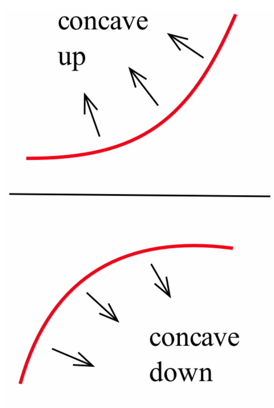
## 3.4 *The Second Derivative and the Shape of $f$*

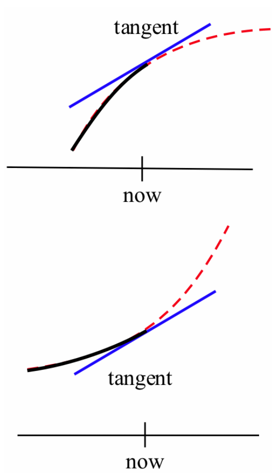
The first derivative of a function provides information about the shape of the function, so the second derivative of a function provides information about the shape of the first derivative, which in turn will provide additional information about the shape of the original function $f$.

In this section we investigate how to use the second derivative (and the shape of the first derivative) to reach conclusions about the shape of the original function. The first derivative tells us whether the graph of $f$ is increasing or decreasing. The second derivative will tell us about the "concavity" of $f$: whether $f$ is curving upward or downward.




### *Concavity*

Graphically, a function is **concave up** if its graph is curved with the opening upward (see margin); similarly, a function is **concave down** if its graph opens downward. The concavity of a function can be important in applied problems and can even affect billion-dollar decisions.




**An Epidemic**: Suppose you, as an official at the CDC, must decide whether current methods are effectively fighting the spread of a disease — or whether more drastic measures are required. In the margin figure, $f(x)$ represents the number of people infected with the disease at time $x$ in two different situations. In both cases the number of people with the disease, $f(\text{now})$, and the rate at which new people are getting sick, $f'(\text{now})$, are the same. The difference is the concavity of $f$, and that difference might have a big effect on your decision. In (a), $f$ is concave down at "now," and it appears that the current methods are starting to bring the epidemic under control; in (b), $f$ is concave up, and it appears that the epidemic is growing out of control.

Usually it is easy to determine the concavity of a function by examining its graph, but we also need a definition that does not require a graph of the function, a definition we can apply to a function described by a formula alone.

**Definition**: Let $f$ be a differentiable function.

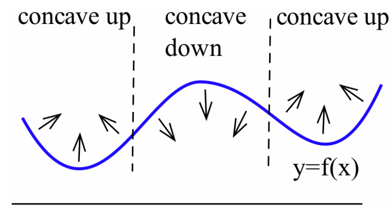
- $f$ is **concave up** at $a$ if the graph of $f$ is above the tangent line $L$ to $f$ for all $x$ close to (but not equal to) $a$:

$$f(x) > L(x) = f(a) + f'(a)(x-a)$$

- $f$ is **concave down** at $a$ if the graph of $f$ is below the tangent line $L$ to $f$ for all $x$ close to (but not equal to) $a$:

$$f(x) < L(x) = f(a) + f'(a)(x-a)$$

The margin figure shows the concavity of a function at several points. The next theorem provides an easily applied test for the concavity of a function given by a formula.

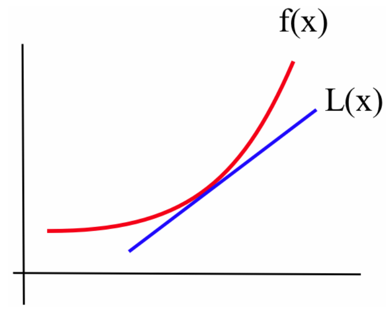



**The Second Derivative Condition for Concavity**:
Let $f$ be a twice differentiable function on an interval $I$.

(a) $f''(x) > 0$ on $I \;\Rightarrow\; f'(x)$ increasing on $I \;\Rightarrow\; f$ concave up on $I$

(b) $f''(x) < 0$ on $I \;\Rightarrow\; f'(x)$ decreasing on $I \;\Rightarrow\; f$ concave down on $I$

(c) $f''(a) = 0 \;\Rightarrow$ no information
($f(x)$ may be concave up or concave down or neither at $a$)




*Proof.* (a) Assume that $f''(x) > 0$ for all $x$ in $I$, and let $a$ be any point in $I$. We want to show that $f$ is concave up at $a$, so we need to prove that the graph of $f$ (see margin) is above the tangent line to $f$ at $a$: $f(x) > L(x) = f(a) + f'(a)(x-a)$ for $x$ close to $a$. Assume that $x$ is in $I$ and apply the Mean Value Theorem to $f$ on the interval with endpoints $a$ and $x$: there is a number $c$ between $a$ and $x$ so that

$$f'(c) = \frac{f(x) - f(a)}{x - a} \;\Rightarrow f(x) = f(a) + f'(c)(x-a)$$

Because $f'' > 0$ on $I$, we know that $f'' > 0$ between $a$ and $x$, so the Second Shape Theorem tells us that $f'$ is increasing between $a$ and $x$. We will consider two cases: $x > a$ and $x < a$.

