Stokes' Theorem

The circulation–curl form of Green's Theorem (section 15.5) says if $\mathbf{F} = \langle M, N \rangle$ is a 2D vector field and C is a simple, closed, piecewise smooth curve enclosing a region R then the integral of the curl of **F** on R is equal to the circulation of **F** around C (with a positive orientation): (Fig. 1)

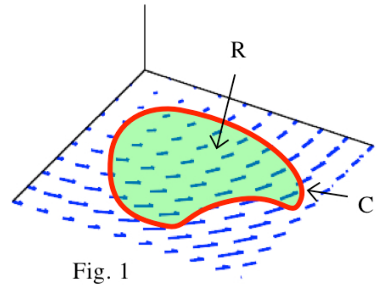


Fig. 1

$$\iint_R \text{curl } \mathbf{F} \; dA = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \oint_{\mathbf{C}} \mathbf{F} \bullet \mathbf{T} \; ds = \text{circulation of } \mathbf{F} \text{ around C .}$$

Stokes' Theorem moves this result into 3D, and it has some very important consequences. If we think of Green's Theorem as applying to a flat soap film then we can think of Stokes' Theorem as giving the same result if we blow gently to create a soap bubble, a surface S in 3D. (Fig. 2)

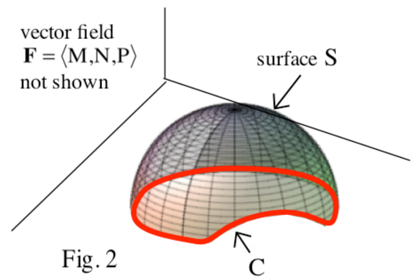


Fig. 2

Stokes' Theorem

If S is a connected, simply–connected, piecewise-smooth surface in 3D with piecewise–smooth boundary curve C, and **F** has continuous partial derivatives,

then $\displaystyle\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S} = \iint_S (\nabla \mathrm{x} \mathbf{F}) \bullet \mathbf{n} \; dS = \oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C \mathbf{F} \bullet \mathbf{T} \; ds = \text{circulation around C}$

Among the consequences of Stokes' Theorem:

* It allows us to trade 2D and 3D integrals – sometimes one of those is much easier than the other.
* It says that the integral of the curl of a vector field only depends on its values on the boundary.
* It says that the integral of the curl over a closed surface (like a sphere) is 0 since a closed surface has no boundary curve.
* It allows us to prove some of Maxwell's equations from physics (section 15.11).

The general proof of Stokes' Theorem is complicated. A proof of an easy special case is given next, and a proof for the common special case when the surface S has the form in Fig. 3 with $S = \{(x, y, z) : \; z = z(x, y) \text{ is a function of x and y}\}$ is given in the Appendix.

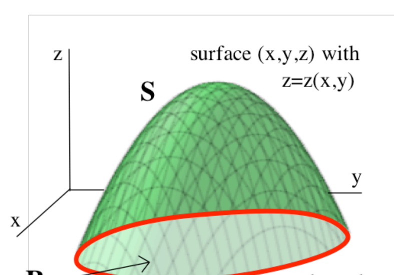


Fig. 3

Proof for an easy special case: S consists of a finite number of flat panels (Fig. 4) not necessarily in the same plane.

Then we can apply Green's Theorem to each panel and add the results together. This is very similar to the "finite Green's Theorem" we saw in section 15.5. The circulations along all of the interior edges cancel since adjacent panels have equal circulations going in opposite directions, and the only circulations remaining are those along the boundary of the region S. If S is the union of sub-regions $S_1, S_2, S_3 \ldots S_n$, then

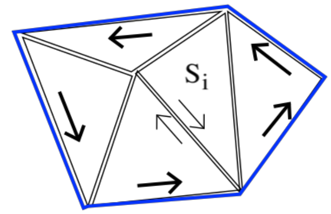


Fig. 4

$$\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S} = \sum_{i=1}^{n} \iint_{S_i} \text{curl } \mathbf{F} \bullet d\mathbf{S} = \sum_{i=1}^{n} \int_{C_i} \mathbf{F} \bullet \mathbf{T} \, dt = \int_C \mathbf{F} \bullet \mathbf{T} \, dt \, .$$

Example 1: Use Stokes' theorem to evaluate $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S}$ where S is the hemisphere bounded by $x^2 + y^2 + z^2 = 9$ with $z \geq 0$ (Fig. 5) for the vector field $\mathbf{F} = \langle y, -x, 0 \rangle$.

