14.6 The Chain Rule(s)

You first encountered the Chain Rule in Section 2.4 and have used it thousands of times since then. If $y = F(x)$ is a differentiable function of x and $x = g(t)$ is a differentiable function of t, then the composite function $\varphi(t) = F(g(t))$ is also differentiable and:

You might remember this pattern as "derivative of the outside function, evaluated at the inside function, times the derivative of the inside function."

$$\varphi'(t) = F'(g(t)) \cdot g'(t)$$

Often the Leibniz version of the Chain Rule comes in handy:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

We now investigate compositions involving multivariable functions.

The Chain Rule for Gradients

In Section 14.5 we stated the Chain Rule for Gradients, which involves the composition of a function of one variable (on the "outside") with a multivariable function (on the "inside").

Example 1. If $F(t) = \sqrt{t}$ and $g(x,y,z) = x^2 + y^2 + z^2$, let $f(x,y,z) = F(g(x,y,z))$ and compute $\nabla f(x,y,z)$.

Solution. We can write $f(x,y,z) = F(g(x,y,z))$ as:

$$f(x,y,z) = F(x^2+y^2+z^2) = \sqrt{x^2+y^2+z^2} = \left(x^2+y^2+z^2\right)^{\frac{1}{2}}$$

and then compute the partial derivative of f with respect to x:

$$f_x(x,y,z) = \frac{1}{2}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$f_y(x,y,z) = \frac{y}{\sqrt{x^2+y^2+z^2}}$$
$$f_z(x,y,z) = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

The other partial derivatives are quite similar (see margin), so:

$$\nabla f(x,y,z) = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$$

which we can rewrite as: $\nabla f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}} \cdot \langle x,y,z \rangle$. ◀

You might have noticed in the previous Example that $F'(t) = \dfrac{1}{2\sqrt{t}}$ and that $\nabla g(x,y,z) = \langle 2x, 2y, 2z \rangle$ so that:

$$\begin{aligned} \nabla\left[F(g(x,y,z)\right] = \nabla f(x,y,z) &= \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot \langle x,y,z \rangle \\ &= \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot \langle 2x,2y,2z \rangle \\ &= F'(g(x,y,z)) \cdot \nabla g(x,y,z) \end{aligned}$$

hence $\nabla\left[F(g(\mathbf{x})\right] = F'(g(\mathbf{x})) \cdot \nabla g(\mathbf{x})$, a result you were asked to prove in Problem 42 of Section 14.5.

The proof involves applying the one-variable Chain Rule to compute each partial derivative of $F(g(\mathbf{x}))$.

Chain Rule for Gradients: If $F(t)$ is differentiable at $t = g(\mathbf{a})$ and $g(\mathbf{x})$ is differentiable at $\mathbf{x} = \mathbf{a}$, then:

$$\nabla\left[F(g(\mathbf{a})\right] = F'(g(\mathbf{a})) \cdot \nabla g(\mathbf{a})$$

Similar to the one-variable Chain Rule, the Chain Rule for Gradients says that the gradient of the composition $F(g(\mathbf{x}))$ is "the derivative of the outside function, evaluated at the inside function, times the gradient of the inside function"; here "times" indicates a dot product.

Example 2. If $f(x,y) = \sin\left(x^3y + 7xy^2\right)$, compute $\nabla f(x,y)$.

Solution. Write $F(t) = \sin(t)$ and $g(x,y) = x^3y + 7xy^2$ so that $f(x,y) = F(g(x,y))$. Then $F'(t) = \cos(t)$ and $\nabla g(x,y) = \left\langle 3x^2y + 7y^2, x^3 + 14xy\right\rangle$, hence the Chain Rule for Gradients says:

$$\begin{aligned}\nabla f(x,y) &= F'(g(x,y)) \cdot \nabla g(x,y) \\ &= \cos\left(x^3y + 7xy^2\right) \cdot \left\langle 3x^2y + 7y^2, x^3 + 14xy\right\rangle\end{aligned}$$

You should check that you get the same answer by computing f_x and f_y directly (without using the Chain Rule for Gradients). ◀

Practice 1. If $\varphi(x,y,z) = \arcsin(3x + 4y - 7z)$, compute $\nabla\varphi(x,y,z)$.