- If $x > a$ then $x - a > 0$ and $c$ is in the interval $[a, x]$ so $a < c$. Because $f'$ is increasing, $a < c \Rightarrow f'(a) < f'(c)$. Multiplying each side of this last inequality by the positive quantity $x - a$ yields $f'(a)(x-a) < f'(c)(x-a)$. Adding $f(a)$ to each side of this last inequality, we have:

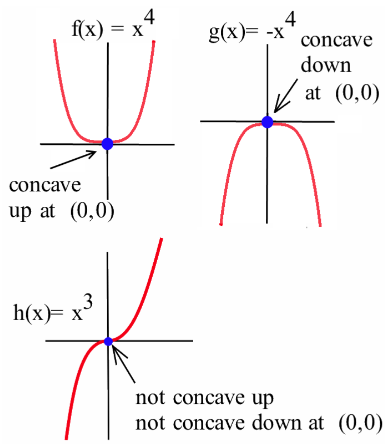
$$L(x) = f(a) + f'(a)(x-a) < f(a) + f'(c)(x-a) = f(x)$$

- If $x < a$ then $x - a < 0$ and $c$ is in the interval $[x, a]$ so $c < a$. Because $f'$ is increasing, $c < a \Rightarrow f'(c) < f'(a)$. Multiplying each side of this last inequality by the negative quantity $x - a$ yields $f'(c)(x-a) > f'(a)(x-a)$ so:

$$f(x) = f(a) + f'(c)(x-a) > f(a) + f'(a)(x-a) = L(x)$$

In each case we see that $f(x)$ is above the tangent line $L(x)$.

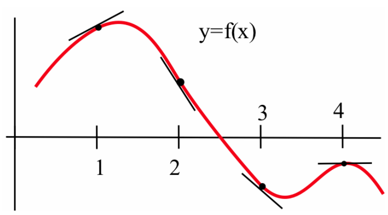
(b) The proof of this part is similar.




(c) Let $f(x) = x^4$, $g(x) = -x^4$ and $h(x) = x^3$ (see margin). The second derivative of each of these functions is zero at $a = 0$, and at $(0, 0)$ they all have the same tangent line: $L(x) = 0$ (the $x$-axis). However, at $(0, 0)$ they all have different concavity: $f$ is concave up, while $g$ is concave down and $h$ is neither concave up nor concave down. ☐

**Practice 1.** Use the graph of $f$ in the lower margin figure to finish filling in the table with "$+$" for positive, "$-$" for negative or "0."

| $x$ | $f(x)$ | $f'(x)$ | $f''(x)$ | concavity |
|---|---|---|---|---|
| 1 | $+$ | $+$ | $-$ | down |
| 2 | $+$ | | | |
| 3 | $-$ | | | |
| 4 | | | | |




## *"Feeling" the Second Derivative*

Earlier we saw that if a function $f(t)$ represents the position of a car at time $t$, then $f'(t)$ gives the velocity and $f''(t)$ the acceleration of the car at the instant $t$.

If we are driving along a straight, smooth road, then what we *feel* is the acceleration of the car:

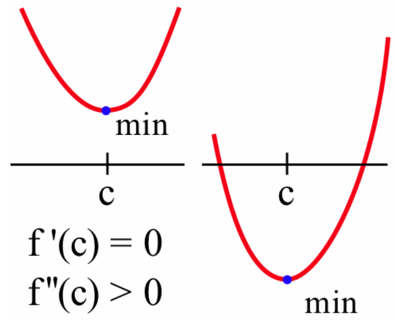
- a large positive acceleration feels like a "push" toward the back of the car
- a large negative acceleration (a deceleration) feels like a "push" toward the front of the car
- an acceleration of 0 for a period of time means the velocity is constant and we do not feel pushed in either direction

In a moving vehicle it is possible to measure these "pushes," the acceleration, and from that information to determine the velocity of the vehicle, and from the velocity information to determine the position. Inertial guidance systems in airplanes use this tactic: they measure front–back, left–right and up–down acceleration several times a second and then calculate the position of the plane. They also use computers to keep track of time and the rotation of the earth under the plane. After all, in six hours the Earth has made a quarter of a revolution, and Dallas has rotated more than 5,000 miles!