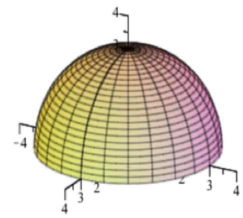


Fig. 5

Solution: By Stokes' theorem $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S} = \int_C \mathbf{F} \bullet d\mathbf{r}$ where C is the bounding circle parameterized by $\mathbf{r}(t) = \langle 3 \cdot \cos(t), 3 \cdot \sin(t), 0 \rangle$. Then

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \mathbf{F} \bullet \mathbf{T} \, dt = \int_{t=0}^{2\pi} \langle 3 \cdot \sin(t), -3 \cdot \cos(t), 0 \rangle \bullet \langle -3 \cdot \sin(t), 3 \cdot \cos(t), 0 \rangle \, dt$$

$$= \int_{t=0}^{2\pi} -9 \cdot \sin^2(t) - 9 \cdot \cos^2(t) \, dt = \int_{t=0}^{2\pi} -9 \, dt = (-9)(2\pi) = -18\pi \, .$$

It is also possible to evaluate $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S}$ directly, but more difficult.

In this case the line integral around C was easier to evaluate than the surface integral on S.

Practice 1: Use Stokes' theorem to evaluate $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S} = \int_C \mathbf{F} \bullet d\mathbf{r}$ where S is the paraboloid bounded by $2x^2 + 2y^2 + z = 18$ with $z \geq 0$ (Fig. 6) for the vector field $\mathbf{F} = \langle y, -x \rangle$.

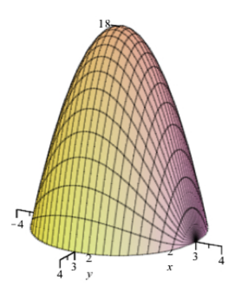


Fig. 6

Example 2: (a) Evaluate the line integral $\int_C \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F} = \left\langle z, -z, x^2 - y^2 \right\rangle$ and C consists of the 3 line segments that bound the plane z=8–4x–2y in the first octant oriented as in Fig. 7.

(b) Evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$ for the line segment from (2, 0, 0) to (0, 4, 0).

Solution: (a) Rather than parameterizing the 3 line segments and evaluating the line integral along each of them, we can use Stoke's theorem and instead evaluate $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S} = \iint_S (\nabla \mathrm{x} \mathbf{F}) \bullet \mathbf{n} \ dS$.

The equation of the triangle is $4x + 2y + z = 8$ so $z = 8 - 4x - 2y$ and $\mathbf{r}_u \mathrm{x} \mathbf{r}_v = \langle 4, 2, 1 \rangle$

$$\nabla \mathrm{x} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial \mathbf{x}} & \frac{\partial}{\partial \mathbf{y}} & \frac{\partial}{\partial \mathbf{z}} \\ z & -z & x^2 - y^2 \end{vmatrix} = \langle 1 - 2y, \ 1 - 2x, \ 0 \rangle$$ Then

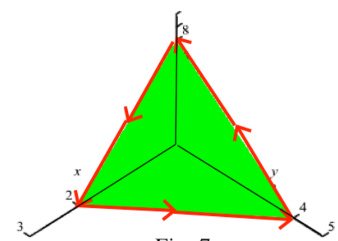


Fig. 7

$$\iint_S (\nabla \mathrm{x} \mathbf{F}) \bullet \mathbf{n} \ dS = \iint_S \langle 1 - 2y, \ 1 - 2x, \ 0 \rangle \bullet \langle 4, \ 2, \ 1 \rangle \ dS$$

$$= \int_{x=0}^{2} \int_{y=0}^{4-2x} 6 - 8y - 4x \ dy \ dx = \text{(a standard double integral)} = -120 \ .$$

(b) C is parameterized by $\mathbf{r}(t) = \langle 2 - t/2, \ t, \ 0 \rangle$ for $0 \le t \le 4$ so $\mathbf{r}'(t) = \langle -1/2, \ 1, \ 0 \rangle$.

