11.2 Calculus in Polar Coordinates

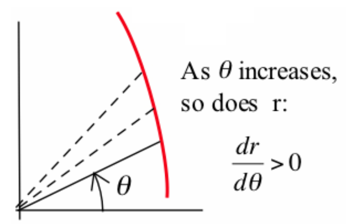
The previous section introduced the polar coordinate system and discussed how to plot points, how to create graphs of functions (from data, a rectangular graph or a formula) and how to convert back and forth between the polar and rectangular systems. This section examines calculus in polar coordinates: rates of change, slopes of tangent lines, areas and lengths of curves.

The results we obtain may appear different than the corresponding results from earlier chapters, but they all follow from the approaches used in the rectangular coordinate system.

Polar Coordinates and Derivatives

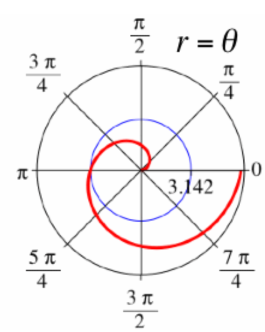
In the rectangular coordinate system, the derivative $\frac{dy}{dx}$ measured both the rate of change of y with respect to x for a function $y = f(x)$ and the slope of the tangent line to the graph of $y = f(x)$. In the polar coordinate system other derivatives also commonly appear, and it is important that you learn to distinguish among them. If $r = g(\theta)$ then:

The sign of $\frac{dr}{d\theta}$ tells us whether r is increasing or decreasing as θ increases.



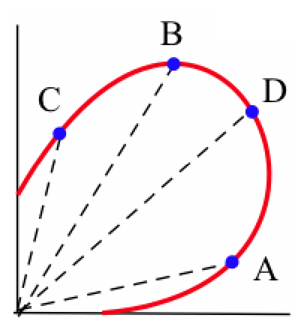
- $\frac{dr}{d\theta} = g'(\theta)$ measures the rate of change of r with respect to θ
- $\frac{dy}{dx}$ gives the slope $\frac{\Delta y}{\Delta x}$ of the tangent line to the graph of $r = g(\theta)$

The derivative of a polar equation $r = g(\theta)$, $\frac{dr}{d\theta} = g'(\theta)$, tells us how r is changing with respect to (increasing) θ. For example, if $\frac{dr}{d\theta} > 0$ then the directed distance r is increasing as θ increases (see margin). However, $\frac{dr}{d\theta} = g'(\theta)$ is *not* the slope of the line tangent to the polar graph of $r = g(\theta)$. For the simple spiral $r = \theta$ (see second margin figure), $\frac{dr}{d\theta} = 1 > 0$ for all values of θ, but the slope of the tangent line, $\frac{dy}{dx}$, is sometimes positive and sometimes negative.



Similarly, $\frac{dx}{d\theta}$ tells us the rate of change of the x-coordinate of the graph with respect to (increasing) θ and $\frac{dy}{d\theta}$ tells us the rate of change of the y-coordinate of the graph with respect to (increasing) θ.

Example 1. State whether the values of $\frac{dx}{d\theta}$, $\frac{dy}{d\theta}$, $\frac{dr}{d\theta}$ and $\frac{dy}{dx}$ are positive (+), negative (−), zero (0) or undefined (U) at the points A and B on the graph in the bottom margin figure.



Solution. As θ increases near A, the x- and y-coordinates of the point on the graph are both increasing, the radius r (the distance from the point to the origin) is increasing, and the slope of the line tangent to the graph is positive, so all four derivatives are positive.

As θ increases near B: the x-coordinate is decreasing, so $\frac{dx}{d\theta} < 0$; the y-coordinate reaches a maximum, so $\frac{dy}{d\theta} = 0$; the radius r is getting

	$\frac{dx}{d\theta}$	$\frac{dy}{d\theta}$	$\frac{dr}{d\theta}$	$\frac{dy}{dx}$
A	+	+	+	+
B	–	0	–	0
C				
D				

smaller, so $\frac{dr}{d\theta} < 0$; and the tangent line is horizontal, so $\frac{dy}{dx} = 0$. We can collect these results in the margin table. ◀

Practice 1. Fill in the margin table for the points labeled C and D.