**Example 1.** The upward acceleration of a rocket was $a(t) = 30$ m/sec$^2$ during the first six seconds of flight, $0 \le t \le 6$. The velocity of the rocket at $t = 0$ was 0 m/sec and the height of the rocket above the ground at $t = 0$ was 25 m. Find a formula for the height of the rocket at time $t$ and determine the height at $t = 6$ seconds.

**Solution.** $v'(t) = a(t) = 30 \Rightarrow v(t) = 30t + K$ for some constant K. We also know $v(0) = 0$ so $30(0) + K = 0 \Rightarrow K = 0$ and this $v(t) = 30t$.

Similarly, $h'(t) = v(t) = 30t \Rightarrow h(t) = 15t^2 + C$ for some constant $C$. We know that $h(0) = 25$ so $15(0)^2 + C = 25 \Rightarrow C = 25$. Thus $h(t) = 15t^2 + 25$ so $h(6) = 15(6)^2 + 25 = 565$ m. ◀




### *$f''$ and Extreme Values of $f$*

The concavity of a function can also help us determine whether a critical point is a maximum or minimum or neither. For example, if a point is at the bottom of a concave-up function then that point is a minimum.

> **The Second Derivative Test for Extremes**:
> Let $f$ be a twice differentiable function.
>
> (a) If $f'(c) = 0$ and $f''(c) < 0$ then $f$ is concave down and has a local maximum at $x = c$.
>
> (b) If $f'(c) = 0$ and $f''(c) > 0$ then $f$ is concave up and has a local minimum at $x = c$.
>
> (c) If $f'(c) = 0$ and $f''(c) = 0$ then $f$ may have a local maximum, a local minimum or neither at $x = c$.

*Proof.* (a) Assume that $f'(c) = 0$. If $f''(c) < 0$ then $f$ is concave down at $x = c$ so the graph of $f$ will be below the tangent line $L(x)$ for values of $x$ near $c$. The tangent line, however, is given by $L(x) = f(c) + f'(c)(x - c) = f(c) + 0(x - c) = f(c)$, so if $x$ is close to $c$ then $f(x) < L(x) = f(c)$ and $f$ has a local maximum at $x = c$.

(b) The proof for a local minimum of $f$ is similar.

(c) If $f'(c) = 0$ and $f''(c) = 0$, then we cannot immediately conclude anything about local maximums or minimums of $f$ at $x = c$. The functions $f(x) = x^4$, $g(x) = -x^4$ and $h(x) = x^3$ all have their first and second derivatives equal to zero at $x = 0$, but $f$ has a local minimum at 0, $g$ has a local maximum at 0, and $h$ has neither a local maximum nor a local minimum at $x = 0$. □

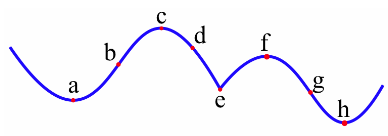
The Second Derivative Test for Extremes is very useful when $f''$ is easy to calculate and evaluate. Sometimes, however, the First Derivative Test — or simply a graph of the function — provides an easier way to determine if the function has a local maximum or a local minimum: it depends on the function and on which tools you have available.

**Practice 2.** $f(x) = 2x^3 - 15x^2 + 24x - 7$ has critical numbers $x = 1$ and $x = 4$. Use the Second Derivative Test for Extremes to determine whether $f(1)$ and $f(4)$ are maximums or minimums or neither.

*Inflection Points*

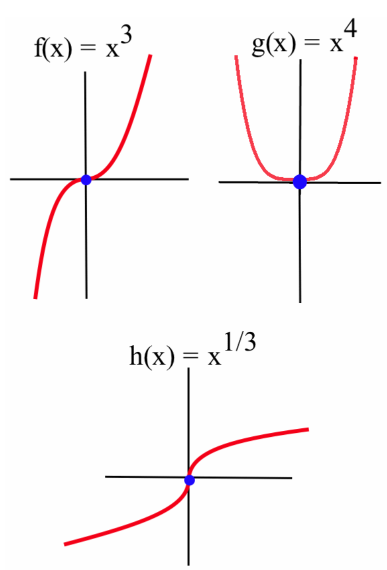
Maximums and minimums typically occur at places where the second derivative of a function is positive or negative, but the places where the second derivative is 0 are also of interest.

