v,\ v^2 \rangle$. What is the value of $f(\mathbf{r}(u,v)) \cdot |\mathbf{r}_u \mathsf{x} \mathbf{r}_v|\ \Delta A$ when u=3, v=2, $\Delta u = 0.3$ and $\Delta v = 0.2$?

5. $f(x,y,z) = x^2 + 4y + z$ on the surface $S = \{(x,y,z):\ 0 \le x \le 3,\ 0 \le y \le 2,\ z = 4)$. Evaluate $\iint_S f(x,y,z)\ dS$.

6. $f(x,y,z) = x^2 + 4y + z$ on the surface $S = \{(x,y,z):\ 0 \le x \le 2,\ y = 3,\ 1 \le z \le 4)$. Evaluate $\iint_S f(x,y,z)\ dS$.

7. $f(x,y,z) = y^2$ on the surface $S = \{(x,y,z):\ x + y + z = 4 \text{ in first octant}\}$ Evaluate $\iint_S f(x,y,z)\ dS$.

8. $f(x,y,z) = xy$ on the surface $S = \{(x,y,z):\ z = 1 + x^2 + y^2,\ 0 \le x \le 2,\ 0 \le y \le 2\}$. Evaluate $\iint\limits_S f(x,y,z)\ dS$.

9. $\mathbf{F}(x,y,z) = \langle x,\ 2y,\ 3z\rangle$ on the surface $S = \{(x,y,z):\ 0 \le x \le 3,\ 0 \le y \le 2,\ z = 4)$. Determine the flux of **F** across S. If the units of **F** are liters/second and the units of x, y and z are meters, what are the units of the flux?

10. $\mathbf{F}(x,y,z) = \langle x,\ -y,\ z\rangle$ on the surface $S = \{(x,y,z):\ z = x^2 + y^2,\ 0 \le x \le 2,\ 0 \le y \le 2\}$ Determine the flux of **F** across S. If the units of **F** are grams/meter^2 and the units of x, y and z are meters, what are the units of the flux?

11. $\mathbf{F}(x,y,z) = \langle x,\ y,\ z\rangle$ on the elliptical cylinder $S = \{(x,y,z):\ x^2 + 4y^2 = 4,\ 0 \le z \le 3\}$ Determine the flux of **F** across S.

12. $\mathbf{F}(x,y,z) = \langle 0,\ 0,\ K\rangle$ on the paraboloid $S = \{(x,y,z):\ x^2 + y^2 \le A^2,\ z = A^2 - x^2 - y^2\}$ Determine the flux of **F** across S.

13. Suppose **F** is the same as in Problem 12 but now S is the "stretched" paraboloid $S = \{(x,y,z):\ x^2 + y^2 \le A^2,\ z = C(A^2 - x^2 - y^2)\}$. Determine the flux of **F** across S.

Practice Answers

Practice 1: From Example 1, $|\mathbf{r}_u \mathbf{x} \mathbf{r}_v| = u\sqrt{2}$ and $x = u\cdot\cos(v)$ so

$$\iint\limits_S (2+x)\ dS = \int_{v=0}^{2\pi}\int_{u=0}^{2} [2 + u\cdot\cos(v)]\cdot(u\sqrt{2})\ du\ dv = \int_{v=0}^{2\pi}\left(u^2\sqrt{2} + \frac{1}{3}u^3\sqrt{2}\cdot\cos(v)\right)\Bigg|_{u=0}^{2} dv = 8\sqrt{2}\pi\ .$$

Practice 2: S can be parameterized by $\mathbf{r}(u,v) = \langle 3\cdot\cos(u),\ 3\cdot\sin(u),\ v\rangle$ with $0 \le u \le 2\pi$ and $0 \le v \le 2$. Then

$r_u = \langle -3\cdot\sin(u),\ 3\cdot\cos(u),\ 0\rangle$, $r_v = \langle 0,\ 0,\ 1\rangle$, $\mathbf{r}_u \mathbf{x} \mathbf{r}_v = \langle 3\cos(u),\ 3\sin(u),\ 0\rangle$ and $|r_u x r_v| = 3$.