Slopes of Tangent Lines

If you know that $r = f(\theta)$ for some differentiable function f, you can calculate $\frac{dy}{dx}$, the slope of the tangent line to the graph of $r = f(\theta)$, by using the polar–rectangular conversion formulas and the Chain Rule:

$$\frac{dy}{d\theta} = \frac{dy}{dx} \cdot \frac{dx}{d\theta} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

so to find the slope of the tangent line we need to compute $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$. From the polar–rectangular conversion formulas, we know that:

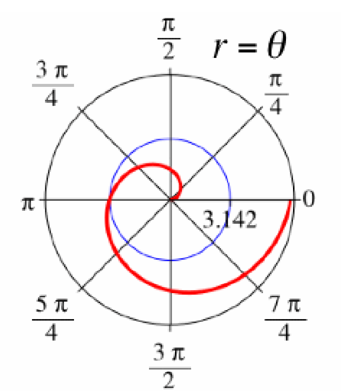
$$x = r\cos(\theta) = f(\theta) \cdot \cos(\theta) \Rightarrow \frac{dx}{d\theta} = -f(\theta) \cdot \sin(\theta) + f'(\theta) \cdot \cos(\theta)$$

$$y = r\sin(\theta) = f(\theta) \cdot \sin(\theta) \Rightarrow \frac{dy}{d\theta} = f(\theta) \cdot \cos(\theta) + f'(\theta) \cdot \sin(\theta)$$

and hence:

$$\frac{dy}{dx} = \frac{f(\theta) \cdot \cos(\theta) + f'(\theta) \cdot \sin(\theta)}{-f(\theta) \cdot \sin(\theta) + f'(\theta) \cdot \cos(\theta)}$$

This result may be difficult to memorize, but you should be able to remember how to obtain the result using the conversion formulas, the Product Rule and the Chain Rule.



Example 2. Find the slopes of the lines tangent to the spiral $r = \theta$ at the points $P\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $Q(\pi, \pi)$.

Solution. Proceeding as above:

$$x = \theta \cdot \cos(\theta) \quad \Rightarrow \quad \frac{dx}{d\theta} = -\theta \cdot \sin(\theta) + \cos(\theta)$$

$$y = \theta \cdot \sin(\theta) \quad \Rightarrow \quad \frac{dy}{d\theta} = \theta \cdot \cos(\theta) + \sin(\theta)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\theta \cdot \cos(\theta) + \sin(\theta)}{-\theta \cdot \sin(\theta) + \cos(\theta)}$$

At the point P, $\theta = \frac{\pi}{2}$, so:

$$\frac{dy}{dx} = \frac{\frac{\pi}{2} \cdot 0 + 1}{-\frac{\pi}{2} \cdot 1 + 0} = -\frac{2}{\pi}$$

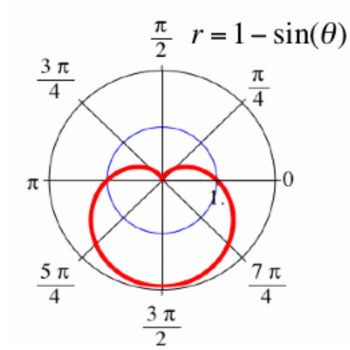
At the point Q, $\theta = \pi$, so:

$$\frac{dy}{dx} = \frac{\pi \cdot (-1) + 0}{-\pi \cdot 0 - 1} = \pi$$

The function $r = \theta$ is steadily increasing, but the slope of the line tangent to the polar graph can be negative or positive or 0 or even undefined (where?). ◀

Practice 2. Find equations for the lines tangent to the graph of $r = \theta$ at points P and Q in the preceding Example.

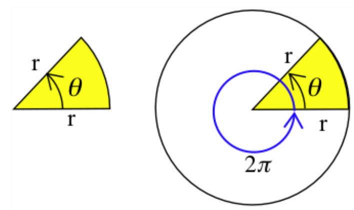
Practice 3. Compute the slopes of the lines tangent to the cardioid $r = 1 - \sin(\theta)$ (see margin for graph) when $\theta = 0$, $\frac{\pi}{4}$ and $\frac{\pi}{2}$.