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_{t=0}^{4} \mathbf{F} \bullet \mathbf{r}' \ dt = \int_{t=0}^{4} \left\langle 0, \ 0, \ (2 - t/2)^2 - t^2 \right\rangle \bullet \langle -1/2, \ 1, \ 0 \rangle \ dt = \int_{t=0}^{4} 0 \ dt = 0 \ .$$

Practice 2: Calculate the circulation of $\mathbf{F} = \langle xy, \ xz, \ -2yz \rangle$ around the curve C that consists of the 3 line segments that bound the plane x+2y+2z=4 in the first octant oriented as in Fig. 8.

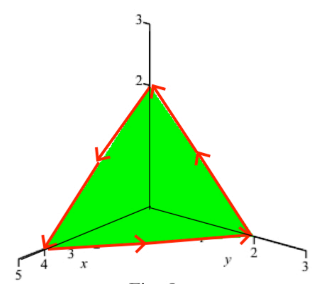


Fig. 8

Example 3: Use Stokes' theorem to evaluate $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S}$ where S is the "cap" on the hemisphere bounded by $x^2 + y^2 + z^2 = 25$ with $z \ge 3$ (Fig. 9) for the vector field $\mathbf{F} = \langle z - y, \ x, \ -x \rangle$.

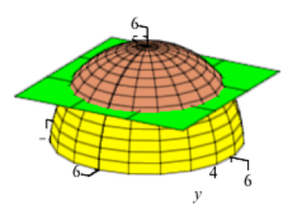


Fig. 9

Solution: $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S} = \int_C \mathbf{F} \bullet d\mathbf{r}$ where C is the circle $\mathbf{r}(t) = \langle 4 \cdot \cos(t), \ 4 \cdot \sin(t), \ 3 \rangle$.

Then $\int_t \mathbf{F} \bullet \mathbf{r}' \ dr = \int_{t=0}^{2\pi} \langle 3 - 4 \cdot \sin(t), \ 4 \cdot \cos(t), \ -4 \cdot \cos(t) \rangle \bullet \langle -4 \cdot \sin(t), \ 4 \cdot \cos(t), \ 0 \rangle \ dr$

$$= \int_{t=0}^{2\pi} -12 \cdot \sin(t) + 16 \cdot \sin^2(t) + 16\cos^2(t) \ dt = 12\cos(t) + 16t \Big|_0^{2\pi} = 32\pi \ .$$

Practice 3: Use Stokes' theorem to evaluate $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S}$ where F is the same as in Example 3, but now S is the cap with $z \geq 4$ on the same hemisphere.

Example 4: Evaluate $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S}$ when S is the surface of the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

Solution: Since S is a piecewise **closed** smooth surface, then S has no boundary curve C and $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S} = 0$.

Practice 4: Evaluate $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S}$ when S is the ellipsoid $x^2 + 2y^2 + 4z^2 = 16$.

If S has holes

If the surface S has a hole (Fig. 10) then the boundary of S has an additional boundary curve, and we can treat that new boundary in the same way we treated the boundary of a hole using Green's Theorem. We can create a single boundary for S by adding a path along S to the hole and then back from the hole (Fig. 11). Then the integral along this total new path will be the sum of the counterclockwise integrals around the outer boundary of S minus the sum of the counterclockwise integral around the hole. The integrals along the added paths sum to 0 since they are traveled once in each direction. Remember, for a counterclockwise orientation the region is always on our left hand side.