The Chain Rule for Paths

Imagine a bug crawling along the graph of $z = f(x,y) = 100 + x^3y^2$ in such a way that the path of the bug on the "map" of the surface in the xy-plane traces out an ellipsoid with $x(t) = 3\cos(t)$ and $y = 2\sin(t)$. The actual path of the bug in $\mathbb{R}^3$ along with surface is a 3D curve with $x(t) = 3\cos(t)$, $y(t) = 2\sin(t)$ and $z(t)$ given by:

$$f(x(t), y(t)) = 100 + (x(t))^3(y(t))^2 = 100 + (3\cos(t))^3(2\sin(t))^2$$

The function $z(t)$ gives the height of the bug above the xy-plane at time t. If we want to know the rate of change of the bug's height with respect to time, we can differentiate $z(t)$:

$$z'(t) = 3(3\cos(t))^2(2\sin(t))^2[-3\sin(t)] + 2(3\cos(t))^3(2\sin(t))[\cos(t)]$$

But $z(t)$ is the composition of $f(x,y) = 100 + x^3y^2$ and the functions $x(t) = 3\cos(t)$ and $y = 2\sin(t)$. Noting that $\nabla f(x,y) = \left\langle 3x^2y^2, 2x^3y\right\rangle$, $x'(t) = -3\sin(t)$ and $y'(t) = 2\cos(t)$, we can rewrite $z'(t)$ as:

$$\begin{aligned}z'(t) &= \left\langle 3[x(t)]^2[y(t)]^2, 2[x(t)]^3 \cdot y(t)\right\rangle \cdot \left\langle x'(t), y'(t)\right\rangle \\ &= \nabla f(x(t), y(t)) \cdot \left\langle x'(t), y'(t)\right\rangle\end{aligned}$$

This could be a coincidence, but in fact the pattern holds true in general.

Chain Rule for Paths: If $f(\mathbf{x})$ is differentiable at $\mathbf{x} = \mathbf{r}(a)$ and $\mathbf{r}(t)$ is differentiable at $t = a$, then:

$$\varphi(t) = f(\mathbf{r}(t)) \quad \Rightarrow \quad \varphi'(a) = \nabla f(\mathbf{r}(a)) \cdot \mathbf{r}'(a)$$

Similar to the other Chain Rules, the Chain Rule for Paths says the derivative of the composition $f(\mathbf{r}(t))$ is "the gradient of the outside function, evaluated at the inside function, times the derivative of the inside function."

Proof. To simplify the proof, assume that $\mathbf{r}(t) = \langle u(t), v(t) \rangle$ is differentiable at $t = a$ and that $f(x, y)$ is C^1 on an open disk $\mathcal{D}$ centered at $(u(a), v(a))$. Using the definition of the derivative:

Assuming f is C^1 (rather than merely differentiable) avoids some messy algebra involving the definition of differentiability. Generalizing the proof so that $\mathbf{r}(t)$ can be a curve in $\mathbb{R}^n$ simply requires n applications of the Mean Value Theorem (rather than two) later in the proof.

$$\begin{aligned}\varphi'(a) &= \lim_{h \to 0} \frac{\varphi(a+h) - \varphi(a)}{h} \\ &= \lim_{h \to 0} \frac{f\left(u(a+h), v(a+h)\right) - f\left(u(a), v(a)\right)}{h}\end{aligned}$$

By "adding 0," we can rewrite the numerator as:

$$\begin{aligned}&f\left(u(a+h), v(a+h)\right) - f\left(u(a), v(a+h)\right) \\ &\qquad + f\left(u(a), v(a+h)\right) - f\left(u(a), v(a)\right)\end{aligned}$$

Because $h \to 0$, we can assume h is small enough that $(u(a+h), v(a+h))$ is in $\mathcal{D}$.