Areas in Polar Coordinates

The formulas for computing areas in rectangular and polar coordinates may appear quite different, but we obtain them the same way: partition a region into pieces, compute (approximate) areas of those pieces, add the small areas together to get a Riemann sum, and take the limit of that sum to get a definite integral. The chief difference here is the shape of the pieces: we use thin, almost-rectangular pieces in the Cartesian system and thin almost-sectors (pieces of pie) in the polar system.

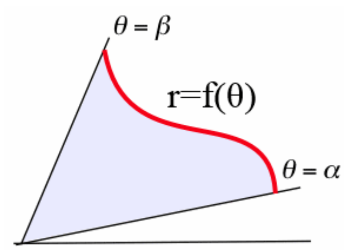
We can obtain the formula for the area of a sector of a circle using proportions (see margin):



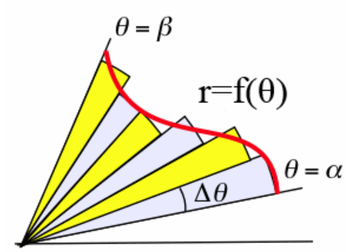
$$\frac{\text{area of sector}}{\text{area of whole circle}} = \frac{\text{sector angle}}{\text{angle of whole circle}} = \frac{\theta}{2\pi}$$

$$\Rightarrow \text{ area of sector} = \frac{\theta}{2\pi}(\text{area of whole circle}) = \frac{\theta}{2\pi}\left(\pi r^2\right) = \frac{1}{2}r^2\theta$$

Given a region bounded by the polar curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$, partition the θ-domain into n small pieces of angular "width" $\Delta\theta$. For the k-th polar "slice", choose an angle θ_k in that slice and approximate the area of the k-th slice with a sector of radius $f(\theta_k)$ and angle $\Delta\theta$. The area of this sector is $\frac{1}{2}\left[f(\theta_k)\right]^2 \Delta\theta$, so the approximate area of the region is given by the Riemann sum:



$$\sum_{k=1}^{n} \frac{1}{2}\left[f(\theta_k)\right]^2 \Delta\theta \quad \longrightarrow \quad \int_{\alpha}^{\beta} \frac{1}{2}\left[f(\theta)\right]^2 \, d\theta$$

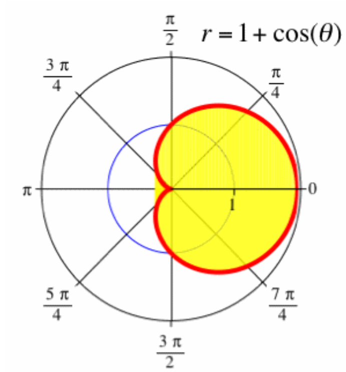


We can guarantee the convergence of the Riemann sum to the integral by requiring that $f(\theta)$ is continuous for $\alpha \le \theta \le \beta$.

Area in Polar Coordinates

If $f(\theta)$ is continuous on $[\alpha, \beta]$, the area of the region bounded by $r = f(\theta)$ and radial lines at angles $\theta = \alpha$ and $\theta = \beta$ is given by:

$$\int_{\alpha}^{\beta} \frac{1}{2}\left[f(\theta)\right]^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

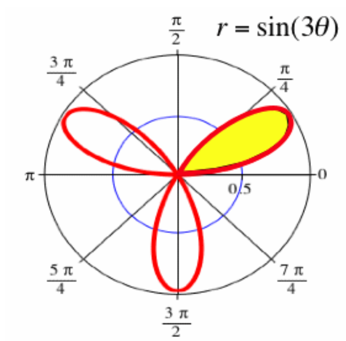


Example 3. Find the area of the region inside the cardioid $r = 1 + \cos(\theta)$ (see margin for graph).