$f(\mathbf{r}(u,v)) = 3\cdot\cos(u)$ so $\iint\limits_S f(x,y,z)\ dS = \int_0^{2\pi}\int_0^{2} 3\cdot\cos(u)\cdot 3\ dv\ du = \int_0^{2\pi} 18\cdot\cos(u)\ dv = 0$.

Practice 3: From Example 2, $|\mathbf{r}_u \mathbf{x} \mathbf{r}_v| = u\sqrt{2}$ and $1 + y = 1 + u\cdot\sin(v)$ so

$$\iint\limits_S (1+y)\ dS = \int_{v=0}^{2\pi}\int_{u=0}^{1} [1 + u\cdot\sin(v)]\cdot(u\sqrt{2})\ du\ dv = \int_{v=0}^{2\pi}\left(\frac{1}{2}u^2\sqrt{2} - \frac{1}{3}u^3\sqrt{2}\cdot\cos(v)\right)\Bigg|_{u=0}^{1} dv = \sqrt{2}\pi\ .$$

Practice 4: For $\mathbf{F} = \langle 0,\ -3,\ 0\rangle$, $\iint\limits_S \mathbf{F}\cdot\mathbf{n}\ d\mathbf{S} = \iint\limits_{\mathbf{R}} \langle 0,\ -3,\ 0\rangle \bullet \left\langle \frac{1}{2},\ \frac{1}{3},\ 1\right\rangle dA = \iint\limits_{\mathbf{R}} 1\ dA = 24$.

For $\mathbf{F} = \langle 1,\ 2,\ 3\rangle$, $\iint\limits_S \mathbf{F}\cdot\mathbf{n}\ d\mathbf{S} = \iint\limits_{\mathbf{R}} \langle 1,\ 2,\ 3\rangle \bullet \left\langle \frac{1}{2},\ \frac{1}{3},\ 1\right\rangle dA = \iint\limits_{\mathbf{R}} \left(\frac{25}{6}\right) dA = 100$.

Practice 5: The surface S can be parameterized by $\mathbf{r}(u,v) = \langle u, v, f(u,v)\rangle$.

Then $\mathbf{r}_u = \langle 1, 0, f_u\rangle$, $\mathbf{r}_v = \langle 0, 1, f_v\rangle$ and $\mathbf{r}_u \times \mathbf{r}_v = \langle -f_u, -f_v, 1\rangle$ so

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, d\mathbf{S} = \iint_{\mathbf{R}} \mathbf{F}\bullet(\mathbf{r}_u \times \mathbf{r}_v)\ dA = \iint_{\mathbf{R}} \langle M, N, P\rangle \bullet \langle -f_u, -f_v, 1\rangle\ dA = \iint_R \left\{-P\cdot f_x - Q\cdot f_y + P\right\} dA\ .$$

Practice 6: As in the example, S can be parameterized by $x(u,v) = \sin(v)\cdot\cos(u)$, $y(u,v) = \sin(v)\cdot\sin(u)$ and $z(u,v) = \cos(v)$ with $0 \le u \le 2\pi$ and $0 \le v \le \pi/2$. Then $F(x,y,z) = \langle \cos(v), \sin(v)\cdot\cos(u), \sin(v)\cdot\sin(u)\rangle$ and

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, d\mathbf{S} = \iint_{\mathbf{R}} \mathbf{F}\bullet(\mathbf{r}_u \times \mathbf{r}_v)\ dA$$

$$= \iint_{\mathbf{R}} \langle \cos(v), \sin(v)\cdot\cos(u), \sin(v)\cdot\sin(u)\rangle \bullet \left\langle \sin^2(v)\cdot\cos(u), \sin^2(v)\cdot\sin(u), \sin(v)\cdot\cos(v)\right\rangle\ dA$$

$$= \int_{v=0}^{\pi/2}\int_{u=0}^{2\pi} \cos(v)\cdot\sin^2(v)\cdot\cos(u) + \sin^3(v)\cdot\cos(u)\cdot\sin(u) + \sin^2(v)\cdot\sin(u)\cdot\cos(v)\ du\ dv\ .$$

But $\int_{u=0}^{2\pi} (\text{each term})\ du = 0$ so the flux =0.