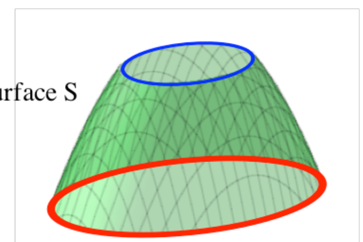


Fig. 10

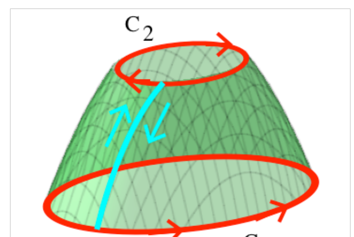


Fig. 11

Example 5: Evaluate $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S}$ for $\mathbf{F} = \langle xy,\ x + z^2,\ y^3 \rangle$ with

$$S = \{(x, y, z) :\ x^2 + y^2 + z^2 = 25,\ 0 \leq z \leq 4\}$$

Solution: This is the situation in Fig. 11. C_1 is parameterized by $\mathbf{r}_1(t) = \langle 5 \cdot \cos(t),\ 5 \cdot \sin(t),\ 0 \rangle$ and C_2 by $\mathbf{r}_2(t) = \langle 3 \cdot \cos(t),\ 3 \cdot \sin(t),\ 4 \rangle$ for $0 \leq t \leq 2\pi$ (both are counterclockwise).

$$\int_{C_1} \mathbf{F} \bullet d\mathbf{r} = \int_0^{2\pi} \langle 25 \cdot \sin(t) \cdot \cos(t),\ 5 \cdot \cos(t) + 0,\ 125 \cdot \sin^3(t) \rangle \bullet \langle -5\sin(t),\ 5\cos(t),\ 0 \rangle\, dt$$

$$= \int_0^{2\pi} -125 \cdot \sin^2(t) \cdot \cos(t) + 25 \cdot \cos^2(t) + 0\, dt = 25\pi .$$

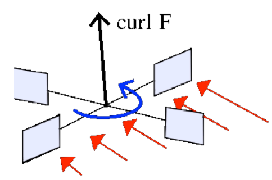
$$\int_{C_2} \mathbf{F} \bullet d\mathbf{r} = \int_0^{2\pi} \left\langle 9\cdot\sin(t)\cdot\cos(t),\ 3\cdot\cos(t)+4,\ 27\cdot\sin^3(t) \right\rangle \bullet \langle -3\cdot\sin(t),\ 3\cdot\cos(t),\ 0\rangle\ dt$$

$$= \int_0^{2\pi} -27\cdot\sin^2(t)\cdot\cos(t) + 9\cdot\cos^2(t) + 12\cdot\cos(t) + 0\ dt = 9\pi\ .$$

So $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S} = \int_{C_1} \mathbf{F} \bullet d\mathbf{r} - \int_{C_2} \mathbf{F} \bullet d\mathbf{r} = 16\pi$.

Practice 5: Evaluate $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S}$ for $\mathbf{F} = \left\langle xy,\ x+z^2,\ y^3 \right\rangle$ with $S = \{(x,y,z):\ x^2+y^2+z^2 = 25,\ 0 \le z \le 3\}$.

Meaning of the curl



In section 15.6 we claimed that the curl vector had two important properties:

* the magnitude of the curl gives the rate of the fluid's rotation, and
* the direction of the curl is normal to the plane of greatest circulation and points in the direction so that the circulation at the point has a right hand orientation (Fig. 12).

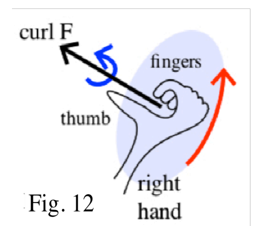


Fig. 12

If we have a small paddle wheel at point P and tilt it in different directions, then the claims say that the wheel will spin fastest with a right-hand orientation when the axis points in the direction of the curl vector.

Now we can use Stokes' Theorem to justify those claims.

Let P be a point in the vector field **F**, and let **u** be any unit vector. Suppose S is a small disk that has center at point P and radius r and that lies in the plane determined by P and u . S has a boundary circle C oriented positively so that S is always on the left side as we move along C.