Solution. A straightforward application of the area formula yields:

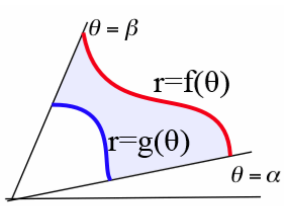
$$\int_0^{2\pi} \frac{1}{2}\left[1 + \cos(\theta)\right]^2\, d\theta = \int_0^{2\pi} \left[\frac{1}{2} + \cos(\theta) + \frac{1}{2}\cos^2(\theta)\right] d\theta$$
$$= \int_0^{2\pi} \left[\frac{1}{2} + \cos(\theta) + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)\right] d\theta$$
$$= \int_0^{2\pi} \left[\frac{3}{4} + \cos(\theta) + \frac{1}{4}\cos(2\theta)\right] d\theta$$
$$= \left[\frac{3}{4}\theta + \sin(\theta) + \frac{1}{8}\sin(2\theta)\right]_0^{2\pi} = \frac{3\pi}{2}$$

We could also have exploited the symmetry of the region, integrating instead from 0 to π (to get the area of the "top half" of the region) and then doubling the result. ◀



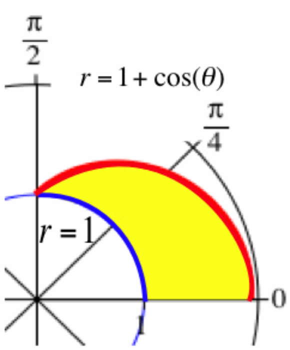
Practice 4. Find the area of the region inside one "petal" of the rose $r = \sin(3\theta)$ (see margin for graph).

We can also calculate the area *between* curves in polar coordinates. The area of the region (see margin) between the continuous curves $r = f(\theta)$ and $r = g(\theta)$ for $\alpha \le \theta \le \beta$, if $f(\theta) > g(\theta)$ on this interval, is:



$$\int_\alpha^\beta \frac{1}{2}\left[(f(\theta))^2 - (g(\theta))^2\right] d\theta = \int_\alpha^\beta \frac{1}{2}\left[r_{\text{outer}}^2 - r_{\text{inner}}^2\right] d\theta$$

It is a good idea to sketch the graphs of the curves to help determine the endpoints of integration.



Example 4. Find the area of the shaded region in the margin figure.

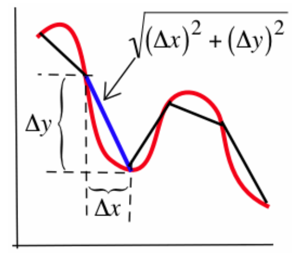
Solution. The shaded region lies in the first quadrant, so $\alpha = 0$ and $\beta = \frac{\pi}{2}$. On that interval, $\cos(\theta) \ge 0 \Rightarrow 1 + \cos(\theta) \ge 1$, so $f(\theta) = 1 + \cos(\theta)$ generates the outer curve and $g(\theta) = 1$ generates the inner curve. The area of the region between these curves is therefore:

$$\int_0^{\frac{\pi}{2}} \frac{1}{2}\left[(1 + \cos(\theta))^2 - (1)^2\right] d\theta = \int_0^{\frac{\pi}{2}} \left[\cos(\theta) + \frac{1}{2}\cos^2(\theta)\right] d\theta$$
$$= \int_0^{\frac{\pi}{2}} \left[\cos(\theta) + \frac{1}{4} + \frac{1}{4}\cos(2\theta)\right] d\theta = \left[\sin(\theta) + \frac{1}{4}\theta + \frac{1}{8}\sin(2\theta)\right]_0^{\frac{\pi}{2}}$$

which evaluates to $1 + \frac{\pi}{8} \approx 1.393$. ◀

Practice 5. Find the area of the region outside the cardioid $1 + \cos(\theta)$ and inside the circle $r = 2$.