> **Definition**:
>
> An **inflection point** is a point on the graph of a function where the concavity of the function changes, from concave up to concave down or from concave down to concave up.




**Practice 3.** Which of the labeled points in the margin figure are inflection points?

To find the inflection points of a function we can use the second derivative of the function. If $f''(x) > 0$, then the graph of $f$ is concave up at the point $(x, f(x))$ so $(x, f(x))$ is not an inflection point. Similarly, if $f''(x) < 0$ then the graph of $f$ is concave down at the point $(x, f(x))$ and the point is not an inflection point. The only points left that can possibly be inflection points are the places where $f''(x) = 0$ or where $f''(x)$ does not exist (in other words, where $f'$ is not differentiable). To find the inflection points of a function we need only check the points where $f''(x)$ is 0 or undefined. If $f''(c) = 0$ or is undefined, then the point $(c, f(c))$ **may or may not** be an inflection point—we need to check the concavity of $f$ on each side of $x = c$. The functions in the next example illustrate what can happen at such a point.




**Example 2.** Let $f(x) = x^3$, $g(x) = x^4$ and $h(x) = \sqrt[3]{x}$ (see margin). For which of these functions is the point $(0, 0)$ an inflection point?

**Solution.** Graphically, it is clear that the concavity of $f(x) = x^3$ and $h(x) = \sqrt[3]{x}$ changes at $(0, 0)$, so $(0, 0)$ is an inflection point for $f$ and $h$. The function $g(x) = x^4$ is concave up everywhere, so $(0, 0)$ is not an inflection point of $g$.

$f(x) = x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow f''(x) = 6x$ so the only point at which $f''(x) = 0$ or is undefined ($f'$ is not differentiable) is at $x = 0$. If $x < 0$ then $f''(x) < 0$ so $f$ is concave down; if $x > 0$ then $f''(x) > 0$ so f is concave up. Thus at $x = 0$ the concavity of $f$ changes so the point $(0, f(0)) = (0, 0)$ is an inflection point of $f(x) = x^3$.

$g(x) = x^4 \Rightarrow g'(x) = 4x^3 \Rightarrow g''(x) = 12x^2$ so the only point at which $g''(x) = 0$ or is undefined is at $x = 0$. But $g''(x) > 0$ (so $g$ is concave up) for any $x \neq 0$. Thus the concavity of $g$ never changes, so the point $(0, g(0)) = (0, 0)$ is not an inflection point of $g(x) = x^4$.

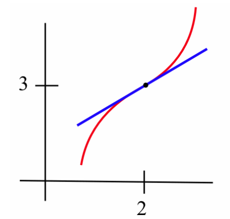
$h(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \Rightarrow h'(x) = \dfrac{1}{3}x^{-\frac{2}{3}} \Rightarrow h''(x) = -\dfrac{2}{9}x^{-\frac{5}{3}}$ so $h''$ is not defined if $x = 0$ (and $h''(x) \neq 0$ elsewhere); $h''(\text{negative number}) > 0$

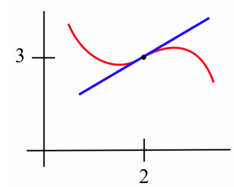
and $h''(\text{positive number}) < 0$, so $h$ changes concavity at $(0, 0)$ and $(0, 0)$ is an inflection point of $h(x) = \sqrt[3]{x}$. ◀

**Practice 4.** Find all inflection points of $f(x) = x^4 - 12x^3 + 30x^2 + 5x - 7$.

**Example 3.** Sketch a graph of a function with $f(2) = 3$, $f'(2) = 1$ and an inflection point at $(2, 3)$.

**Solution.** Two solutions appear in the margin. ◀

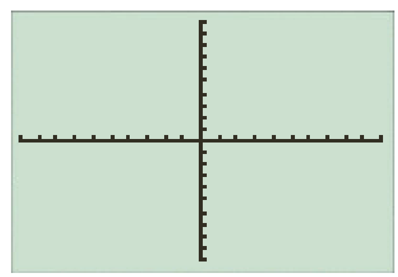



## *Using $f'$ and $f''$ to Graph $f$*

Today you can easily graph most functions of interest using a graphing calculator—and create even nicer graphs using an app on your phone or a Web-based graphing utility. Earlier generations of calculus students did not have these tools, so they relied on calculus to help them draw graphs of unfamiliar functions by hand. While you can create a graph in seconds that your predecessors may have labored over for half an hour or longer, you can still use calculus to help you select an appropriate graphing "window," and to be confident that your window has not missed any points of interest on the graph of a function.