Since S is small, the value of $\nabla \mathbf{x} \mathbf{F}$ is almost constant on S and so

$$\iint_{\mathbf{S}} (\nabla \mathbf{x}\mathbf{F}) \bullet \mathbf{u}\ dS \approx (\nabla \mathbf{x}\mathbf{F}) \bullet \mathbf{u} \iint_{\mathbf{S}} 1\ dS = (\nabla \mathbf{x}\mathbf{F}) \bullet \mathbf{u}\ (\text{area of S}) = |\nabla \mathbf{x}\mathbf{F}|\cdot\cos(\theta)\cdot(\pi\cdot r^2)$$

But by Stokes' Theorem, $\iint_{\mathbf{S}} (\nabla \mathbf{x}\mathbf{F}) \bullet \mathbf{u}\ dS = \oint_{C} \mathbf{F} \bullet \mathbf{T}\ dt = \text{circulation of F around C}$ so

$|\nabla \mathbf{x}\mathbf{F}|\cdot\cos(\theta) = \dfrac{\text{circulation of } \mathbf{F} \text{ around C}}{\pi\cdot r^2}$ which is maximum when $\theta = 0$ and **u** has the same direction as $\nabla \mathbf{x}\mathbf{F}$.

Together these statements say that an axis in the direction of $\nabla\text{x}\mathbf{F}$ gives the maximum circulation, and the magnitude of $\nabla\text{x}\mathbf{F}$ is the maximum rate of circulation per unit of area.

In section 15.6 we also stated the following theorem and said that a proof needed to wait until we had Stokes' Theorem.

Theorem: If **F** is defined and has continuous partial derivatives at every point in 3D and curl **F** = 0,
then **F** is a conservative field.

Proof: With Stokes' Theorem this is easy. If curl $\mathbf{F} = 0$ then $0 = \iint_S \text{curl } \mathbf{F} \, d\mathbf{S}$ for every simply-connected region S so by Stokes' Theorem $\int_C \mathbf{F} \bullet \mathbf{T} \, ds = \iint_S \text{curl } \mathbf{F} \, d\mathbf{S} = \mathbf{0}$ for every simple, closed, piece-wise smooth curve C. That means that **F** is path independent and conservative.

Problems

For problems 1 to 12 use Stoke's Theorem to find the circulation of vector field **F** around the positively oriented curve .

1. $\mathbf{F} = \langle y, 2x, -z^2 \rangle$ and C is the ellipse $x^2 + 4y^2 = 4$.

2. $\mathbf{F} = \langle y, 2x, -z^2 \rangle$ and C is the circle $x^2 + y^2 = 9$.

3. $\mathbf{F} = \langle z, y^2, xy \rangle$ and C is the boundary of the triangle 2x+2y+2z=6 in the first octant.

4. $\mathbf{F} = \langle yz, xz, xy \rangle$ and C is the boundary of a simple closed curve in the yz=plane.

5. $\mathbf{F} = \langle z - y, x - z, x - y \rangle$ and C is the boundary of a simple closed curve in the x+y+z=5 plane.

6. $\mathbf{F} = \langle x + y^2, 3x, 2z \rangle$ and C is the boundary of the rectangle $R = \{(x,y,z) : 0 \le x \le 3, 1 \le y \le 3, z = 0\}$ oriented in the counterclockwise direction.

7. $\mathbf{F} = \left\langle y^2, 3x + z, 2z + y \right\rangle$ and C is the boundary of the circle $R = \{(x,y,z) : x^2 + y^2 \le 4, z = 0\}$ oriented in the counterclockwise direction.

8. $\mathbf{F} = \left\langle -y^2, x + z, z^2 + y \right\rangle$ and C is the boundary of the circle $R = \{(x,y,z) : x^2 + y^2 \le 9, z = 2\}$ oriented in the counterclockwise direction.

9. $\mathbf{F} = \langle y, -x, z \rangle$ and $S = \{(x,y,z) : x^2 + y^2 + z^2 = 16, 0 \le z\}$ is a hemisphere.

10. $\mathbf{F} = \langle y, -x, z \rangle$ and $S = \{(x,y,z) : x^2 + y^2 + z^2 = 16, 0 \le y\}$ is a hemisphere.