Applying the Mean Value Theorem to the first two terms:

$$\begin{aligned}&f\left(u(a+h), v(a+h)\right) - f\left(u(a), v(a+h)\right) \\ &\qquad = f_x\left(c(h), v(a+h)\right) \cdot [u(a+h) - u(a)]\end{aligned}$$

for some number $c(h)$ with $u(a) < c(h) < u(a+h)$. Similarly, there is a number $\gamma(h)$ with $v(a) < \gamma(h) < v(a+h)$ so that:

$$f\left(u(a), v(a+h)\right) - f\left(u(a), v(a)\right) = f_y\left(u(a), \gamma(h)\right) \cdot [v(a+h) - v(a)]$$

We know f_x and f_y are continuous on $\mathcal{D}$ because we assumed f was C^1 there. We know u and v are continuous at $t = a$ because they are differentiable there.

Because f_x, f_y, u and v are continuous:

$$\begin{aligned}\lim_{h \to 0} f_x\left(c(h), v(a+h)\right) \cdot \frac{u(a+h) - u(a)}{h} &= f_x\left(u(a), v(a)\right) \cdot u'(a) \\ \lim_{h \to 0} f_y\left(u(a), \gamma(h)\right) \cdot \frac{v(a+h) - v(a)}{h} &= f_y\left(u(a), v(a)\right) \cdot v'(a)\end{aligned}$$

so $\varphi'(a) = \nabla f\left(u(a), v(a)\right) \cdot \langle u'(a), v'(a) \rangle$, as required. □

Example 3. Find the rate of change of $f(x, y) = xy$ with respect to t along the path $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ when $t = \frac{\pi}{3}$.

Solution. $\mathbf{r}'(t) = \langle -\sin(t), \cos(t) \rangle \Rightarrow \mathbf{r}'\left(\frac{\pi}{3}\right) = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ and $\nabla f(x, y) = \langle y, x \rangle \Rightarrow \nabla f\left(\mathbf{r}\left(\frac{\pi}{3}\right)\right) = \nabla f\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$, so:

$$\nabla f\left(\mathbf{r}\left(\frac{\pi}{3}\right)\right) \cdot \mathbf{r}'\left(\frac{\pi}{3}\right) = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle \cdot \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle = 0$$

according to the Chain Rule for Paths. ◀

Practice 2. Find the rate of change of $f(x, y) = x^4 y^3 + 3x^2 y$ with respect to t along the path $\mathbf{r}(t) = \left\langle t^2 + t - 5, t^3 - 2t^2 - t + 4 \right\rangle$ when $t = 1$.

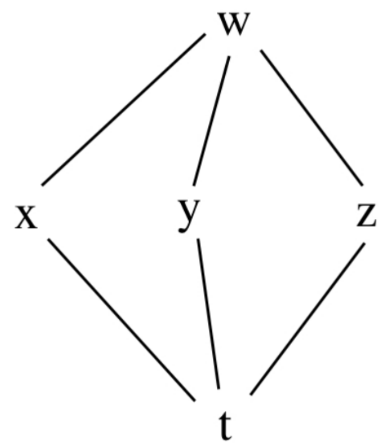
If we write $w = f(x, y)$ with $x = u(t)$ and $y = v(t)$, then the Chain Rule for Paths says that: $w'(t) = \nabla f(u(t), v(t)) \cdot \langle u'(t), v'(t) \rangle$, which we can rewrite as:

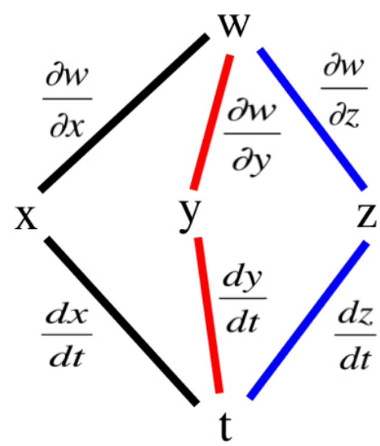
$$w'(t) = f_x(u(t), v(t)) \cdot u'(t) + f_y(u(t), v(t)) \cdot v'(t)$$

or, using Leibniz notation: $\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$. And if f is a function of three variables, we can write:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

To help keep track of the correct pattern for the Leibniz version of the Chain Rule, use a tree diagram (see margin): here w depends on x, y and z, while x depends on t, y depends on t and z depends on t. Then multiply the corresponding derivatives along each of the paths and add the results.