**Example 4.** Create a graph of $f(x) = xe^{-9x^2}$ that shows all local and global extrema and all inflection points.



**Solution.** If you graph $f(x)$ on a calculator using the standard window ($-10 \leq x \leq 10$ and $-10 \leq y \leq 10$) you will likely see nothing other than the coordinate axes (see margin). You might consult a table of values for the function to help adjust the window, but this trial-and-error technique will still not guarantee that you have displayed all points of interest. Computing the first derivative of $f$, we get:

$$f'(x) = x\left[-18xe^{-9x^2}\right] + e^{-9x^2} \cdot 1 = \left[1 - 18x^2\right]e^{-9x^2}$$

which is defined for all values of $x$; $f'(x) = 0 \Rightarrow 1 - 18x^2 = 0 \Rightarrow x^2 = \frac{1}{18} \Rightarrow x = \pm\frac{1}{3\sqrt{2}}$, so the only critical numbers are $x = -\frac{1}{3\sqrt{2}}$ and $x = \frac{1}{3\sqrt{2}}$. Computing the second derivative of $f$, we get:

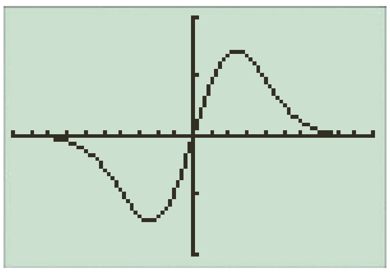
$$\begin{aligned} f''(x) &= \left[1 - 18x^2\right] \cdot \left[-18xe^{-9x^2}\right] + e^{-9x^2} \cdot [-36x] \\ &= \left[324x^3 - 54x\right]e^{-9x^2} = 54x\left[6x^2 - 1\right]e^{-9x^2} \end{aligned}$$

We can check that $f''\left(-\frac{1}{3\sqrt{2}}\right) = 12\sqrt{\frac{2}{e}} > 0$, so $f$ must have a local minimum at $x = -\frac{1}{3\sqrt{2}}$; similarly, $f''\left(\frac{1}{3\sqrt{2}}\right) = -12\sqrt{\frac{2}{e}} < 0$, so $f$ must have a local maximum at $x = \frac{1}{3\sqrt{2}}$.

Furthermore, $f''(x) = 0$ only when $x = 0$ or when $6x^2 - 1 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{6}}$, so these three values are candidates for locations of inflection points of $f$. Noting that:

$$-1 < -\frac{1}{\sqrt{6}} < -\frac{1}{3\sqrt{2}} < 0 < \frac{1}{3\sqrt{2}} < \frac{1}{\sqrt{6}} < 1$$

and that $f''(-1) = -270e^{-9} < 0$ and $f''\left(-\frac{1}{3\sqrt{2}}\right) = 12\sqrt{\frac{2}{e}} > 0$, we observe that $f$ is concave down to the left of $x = -\frac{1}{\sqrt{6}}$ and concave up to the right of $x = -\frac{1}{\sqrt{6}}$, so $f$ does in fact have an inflection point at $x = -\frac{1}{\sqrt{6}}$. Likewise, $f''\left(\frac{1}{3\sqrt{2}}\right) = -12\sqrt{\frac{2}{e}} < 0$ and $f''(1) = 270e^{-9} > 0$, so $f''(x)$ switches sign at $x = 0$ and at $x = \frac{1}{\sqrt{6}}$, and therefore $f(x)$ changes concavity at those points as well.



We have now identified two local extrema of $f$ and three inflection points of $f$. Equally important, we have used calculus to show that these five points of interest are the *only* places where extrema or inflection points can occur. If we create a graph of $f$ that includes these five points, our graph is guaranteed to include all "interesting" features of the graph of $f$. A window with $-1 \le x \le 1$ and $-0.2 < y < 0.2$ (because the local extreme values are $f\left(\pm\frac{1}{2\sqrt{3}}\right) \approx \pm 0.14$) should provide a graph (see margin) that includes all five points of interest. ◀

**Practice 5.** Compute the first and second derivatives of the function $g(x) = x^4 + 4x^3 - 90x^2 + 13$, locate all extrema and inflection points of $g(x)$, and create a graph of $g(x)$ that shows these points of interest.