Arclength in Polar Coordinates



The formulas for calculating the lengths of curves in rectangular and polar coordinates look a bit different, but we can obtain both from the Pythagorean Theorem, following the method we used in Section 5.3 (see margin figure):

$$\text{length} \approx \sum_{k=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sum_{k=1}^{n} \sqrt{\left(\frac{\Delta x}{\Delta \theta}\right)^2 + \left(\frac{\Delta y}{\Delta \theta}\right)^2} \cdot \Delta\theta$$

If $r = f(\theta)$, a differentiable function of θ, then $x = f(\theta) \cdot \cos(\theta)$ and $y = f(\theta) \cdot \sin(\theta)$ will both be differentiable functions of θ so that $\frac{\Delta x}{\Delta \theta} \to \frac{dx}{d\theta}$ and $\frac{\Delta y}{\Delta \theta} \to \frac{dy}{d\theta}$ as $\Delta\theta \to 0$. Hence:

$$\sum_{k=1}^{n} \sqrt{\left(\frac{\Delta x}{\Delta \theta}\right)^2 + \left(\frac{\Delta y}{\Delta \theta}\right)^2} \cdot \Delta\theta \longrightarrow \int_{\theta=\alpha}^{\theta=\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Building on results from our earlier derivative computations:

$$\frac{dx}{d\theta} = -f(\theta)\sin(\theta) + f'(\theta)\cos(\theta) \Rightarrow \left(\frac{dx}{d\theta}\right)^2 = [f(\theta)]^2 \sin^2(\theta) - 2f(\theta)f'(\theta)\sin(\theta)\cos(\theta) + \left[f'(\theta)\right]^2 \cos^2(\theta)$$

$$\frac{dy}{d\theta} = f(\theta)\cos(\theta) + f'(\theta)\sin(\theta) \Rightarrow \left(\frac{dy}{d\theta}\right)^2 = [f(\theta)]^2 \cos^2(\theta) + 2f(\theta)f'(\theta)\sin(\theta)\cos(\theta) + \left[f'(\theta)\right]^2 \sin^2(\theta)$$

Adding these quantities and applying a square root yields:

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}$$

providing a compact formula for arclength in polar coordinates.

Arclength in Polar Coordinates
If $r = f(\theta)$, a differentiable function for $\alpha \le \theta \le \beta$, then the length of the graph of $r = f(\theta)$ between $\theta = \alpha$ and $\theta = \beta$ is:

$$\int_{\theta=\alpha}^{\theta=\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2}\, d\theta$$

Example 5. Find the length of the polar curve $r = \sqrt{\theta}$ for $\pi \le \theta \le 2\pi$.

Solution. $r = \sqrt{\theta} \Rightarrow \frac{dr}{d\theta} = \frac{1}{2\sqrt{\theta}}$ so the length is given by:

$$\int_{\pi}^{2\pi} \sqrt{\left(\sqrt{\theta}\right)^2 + \left(\frac{1}{2\sqrt{\theta}}\right)^2}\, d\theta = \int_{\pi}^{2\pi} \sqrt{\theta + \frac{1}{4\theta}}\, d\theta \approx 6.8287$$

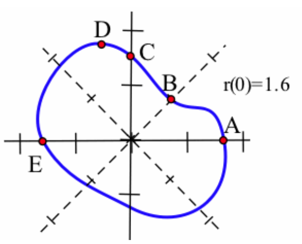
Like most integrals arising from arclength computations, we are unable find an antiderivative of the integrand and compute an exact value for the length, so we resort to technology to provide an approximate numerical answer. ◀

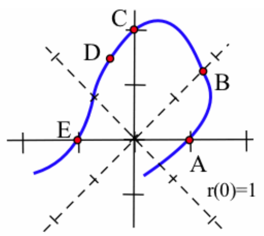
11.2 Problems

In problems 1–4, fill in the table below to indicate whether the values of the indicated derivatives are are positive (+), negative (−), zero (0) or undefined (U) at each point.

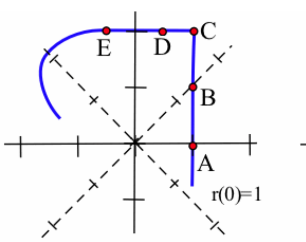
	$\frac{dx}{d\theta}$	$\frac{dy}{d\theta}$	$\frac{dr}{d\theta}$	$\frac{dy}{dx}$
A				
B				
C				
D				
E				

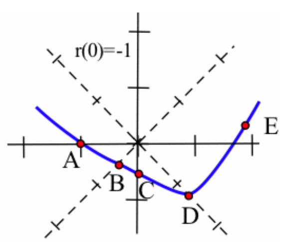
1. See figure below left.
2. See figure below.