Example 4. Find $\frac{df}{dt}$ for $f(x,y,z) = xy + z$ along the helix $x(t) = \cos(t)$, $y(t) = \sin(t)$, $z(t) = t$ when $t = 0$.

Solution. Computing the various derivatives, $\frac{\partial f}{\partial x} = y$, $\frac{\partial f}{\partial y} = x$, $\frac{\partial f}{\partial z} = 1$, $\frac{dx}{dt} = -\sin(t)$, $\frac{dy}{dt} = \cos(t)$ and $\frac{dz}{dt} = 1$, so:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} = (y)\,(-\sin(t)) + (x)\,(\cos(t)) + (1)(1)$$

If $t = 0$ then $x(0) = 1$, $y(0) = 0$ and $z(0) = 0$, hence $\left.\frac{df}{dt}\right|_{t=0} = (0)(0) + (1)(1) + (1)(1) = 2$. ◀

Practice 3. Find $\frac{df}{dt}$ for the functions in Example 4 when $t = \frac{\pi}{3}$.

Practice 4. In Example 4, if the units of t are seconds, the units of x, y and z are meters and the units of f are °C, what are the units of $f'(t)$?

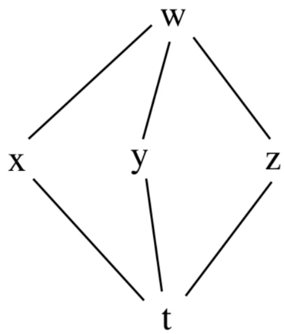
More General Chain Rules

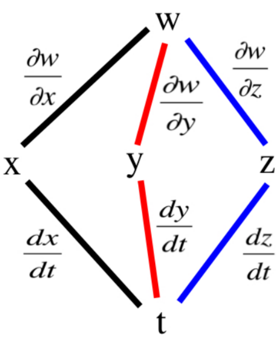
The Chain Rule for Paths lets you find the rate of change of a function $f(x,y)$ with respect to a variable t when x and y are each functions of t. But what if x and y are functions of two (or more) variables?

If $w = f(x,y)$, $x = u(s,t)$ and $y = v(s,t)$ then w is a function of s and t, so you can compute both $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ by treating t (and then s) as a constant and applying the Chain Rule for Paths:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s}$$
$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Example 5. If $w = f(x,y,z)$, $x = \varphi(r,s)$, $y = \gamma(r,s)$, and $z = \psi(r)$, with f, φ, γ and ψ all being differentiable, write valid Leibniz-style chain rule expressions for $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$.





Solution. Referring to the diagram in the margin, there are three paths from w to r:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

and two paths from w to s:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s}$$

As before, drawing a diagram can help keep track of the correct pattern for these more general versions of the multivariable chain rule. ◀

Practice 5. If $T = f(x,y,z)$, $x = \varphi(p,q,r)$, $y = \gamma(p,q)$ and $z = \psi(p,r)$, with f, φ, γ and ψ all being differentiable, write valid Leibniz-style chain rule expressions for $\frac{\partial T}{\partial p}$ and $\frac{\partial T}{\partial q}$.

Example 6. The voltage V in a circuit satisfies Ohm's Law, $V = IR$.

(a) Write a chain rule for expression for $\frac{dV}{dt}$.

(b) If the voltage V is dropping (because a battery is wearing out) and the resistance R is increasing (because the circuit is heating up), at what rate is the current I changing when $R = 500$ ohms, $I = 0.04$ amps, $\frac{dR}{dt} = 0.5$ ohms/sec and $\frac{dV}{dt} = -0.01$ volt/sec?