Most problems in calculus textbooks are set up to make solving these equations relatively straightforward, but in general this will not be the case.

Even with calculus, we will typically need calculators or computers to help solve the equations $f'(x) = 0$ and $f''(x) = 0$ that we use to find critical numbers and candidates for inflection points.

## *3.4 Problems*

In Problems 1–2, each statement describes a quantity $f(t)$ changing over time. For each statement, tell what $f$ represents and whether the first and second derivatives of $f$ are positive or negative.

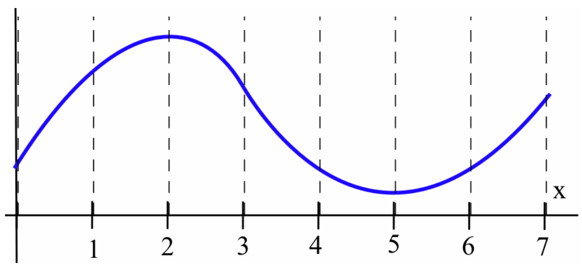
1. (a) "Unemployment rose again, but the rate of increase is smaller than last month."
   (b) "Our profits declined again, but at a slower rate than last month."
   (c) "The population is still rising and at a faster rate than last year."

2. (a) "The child's temperature is still rising, but more slowly than it was a few hours ago."
   (b) "The number of whales is decreasing, but at a slower rate than last year."
   (c) "The number of people with the flu is rising and at a faster rate than last month."

3. Sketch the graphs of functions that are defined and concave up everywhere and have exactly: (a) no roots. (b) 1 root. (c) 2 roots. (d) 3 roots.

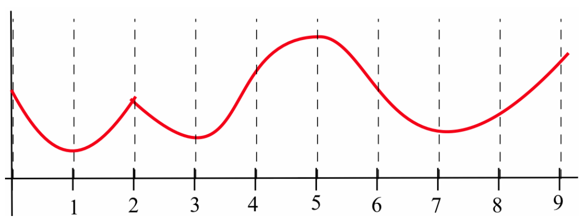
4. On which intervals is the function graphed below:

(a) concave up? (b) concave down?



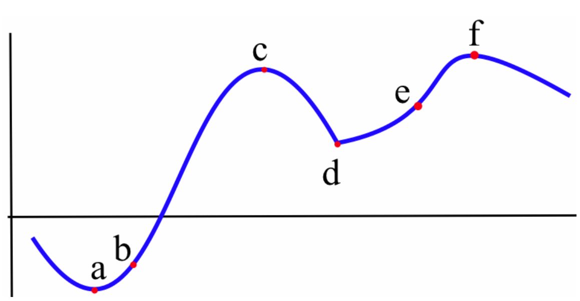

5. On which intervals is the function graphed below:

(a) concave up? (b) concave down?

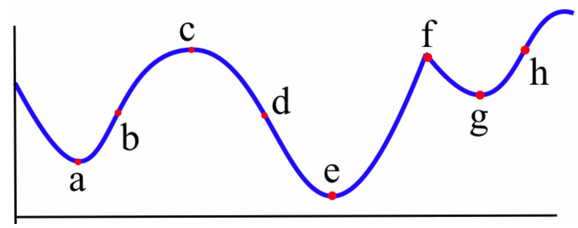



Problems 6–10 give a function and values of $x$ so that $f'(x) = 0$. Use the Second Derivative Test to determine whether each point $(x, f(x))$ is a local maximum, a local minimum or neither.

6. $f(x) = 2x^3 - 15x^2 + 6$; $x = 0, 5$
7. $g(x) = x^3 - 3x^2 - 9x + 7$; $x = -1, 3$
8. $h(x) = x^4 - 8x^2 - 2$; $x = -2, 0, 2$
9. $f(x) = \sin^5(x)$; $x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$
10. $f(x) = x \cdot \ln(x)$; $x = \frac{1}{e}$
11. At which values of $x$ labeled in the figure below is the point $(x, f(x))$ an inflection point?