11. $\mathbf{F} = \langle \sin(x), \cos(y), \sin(z) \rangle$ and S is the solid tour with large radius 3 and small radius 1.

12. $\mathbf{F} = \left\langle xy, x^2 + z^2, y^3 \right\rangle$ and S is the solid cube with vertices $x = \pm 1$, $y = \pm 1$, $z = \pm 1$.

In problems 13 to 18 use Stokes' Theorem to determine which of these fields are conservative.

13. $\mathbf{F} = \langle yz, xz, xy \rangle$

14. $\mathbf{F} = \langle yz + a, xz + b, xy + c \rangle$ a, b and c are constants.

15. $\mathbf{F} = \langle yza, xzb, xyc \rangle$ a, b and c are constants.

16. $\mathbf{F} = \langle y + z, x + z, x + y \rangle$

17. $\mathbf{F} = \langle x + z, y + z, x + y \rangle$

18. $\mathbf{F} = \langle yz, x - y, -x \rangle$

For problems 19 to 26 evaluate $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S}$ for the given field $\mathbf{F}$ and surface S.

19. $\mathbf{F} = \langle z - y, x - z, y - x \rangle$ and $S = \{(x,y,z) : x^2 + y^2 = 16, 0 \le z \le 3\}$ is a cylinder.

20. $\mathbf{F} = \langle y, -x, z \rangle$ and $S = \{(x,y,z) : x^2 + z^2 = 9, 1 \le y \le 5\}$ is a cylinder.

21. $\mathbf{F} = \left\langle x, y^2, z^3 \right\rangle$ and $S = \{(x,y,z) : x^2 + y^2 + z^2 = 25, -3 \le y\}$.

22. $\mathbf{F} = \langle \sin(x), \cos(y), \sin(z) \rangle$ and $S = \{(x,y,z) : x^2 + +y^2 + z^2 = 25, -3 \le z \le 4\}$.

23. $F = \langle -y, z, x \rangle$ and $S = \{(x,y,z) : \dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{25} = 1, 0 \le y\}$ a truncated ellipsoid.

24. $F = \langle -y, z, x \rangle$ and $S = \{(x,y,z) : \dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{25} = 1, 0 \le z\}$ a truncated ellipsoid.

25. $\mathbf{F} = \langle y, z, x \rangle$ and $S = \{(x,y,z) : x^2 + y^2 + z = 25,\ 0 \le z \le 16\}$ a truncated paraboloid.

26. $\mathbf{F} = \langle y, z, x \rangle$ and $S = \{(x,y,z) : x^2 + y^2 + z = 25,\ 0 \le z \le 9\}$.

Practice Answers

Practice 1: The field F and the boundary C (a circle in the xy-plane with radius 3) are the same as in Example 1 so $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S} = \int_C \mathbf{F} \bullet d\mathbf{r} = -18\pi$ just as in Example 1.

Practice 2: By Stokes' theorem $\int_C \mathbf{F} \bullet d\mathbf{r} = \iint_S (\nabla \mathrm{x} \mathbf{F}) \bullet \mathbf{n}\ dS$ and the second integral is easier to evaluate then doing 3 line integrals.. On the triangle $z = 2 - \frac{x}{2} - y$ so $\mathbf{r}_u \mathrm{x} \mathbf{r}_v = \left\langle \frac{1}{2}, 1, 1 \right\rangle$

$$\nabla \mathrm{x} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xz & -2yz \end{vmatrix} = \langle -2z - x,\ 0,\ z - x \rangle \ .$$

But x+2y+2z=4 so $\nabla \mathrm{x} \mathbf{F} = \left\langle -4 + 2y,\ 0,\ 2 - \frac{3}{2}x - y \right\rangle$

Then $\iint_S (\nabla \mathrm{x} \mathbf{F}) \bullet \mathbf{n}\ dS = \iint_S \left\langle -4 + 2y,\ 0,\ 2 - \frac{3}{2}x - y \right\rangle \bullet \left\langle \frac{1}{2}, 1, 1 \right\rangle dS = \int_{x=0}^{4} \int_{y=0}^{2-x/2} -\frac{3}{2} x \, dy \, dx$

$$= -\frac{3}{2} \int_{x=0}^{4} -2x + \frac{1}{2}x^2 \ dx = \left(-\frac{3}{2}\right)\left(-\frac{16}{3}\right) = 8 \ .$$