Solution. (a) $\frac{dV}{dt} = \frac{\partial V}{\partial I} \cdot \frac{\partial I}{\partial t} + \frac{\partial V}{\partial R} \cdot \frac{\partial R}{\partial t}$

(b) $\frac{\partial V}{\partial I} = R = 500$ ohms and $\frac{\partial V}{\partial R} = I = 0.04$ amps so:

$$-0.01 = 500 \cdot \frac{\partial I}{\partial t} + (0.04)(0.5)$$

and solving this equation yields $\frac{\partial I}{\partial t} = -0.00006\,\frac{\text{amps}}{\text{sec}}$. ◀

14.6 Problems

In Problems 1–6, express $\frac{df}{dt}$ as a function of t, then evaluate this derivative at the given value of t.

1. $f(x,y) = x^2 + y^2$, $x = \cos(t)$, $y = \sin(t)$, $t = \pi$
2. $f(x,y) = x^2y^2 + 3x + 4y$, $x = t^2$, $y = 1 + t$, $t = 2$
3. $f(x,y,z) = x^2y + yz + xz$, $x = 3 + 2t$, $y = t^2$, $z = 5t$, $t = 2$
4. $f(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, $x = 1 + 2t$, $y = 2 + 3t$, $z = 3 + 4t$, $t = 1$
5. $f(x,y,z) = 2ye^x - \ln(z)$, $x = \ln\left(t^2 + 1\right)$, $y = \arctan(t)$, $z = e^t$, $t = 1$
6. $f(x,y,z) = xyz + 374$, $x = 2\cos(t)$, $y = 2\sin(t)$, $z = 3t$, $t = \pi$

In Problems 7–8, use these values for $\frac{\partial f}{\partial x}$:

y				
3	7	8	5	13
2	3	5	4	11
1	1	9	7	10
0	5	4	6	2
	0	1	2	3 x

these values for $\frac{\partial f}{\partial y}$:

y				
3	5	2	9	7
2	1	4	5	6
1	6	1	8	3
0	3	7	2	4
	0	1	2	3 x

these values for x and y:

t	1	2	3	4
x	2	0	1	3
y	3	1	0	2

and these values for $\frac{dx}{dt}$ and $\frac{dy}{dt}$:

t	1	2	3	4
$\frac{dx}{dt}$	-1	5	-2	6
$\frac{dy}{dt}$	-3	7	-1	8

7. Calculate $\dfrac{df}{dt}$ when $t = 1$ and $t = 3$.
8. Calculate $\dfrac{df}{dt}$ when $t = 2$ and $t = 4$.
9. If $w = xy + yz + xz$, $x = u + v$, $y = u - v$ and $z = uv$, express $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$ as functions of u and v, then evaluate each at the point $(u, v) = (-2, 0)$.
10. If $w = (x + y + z)^2$, $x = r - s$, $y = \cos(r + s)$ and $z = \sin(r + s$, find $\dfrac{\partial w}{\partial r}$ at the point where $r = 1$ and $s = -1$.
11. If $z = \cos(x + y) + x \cdot \sin(y)$, $x = u + v + 2$ and $y = uv$, find $\dfrac{\partial z}{\partial u}$ at the point $(u, v) = (0, 0)$.
12. The lengths a, b and c of the edges of a rectangular box are changing. At one moment in time, $a = 1$ m, $b = 2$ m, $c = 3$ m, $\frac{da}{dt} = \frac{db}{dt} = 1\,\frac{\text{m}}{\text{sec}}$ and $\frac{dc}{dt} = 3\,\frac{\text{m}}{\text{sec}}$. At this moment:
    (a) At what rate is the box's volume changing?
    (b) At what rate is its surface area changing?
    (c) Are its interior diagonals increasing or decreasing in length?
13. In a certain ideal gas, the pressure P (in kilopascals, kPa), the volume V (in liters, L) and the temperature T (in °Kelvin, K) satisfy the equation $PV = 8.31T$. At what rate is the pressure changing at the moment when the temperature is 310 °K and decreasing at a rate of 0.2 $\frac{°\text{K}}{\text{sec}}$, and the volume is 80 L and increasing at a rate of 0.1 $\frac{\text{L}}{\text{sec}}$?
14. You know: w is a function of x, y and z; x is function of r; y is a function of r and s; z is a function of s and t; and s is a function of t. Use a tree diagram to write a Chain Rule formula for $\dfrac{\partial w}{\partial t}$.