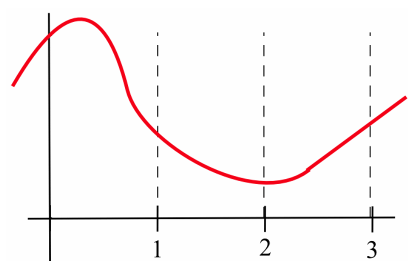



12. At which values of $x$ labeled in the figure below is the point $(x, g(x))$ an inflection point?



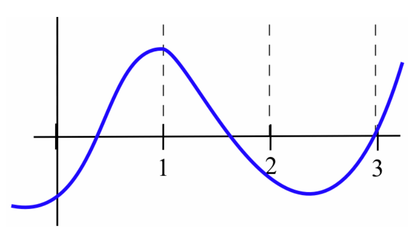

13. How many inflection points can a:
    (a) quadratic polynomial have?
    (b) cubic polynomial have?
    (c) polynomial of degree $n$ have?
14. Fill in the table with "+," "−,"or "0" for the function graphed below.

| $x$ | $f(x)$ | $f'(x)$ | $f''(x)$ |
|---|---|---|---|
| 0 | | | |
| 1 | | | |
| 2 | | | |
| 3 | | | |

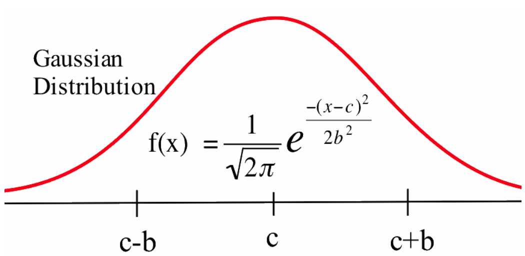



15. Fill in the table with "+," "−,"or "0" for the function graphed below.

| $x$ | $g(x)$ | $g'(x)$ | $g''(x)$ |
|---|---|---|---|
| 0 | | | |
| 1 | | | |
| 2 | | | |
| 3 | | | |



16. Sketch functions $f$ for $x$-values near 1 so that $f(1) = 2$ and:
    (a) $f'(1) > 0$, $f''(1) > 0$
    (b) $f'(1) > 0$, $f''(1) < 0$
    (c) $f'(1) < 0$, $f''(1) > 0$
    (d) $f'(1) > 0$, $f''(1) = 0$, $f''(1^-) < 0$, $f''(1^+) > 0$
    (e) $f'(1) > 0$, $f''(1) = 0$, $f''(1^-) > 0$, $f''(1^+) < 0$
17. Some people like to think of a concave-up graph as one that will "hold water" and of a concave-down graph as one which will "spill water." That description is accurate for a concave-down graph, but it can fail for a concave-up graph. Sketch the graph of a function that is concave up on an interval but will not "hold water."
18. The function $f(x) = \frac{1}{2\pi} e^{-\frac{(x-c)^2}{2b^2}}$ defines the **Gaussian distribution** used extensively in statistics and probability; its graph (see below) is a "bell-shaped" curve.



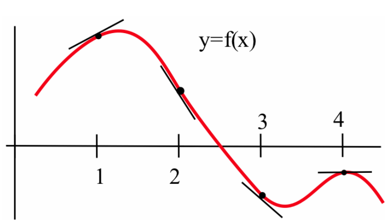

    (a) Show that $f$ has a maximum at $x = c$. (The value $c$ is called the **mean** of this distribution.)
    (b) Show that $f$ has inflection points where $x = c + b$ and $x = c - b$. (The value $b$ is called the **standard deviation** of this distribution.)

In Problems 19–36, locate all critical numbers, local extrema and inflection points of the given function, and use these results to sketch a graph of the function showing all points of interest.

19. $f(x) = x^3 - 21x^2 + 144x - 350$
20. $g(x) = \frac{1}{6}x^3 + x^2 - \frac{45}{2}x + 100$
21. $f(x) = e^{7x} - 5x$
22. $g(x) = e^{7x} - 5x$
23. $f(x) = e^{-3x} + x$
24. $g(x) = e^{-3x} - x$
25. $f(x) = xe^{-3x}$
26. $g(x) = xe^{5x}$
27. $f(x) = x^{\frac{4}{3}} - x^{\frac{1}{3}}$
28. $g(x) = 6x^{\frac{4}{3}} + 3x^{\frac{1}{3}}$
29. $f(x) = \ln\left(1 + x^2\right)$
30. $g(x) = \ln\left(x^2 - 6x + 10\right)$
31. $f(x) = \sqrt[3]{x^2 + 2x + 2}$
32. $g(x) = \sqrt{x^2 + 2x + 2}$
33. $f(x) = x^{\frac{2}{3}}(1 - x)^{\frac{1}{3}}$
34. $g(x) = x^{\frac{1}{3}}(1 - x)^{\frac{2}{3}}$
35. $f(\theta) = \sin(\theta) + \sin^2(\theta)$
36. $g(\theta) = \cos(\theta) - \sin^2(\theta)$