3. See figure below left.
4. See figure below.





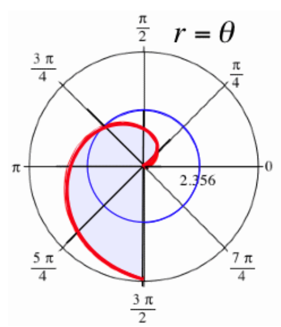
In Problems 5–8, sketch the graph of the given polar equation for $0 \le \theta \le 2\pi$; label the points A, B and C; and calculate the values of $\frac{dr}{d\theta}$ and $\frac{dy}{dx}$ at each of those points.

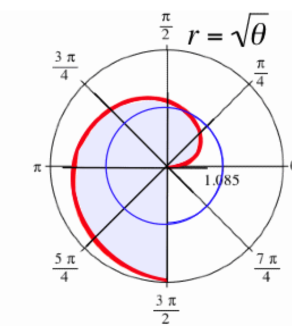
5. $r = 5$; $A\left(5, \frac{\pi}{4}\right)$, $B\left(5, \frac{\pi}{2}\right)$ and $C\,(5, \pi)$
6. $r = 2 + \cos(\theta)$; $A\left(2 + \sqrt{2}, \frac{\pi}{4}\right)$, $B\left(2, \frac{\pi}{2}\right)$ and $C\,(1, \pi)$
7. $r = 1 + \cos^2(\theta)$; $A\,(2, 0)$, $B\left(\frac{3}{2}, \frac{\pi}{4}\right)$ and $C\left(1, \frac{\pi}{2}\right)$
8. $r = \dfrac{6}{2 + \cos(\theta)}$; $A\,(2, 0)$, $B\left(3, \frac{\pi}{2}\right)$ and $C\left(\frac{24 - 6\sqrt{2}}{7}, \frac{\pi}{4}\right)$

9. Graph $r = 1 + 2\cos(\theta)$ for $0 \le \theta \le 2\pi$.
   (a) Show that the graph goes through the origin when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.
   (b) Calculate $\dfrac{dy}{dx}$ when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.
   (c) How can a curve have two different tangent lines (and slopes) at the origin?
10. Graph the cardiod $r = 1 + \sin(\theta)$ for $0 \le \theta \le 2\pi$.
    (a) At what points on the cardioid does $\frac{dx}{d\theta} = 0$?
    (b) At what points does $\frac{dy}{d\theta} = 0$?
    (c) At what points does $\frac{dr}{d\theta} = 0$?
    (d) At what points does $\frac{dy}{dx} = 0$?
11. Show that if a polar graph goes through the origin when the angle is θ_0 (and if $\frac{dr}{d\theta}$ exists there, but is not equal to 0) then the slope of the tangent line at the origin is $\tan(\theta_0)$.

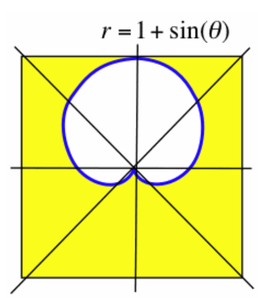
In 12–20, represent the area of the given region as a definite integral. Then evaluate the integral exactly (if possible) or approximate using technology.

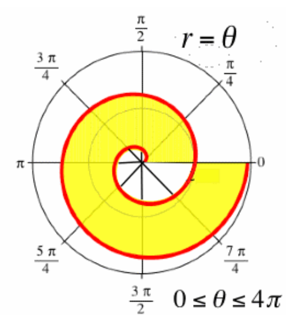
12. The shaded region in the figure below left.