14.6 Practice Answers

1. $\varphi(x, y, z) = \arcsin(3x + 4y - 7z) = F(g(x, y, z))$ where $F(t)\arcsin(t)$ and $g(x, y, z) = 3x + 4y - 7z$, so differentiating yields:

$$F'(t) = \frac{1}{\sqrt{1 - t^2}} \qquad \text{and} \qquad \nabla g(x, y, z) = \langle 3, 4, -7 \rangle$$

hence:

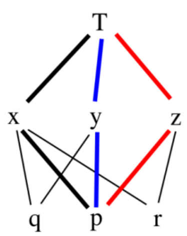
$$\begin{aligned}\nabla \varphi(x, y, z) &= F'(g(x, y, z)) \cdot \nabla g(x, y, z) \\ &= \frac{1}{\sqrt{1 - (3x + 4y - 7z)^2}} \langle 3, 4, -7 \rangle \\ &= \left\langle \frac{3}{\sqrt{1 - (3x + 4y - 7z)^2}}, \frac{4}{\sqrt{1 - (3x + 4y - 7z)^2}}, \frac{-7}{\sqrt{1 - (3x + 4y - 7z)^2}} \right\rangle\end{aligned}$$

2. $\mathbf{r}(t) = \left\langle t^2 + t - 5, t^3 - 2t^2 - t + 4 \right\rangle \Rightarrow \mathbf{r}'(t) = \left\langle 2t + 1, 3t^2 - 4t - 1 \right\rangle$ so $\mathbf{r}(1) = \langle -3, 2 \rangle$ and $\mathbf{r}'(1) = \langle 3, -2 \rangle$ while $f(x,y) = x^4y^3 + 3x^2y \Rightarrow \nabla f(x,y) = \left\langle 4x^3y^3 + 6xy, 3x^4y^2 + 3x^2 \right\rangle \Rightarrow \nabla f(3,-2) = \langle -900, 999 \rangle$:

$$\left.\frac{df}{dt}\right|_{t=1} = \nabla f(\mathbf{r}(1)) \cdot \mathbf{r}'(1) = \nabla f(3,-2) \cdot \mathbf{r}'(1) = $$
$$= \langle -900, 999 \rangle \cdot \langle 3, -2 \rangle = -4698$$

3. If $t = \frac{\pi}{3}$ then $x(\frac{\pi}{3}) = \frac{1}{2}$, $y(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ and $z(\frac{\pi}{3}) = \frac{\pi}{3}$, hence:

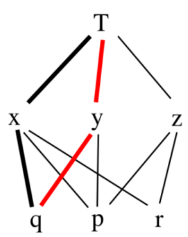
$$\left.\frac{df}{dt}\right|_{t=\frac{\pi}{3}} = (\frac{\sqrt{3}}{2})(-\frac{\sqrt{3}}{2}) + (\frac{1}{2})(\frac{1}{2}) + (1)(1) = \frac{1}{2}$$



4. $\frac{^\circ\mathrm{C}}{\mathrm{m}} \cdot \frac{\mathrm{m}}{\mathrm{sec}} + \frac{^\circ\mathrm{C}}{\mathrm{m}} \cdot \frac{\mathrm{m}}{\mathrm{sec}} + \frac{^\circ\mathrm{C}}{\mathrm{m}} \cdot \frac{\mathrm{m}}{\mathrm{sec}} = \frac{^\circ\mathrm{C}}{\mathrm{sec}}$

5. Referring the tree diagrams in the margin:

$$\frac{\partial T}{\partial p} = \frac{\partial T}{\partial x} \cdot \frac{\partial x}{\partial p} + \frac{\partial T}{\partial y} \cdot \frac{\partial y}{\partial p} + \frac{\partial T}{\partial z} \cdot \frac{\partial x}{\partial p}$$
$$\frac{\partial T}{\partial q} = \frac{\partial T}{\partial x} \cdot \frac{\partial x}{\partial q} + \frac{\partial T}{\partial y} \cdot \frac{\partial y}{\partial q}$$



(Note that z does not depend on q.)