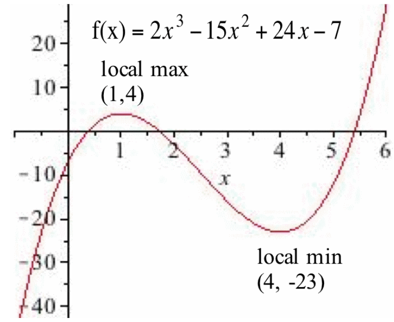
## *3.4 Practice Answers*

1. See the margin figure for reference.




| $x$ | $f(x)$ | $f'(x)$ | $f''(x)$ | concavity |
|---|---|---|---|---|
| 1 | + | + | – | down |
| 2 | + | – | – | down |
| 3 | – | – | + | up |
| 4 | – | 0 | – | down |

2. $f'(x) = 6x^2 - 30x + 24$, which is defined for all $x$. $f'(x) = 0$ if $x = 1$ or $x = 4$ (critical values). $f''(x) = 12x - 30$ so $f''(1) = -18 < 0$ tells us that $f$ is concave down at the critical value $x = 1$, so $(1, f(1)) = (1, 4)$ is a relative maximum; and $f''(4) = 18 > 0$ tells us that $f$ is concave up at the critical value $x = 4$, so $(4, f(4)) = (4, -23)$ is a relative minimum. A graph of $f$ appears in the margin.

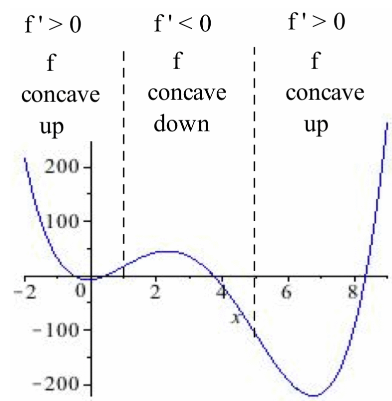



3. The points labeled b and g are inflection points.

4. $f'(x) = 4x^3 - 36x^2 + 60x + 5 \Rightarrow f''(x) = 12x^2 - 72x + 60 = 12(x^2 - 6x + 5) = 12(x-1)(x-5)$ so the only candidates to be inflection points are $x = 1$ and $x = 5$.

   - If $x < 1$ then $f''(x) = 12(\text{neg})(\text{neg}) > 0$
   - If $1 < x < 5$ then $f''(x) = 12(\text{pos})(\text{neg}) < 0$
   - If $5 < x$ then $f''(x) = 12(\text{pos})(\text{pos}) > 0$

   $f$ changes concavity at $x = 1$ and $x = 5$, so $x = 1$ and $x = 5$ are both inflection points. A graph of $f$ appears in the margin.




f(x) = $x^4 - 12x^3 + 30x^2 + 5x - 7$

5. $g(x) = x^4 + 4x^3 - 90x^2 + 13 \Rightarrow g'(x) = 4x^3 + 12x^2 - 180x \Rightarrow g''(x) = 12x^2 + 24x - 180$; because $g'(x)$ and $g''(x)$ are polynomials, they exist everywhere. The critical numbers for $g(x)$ occur where $g'(x) = 0 \Rightarrow 4x^3 + 12x^2 - 180x = 4x(x^2 + 3x - 45) = 4x(x+9)(x-5) = 0 \Rightarrow x = -9$, $x = 0$ or $x = 5$. Using the Second Derivative Test: $g''(-9) = 576 > 0$, so $g(x)$ has a local minimum at $x = -9$; $g''(0) = -180 < 0$, so $g(x)$ has a local maximum at $x = 0$; and $g''(5) = 240 > 0$, so $g(x)$ has a local minimum at $x = 5$.

   Candidates for inflection points occur where $g''(x) = 0$:

$$12x^2 + 24x - 180 = 12(x^2 + 2x - 15) = 12(x-3)(x+5) = 0$$
$$\Rightarrow x = -5 \text{ or } x = 3$$

   Observing that $g''(x) > 0$ for $x < -5$, $g''(x) < 0$ for $-5 < x < 3$ and $g''(x) > 0$ for $x > 3$ confirms that both candidates are in fact inflection points. A graphing window with $-12 \leq x \leq 8$ (this is only one reasonable possibility) should include all points of interest. Checking that $g(-9) = -3632$, $g(0) = 13$ and $g(5) = -1112$ suggests that a graphing window with $-4000 \leq y \leq 1000$ should work (see margin).