13. The shaded region in the figure above right.
14. The shaded region in the figure below left.



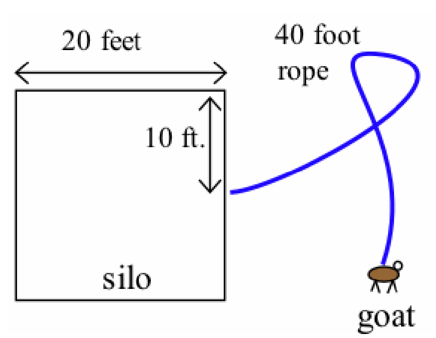


15. The shaded region in the figure above right.

16. The region inside the circle $r = 4\sin(\theta)$.
17. The region in the first quadrant outside the circle $r = 1$ and inside the cardiod $r = 1 + \cos(\theta)$.
18. The region in the second quadrant bounded by $r = \theta$ and $r = \theta^2$.
19. One "petal" of the graph of $r = \sin(3\theta)$.
20. One "petal" of the graph of $r = \sin(5\theta)$.
21. The "peanut" $r = 1.5 + \cos(2\theta)$.
22. The "peanut" $r = a + \cos(2\theta)$ (for $a > 1$).

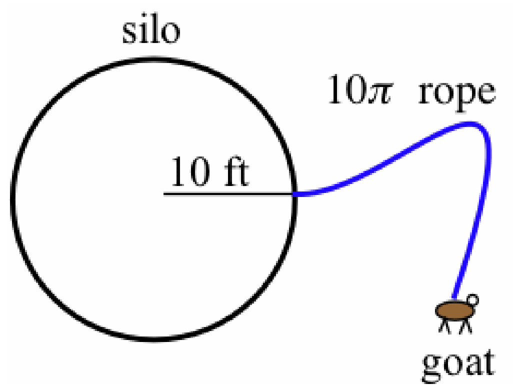
In 23–30, represent the length of the curve as a definite integral. Then evaluate the integral exactly (if possible) or approximate using technology.

23. The spiral $r = \theta$ from $\theta = 0$ to $\theta = 2\pi$.
24. The spiral $r = \theta$ from $\theta = 2\pi$ to $\theta = 4\pi$.
25. The cardioid $r = 1 + \cos(\theta)$.
26. The circle $r = 4\sin(\theta)$ from $\theta = 0$ to $\theta = \pi$.
27. The circle $r = 5$ from $\theta = 0$ to $\theta = 2\pi$.
28. The "peanut" $r = 1.2 + \cos(2\theta)$.
29. One "petal" of $r = \sin(3\theta)$
30. One "petal" of $r = \sin(5\theta)$.
31. **Goat and Square Silo** (This problem does not require calculus.) One end of a 40-foot-long rope is attached to the middle of a wall of a 20-foot-square silo, and the other end is tied to a goat.



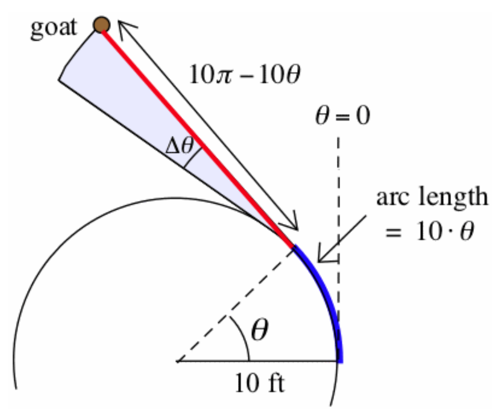
(a) Sketch the region that the goat can reach.
(b) Find the area of the region the goat can reach.
(c) Can the goat reach a region with a bigger area if the rope is tied to the corner of the silo?

32. **Goat and Round Silo** (This problem does require calculus.) One end of a 10π-foot-long rope is attached to the wall of a round silo that has a radius of 10 feet, and the other end is tied to a goat.



(a) Sketch the region that the goat can reach.
(b) Justify that the area of the shaded region shown below, as the goat goes around the silo from having θ feet of rope taut against the silo to having $\theta + \Delta\theta$ feet taut against the silo, is approximately:

$$\frac{1}{2}(10\pi - 10\theta)^2 \cdot \Delta\theta$$



(c) Use the preceding result to help calculate the area of the region that the goat can reach.