Practice 3: Now $\mathbf{r}(t) = \langle 3 \cdot \cos(t),\ 3 \cdot \sin(t),\ 4 \rangle$ so $\mathbf{r}'(t) = \langle -3 \cdot \sin(t),\ 3 \cdot \cos(t),\ 0 \rangle$ and

$$\int_t \mathbf{F} \bullet \mathbf{r}' \ dr = \int_{t=0}^{2\pi} \langle 4 - 3 \cdot \sin(t),\ 4 \cdot \cos(t),\ -3 \cdot \cos(t) \rangle \bullet \langle -3 \cdot \sin(t),\ 3 \cdot \cos(t),\ 0 \rangle \, dr$$

$$= \int_{t=0}^{2\pi} -12 \cdot \sin(t) + 9 \cdot \sin^2(t) + 9\cos^2(t) \ dt = \int_{t=0}^{2\pi} -12 \cdot \sin(t) + 9 \ dt = 18\pi \ .$$

Practice 4: S is a smooth **closed** surface so $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S} = 0$.

Practice 5: C_1 and $\mathbf{r}_1$ are the same as in Example 5 but now C_2 is parameterized by

$\mathbf{r}_2(t) = \langle 4 \cdot \cos(t),\, 4 \cdot \sin(t),\, 3 \rangle$ for $0 \le t \le 2\pi$ (counterclockwise).

$$\int_{C_2} \mathbf{F} \bullet d\mathbf{r} = \int_0^{2\pi} \left\langle 16 \cdot \sin(t) \cdot \cos(t),\, 4 \cdot \cos(t) + 3,\, 27 \cdot \sin^3(t) \right\rangle \bullet \langle -4 \cdot \sin(t),\, 4 \cdot \cos(t),\, 0 \rangle \, dt$$

$$= \int_0^{2\pi} -64 \cdot \sin^2(t) \cdot \cos(t) + 16 \cdot \cos^2(t) + 12 \cdot \cos(t) + 0 \, dt = 16\pi .$$

Then $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S} = \int_{C_1} \mathbf{F} \bullet d\mathbf{r} - \int_{C_2} \mathbf{F} \bullet d\mathbf{r} = 25\pi - 16\pi = 9\pi$.

Appendix: Proof of Stokes' Theorem for a common special case

Stoke's Theorem: $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S} = \iint_S (\nabla \text{x} \mathbf{F}) \bullet \mathbf{n} \ dS = \int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \mathbf{F} \bullet \mathbf{T} \ dt = \text{circulation around C}$

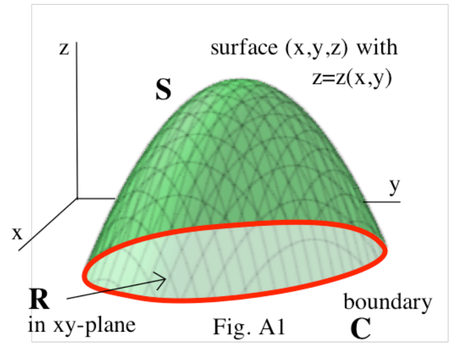


Fig. A1

Common special case: S is a surface of the form (x, y, z(x,y)) (Fig. A1) This requires a lot of calculations and very careful attention to details. Let $\mathbf{F} = \langle P, Q, R \rangle$ and $S = \{(x,y,z) : z = z(x,y) \text{ is a function of x and y}\}$. We will evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$ and $\iint_S (\nabla \text{x} \mathbf{F}) \bullet \mathbf{n} \ dS$ separately and show that they are equal.