11.2 Practice Answers

	$\frac{dx}{d\theta}$	$\frac{dy}{d\theta}$	$\frac{dr}{d\theta}$	$\frac{dy}{dx}$
C	$-$	$-$	$-$	$+$
D	$-$	$+$	$+$	$-$

1. See margin table.

2. At P, $\theta = \frac{\pi}{2}$ and $r = \frac{\pi}{2}$, so $x = r\cos(\theta) = \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) = 0$ and $y = r\sin(\theta) = \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$. From Example 2, we know that $\frac{dy}{dx} = -\frac{2}{\pi}$ at P, so an equation for the tangent line is $y = \frac{\pi}{2} - \frac{2}{\pi}(x-0)$.

   At Q, $\theta = \pi$ and $r = \pi$, so $x = r\cos(\theta) = \pi\cos(\pi) = -\pi$ and $y = r\sin(\theta) = \pi\sin(\pi) = 0$. From Example 2, we know that $\frac{dy}{dx} = \pi$ at Q, so an equation for the tangent line is $y = 0 + \pi(x+\pi)$.

3. With $r = 1-\sin(\theta)$, $x = r\cos(\theta) = (1-\sin(\theta))\cos(\theta)$, so:

$$\begin{aligned}\frac{dx}{d\theta} &= (1-\sin(\theta))(-\sin(\theta)) + \cos(\theta)(-\cos(\theta)) \\ &= -\sin(\theta) + \sin^2(\theta) - \cos^2(\theta) = -\sin(\theta) - \cos(2\theta)\end{aligned}$$

   Similarly, $y = r\sin(\theta) = (1-\sin(\theta))\sin(\theta)$, so:

$$\begin{aligned}\frac{dy}{d\theta} &= (1-\sin(\theta))\cdot\cos(\theta) + \sin(\theta)(-\cos(\theta)) \\ &= \cos(\theta) - 2\sin(\theta)\cos(\theta) = \cos(\theta) - \sin(2\theta)\end{aligned}$$

   Therefore:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos(\theta) - \sin(2\theta)}{-\sin(\theta) - \cos(2\theta)}$$

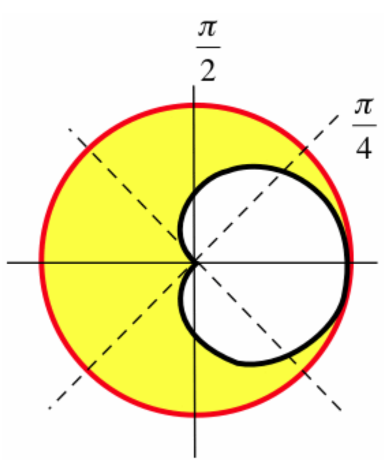
   When $\theta = 0$, $\frac{dy}{dx} = -1$; when $\theta = \frac{\pi}{4}$, $\frac{dy}{dx} = \sqrt{2}-1 \approx 0.414$; and when $\theta = \frac{\pi}{2}$, the derivative is undefined.

Looking at the graph of the cardioid, why should the slope be undefined there?

4. The "petals" of the rose $r = \sin(3\theta)$ intersect at the origin, where $r = 0 \Rightarrow \sin(3\theta) = 0 \Rightarrow 3\theta = k\pi \Rightarrow \theta = k\cdot\frac{\pi}{3}$ for any integer k. The shaded petal corresponds to $0 \le \theta \le \frac{\pi}{3}$, so its area is:

$$\int_0^{\frac{\pi}{3}} \frac{1}{2}\sin^2(3\theta)\,d\theta = \int_0^{\frac{\pi}{3}} \left[\frac{1}{4} - \frac{1}{4}\cos(6\theta)\right]\,d\theta = \left[\frac{1}{4}\theta - \frac{1}{24}\sin(6\theta)\right]_0^{\frac{\pi}{3}}$$

   which equals $\frac{\pi}{12} \approx 0.2618$.



5. See margin for graph. The area of the region enclosed by the circle is $\pi\cdot 2^2 = 4\pi$, while the area of the region enclosed by the cardioid is $\frac{3\pi}{2}$ (using the result of Example 3). The shaded region therefore has area $4\pi - \frac{3\pi}{2} = \frac{5\pi}{2}$.