$\iint_S (\nabla \text{x} \mathbf{F}) \bullet \mathbf{n} \ dS$: Thinking of S as a parameterized surface (but using x and y instead of u and v) for x and y in the xy-region R (Fig. A2), then $\iint_S (\nabla \text{x} \mathbf{F}) \bullet \mathbf{n} \ dS = \iint_R (\nabla \text{x} \mathbf{F}) \bullet (\mathbf{r}_x \text{x} \mathbf{r}_y) \ dA$. $\mathbf{r}_x = \langle 1, 0, z_x \rangle$, $\mathbf{r}_y = \langle 0, 1, z_y \rangle$ and $\mathbf{r}_x \text{x} \mathbf{r}_y = \langle -z_y, -z_y, 1 \rangle$. Then

$$\iint_S (\nabla \text{x} \mathbf{F}) \bullet \mathbf{n} \ dS = \iint_R \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \bullet \langle -z_y, -z_y, 1 \rangle \ dA \ .$$

$\int_C \mathbf{F} \bullet d\mathbf{r}$: This one is more complicated and requires the Chain Rule for functions of several variables.

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \mathbf{F} \bullet \mathbf{r}' \ dt = \int_C \langle P, Q, R \rangle \bullet \langle dx, dy, dz \rangle \ dt = \int_C P \cdot dx + Q \cdot dy + R \cdot dz \ .$$

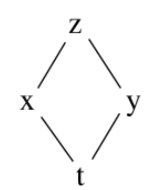


Fig. A2

But z=z(x,y) so (Fig. A2) $dz = z_x \cdot dx + z_y \cdot dy$, and

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C (P + R \cdot z_x) dx + (Q + R \cdot z_y) \cdot dy = \int_C M \, dx + N \, dy \cdot$$

By Green's Theorem $\int_C M \, dx + N \, dy = \iint_R N_x - M_y \, dA$ with $M = P + R \cdot z_x$ and $N = Q + R \cdot z_y$.

$$M_y = \frac{\partial}{\partial y}(P + R \cdot z_x) = P_y + P_z \cdot z_y + z_x \cdot R_y + R \cdot \frac{\partial}{\partial y} z_x$$

$$= P_y + P_z \cdot z_y + R \cdot z_{xy} + z_x \cdot (R_y + R_z \cdot z_y) \ .$$

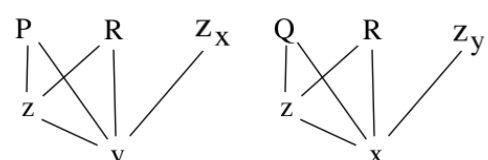


Fig. A3

Similarly, $N_x = \frac{\partial}{\partial x}(Q + R \cdot z_y) = Q_x + Q_z \cdot z_x + R \cdot z_{xy} + z_y \cdot (R_x + R_z \cdot z_x)$.

Then, after substituting and simplifying,

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \iint_R N_x - M_y \, dA = \iint_R z_x(Q_z - R_y) + z_y(R_x - P_z) \; + \; (Q_x - P_y) \; dA$$

$$= \iint_R \left\langle R_y - Q_z, P_z - R_x, Q_x - P_y \right\rangle \bullet \left\langle -z_y, \; -z_y, \; 1 \right\rangle dA = \iint_S (\nabla \mathbf{x} \mathbf{F}) \bullet \mathbf{n} \; dS$$

so Stokes' Theorem is also true for surfaces of the form $S = \{(x,y,z): \; z = z(x,y)\}$.

Suppose we cut a hole in the surface S and attach a smooth "bump" to cover the hole (Fig. A4). Is Stokes' Theorem still true for this new surface consisting of the old S minus the hole plus the bump? You should be able to justify your answer.

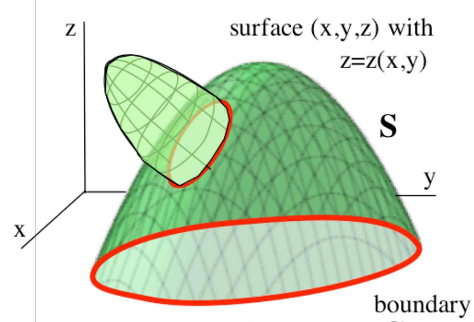


Fig. A4